2017

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Volume 3

# MATHEMATICS FOR JEE (MAIN & ADVANCED)

## CALCULUS


Dr. G S N MURTI
Dr. K P R SASTRY

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WILEY-INDIA

# MATHEMATICS FOR IIT-JEE

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Mathematics for IIT-JEE | CALCULUS

VOL. 3



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John Wiley & Sons Canada Ltd, 22 Worcester Road, Etobicoke, Ontario, Canada, M9W 1L1

First Edition: 2012

ISBN: 978-81-265-2184-5

ISBN: 978-81-265-8060-6 (ebk)

www.wileyindia.com

Printed at: *Sanat Printers, Kundli, Haryana.*

# Dedication

*Dedicated to*
*my maternal uncle and his wife*

**Sri. Adibhatla Sreekantarka Sastry and Smt. Savitramma**

*who took the responsibility of my five years of high school education.*
*They treated me on par with their children.*

Dr. G. S. N. Murti

# Acknowledgments

**1.** I am very thankful to my 12th class students of 2011–2012 batch, namely Mr. Akhil, Mr. Pavan, Mr. Vamsi, Mr. Girish, Mr. Harish and Mr. Bhargav for their proofreading of this Volume III without original manuscript.

**2.** I am very much thankful to my wife Smt. Balamba for her constant encouragement.

Dr. G. S. N. Murti

# Features and Benefits at a Glance

| Feature | Benefit to student |
|---|---|
| **Chapter Opener** | Peaks the student's interest with the chapter opening vignette, definitions of the topic, and contents of the chapter. |
| **Clear, Concise, and Inviting Writing Style, Tone and Layout** | Students are able to **Read** this book, which reduces math anxiety and encourages student success. |
| **Theory and Applications** | Unlike other books that provide very less or no theory, here theory is well matched with solved examples. |
| **Theorems** | Relevant theorems are provided along with proofs to emphasize conceptual understanding. |
| **Solved Examples** | Topics are followed by solved examples for students to practice and understand the concept learned. |
| **Examples** | Wherever required, examples are provided to aid understanding of definitions and theorems. |
| **Quick Look** | Formulae/concepts that do not require extensive thought but can be looked at the last moment. |
| **Try It Out** | Practice problems for students in between the chapter. |
| **Worked Out Problems** | Based on IIT-JEE pattern problems are presented in the form of<br>**Single Correct Choice Type Questions**<br>**Multiple Correct Choice Type Questions**<br>**Matrix-Match Type Questions**<br>**Comprehension-Type Questions**<br>**Assertion–Reasoning Type Questions**<br>**Integer Answer Type Questions**<br>In-depth solutions are provided to all problems for students to understand the logic behind. |
| **Summary** | Key formulae, ideas and theorems are presented in this section in each chapter. |
| **Exercises** | Offer self-assessment. The questions are divided into subsections as per requirements of IIT-JEE. |
| **Answers** | Answers are provided for all exercise questions for student's to validate their solution. |

# Note to the Students

The IIT-JEE is one of the hardest exams to crack for students, for a very simple reason – concepts cannot be learned by rote, they have to be absorbed, and IIT believes in strong concepts. Each question in the IIT-JEE entrance exam is meant to push the analytical ability of the student to its limit. That is why the questions are called brainteasers!

Students find Mathematics the most difficult part of IIT-JEE. We understand that it is difficult to get students to love mathematics, but one can get students to love succeeding at mathematics. In order to accomplish this goal, the book has been written in clear, concise, and inviting writing style. It can be used as a self-study text as theory is well supplemented with examples and solved examples. Wherever required, figures have been provided for clear understanding.

If you take full advantage of the unique features and elements of this textbook, we believe that your experience will be fulfilling and enjoyable. Let's walk through some of the special book features that will help you in your efforts to crack IIT-JEE.

To crack mathematics paper for IIT-JEE the five things to remember are:

**1. Understanding the concepts**

**2. Proper applications of concepts**

**3. Practice**

**4. Speed**

**5. Accuracy**

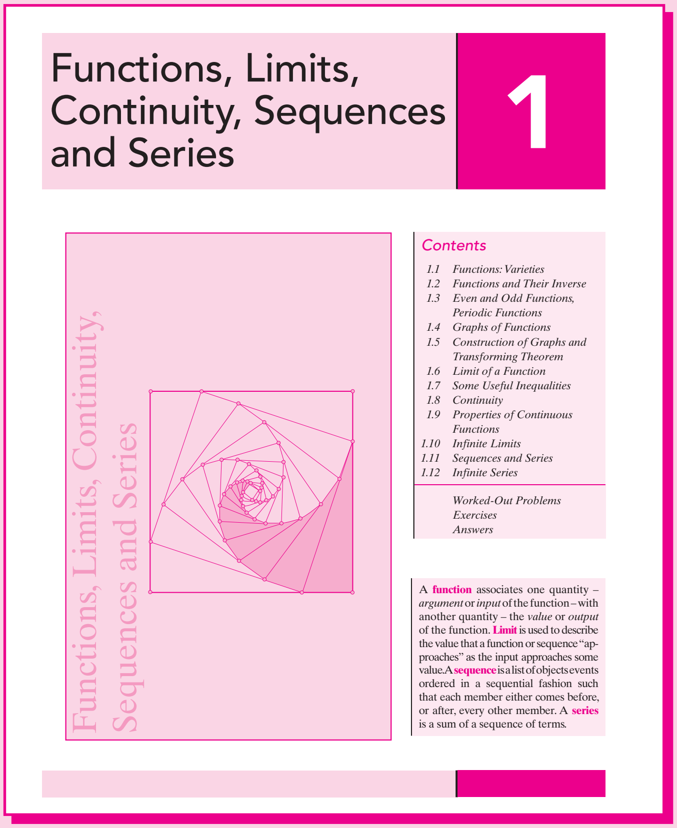
# About the Cover Picture

The picture on the cover is a Double Spiral Staircase at Saint Peters Basilica, Vatican City. It is a classic example of double helix structure. A helix (is a type of smooth space curve, i.e. a curve in three-dimensional space. It has the property that the tangent line at any point makes a constant angle with a fixed line called the axis. Examples of helixes are coil springs and the handrails of spiral staircases. The word helix comes from the Greek word ἕλιξ meaning twisted, curved.

# A. PEDAGOGY

## CHAPTER OPENER

Each chapter starts with an opening vignette, definition of the topic, and contents of the chapter that give you an overview of the chapter to help you see the big picture.



Functions, Limits, Continuity, Sequences and Series

1

Functions, Limits, Continuity, Sequences and Series

Contents

1.1 Functions: Varieties
1.2 Functions and Their Inverse
1.3 Even and Odd Functions, Periodic Functions
1.4 Graphs of Functions
1.5 Construction of Graphs and Transforming Theorem
1.6 Limit of a Function
1.7 Some Useful Inequalities
1.8 Continuity
1.9 Properties of Continuous Functions
1.10 Infinite Limits
1.11 Sequences and Series
1.12 Infinite Series

Worked-Out Problems
Exercises
Answers

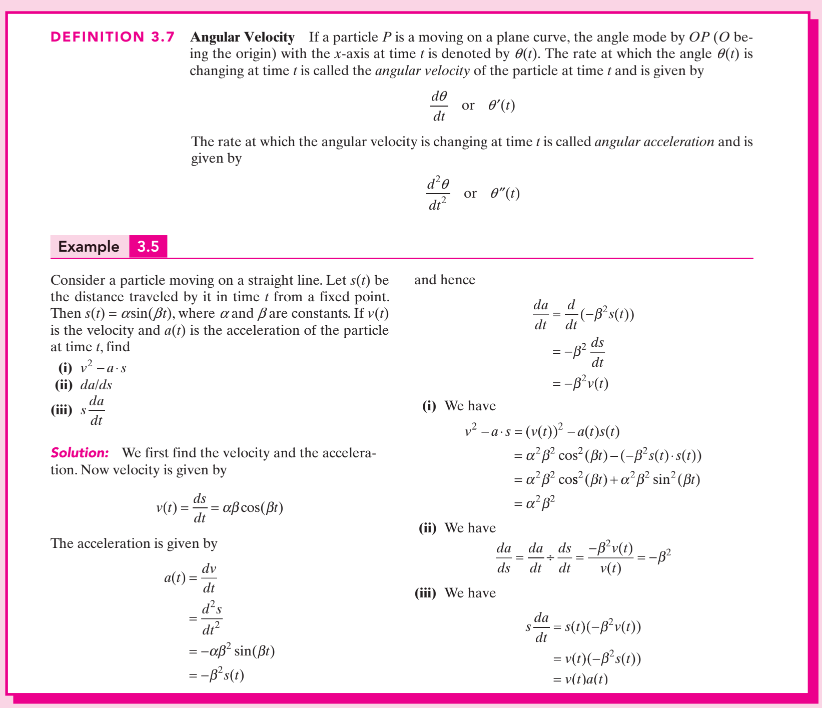
A **function** associates one quantity – *argument* or *input* of the function – with another quantity – the *value* or *output* of the function. **Limit** is used to describe the value that a function or sequence "approaches" as the input approaches some value. A **sequence** is a list of objects events ordered in a sequential fashion such that each member either comes before, or after, every other member. A **series** is a sum of a sequence of terms.

## CLEAR, CONCISE, AND INVITING WRITING

Special attention has been paid to present an engaging, clear, precise narrative in the layout that is easy to use and designed to reduce math anxiety students may have.

## DEFINITIONS

Every new topic or concept starts with defining the concept for students. Related examples to aid the understanding follow the definition.



**DEFINITION 3.7** **Angular Velocity** If a particle $P$ is a moving on a plane curve, the angle mode by $OP$ ($O$ being the origin) with the $x$-axis at time $t$ is denoted by $\theta(t)$. The rate at which the angle $\theta(t)$ is changing at time $t$ is called the *angular velocity* of the particle at time $t$ and is given by

$$\frac{d\theta}{dt} \quad \text{or} \quad \theta'(t)$$

The rate at which the angular velocity is changing at time $t$ is called *angular acceleration* and is given by

$$\frac{d^2\theta}{dt^2} \quad \text{or} \quad \theta''(t)$$

**Example 3.5**

Consider a particle moving on a straight line. Let $s(t)$ be the distance traveled by it in time $t$ from a fixed point. Then $s(t) = \alpha\sin(\beta t)$, where $\alpha$ and $\beta$ are constants. If $v(t)$ is the velocity and $a(t)$ is the acceleration of the particle at time $t$, find

**(i)** $v^2 - a \cdot s$
**(ii)** $da/ds$
**(iii)** $s\dfrac{da}{dt}$

***Solution:*** We first find the velocity and the acceleration. Now velocity is given by

$$v(t) = \frac{ds}{dt} = \alpha\beta\cos(\beta t)$$

The acceleration is given by

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d^2 s}{dt^2} \\ &= -\alpha\beta^2 \sin(\beta t) \\ &= -\beta^2 s(t) \end{aligned}$$

and hence

$$\begin{aligned} \frac{da}{dt} &= \frac{d}{dt}(-\beta^2 s(t)) \\ &= -\beta^2 \frac{ds}{dt} \\ &= -\beta^2 v(t) \end{aligned}$$

**(i)** We have

$$\begin{aligned} v^2 - a \cdot s &= (v(t))^2 - a(t)s(t) \\ &= \alpha^2\beta^2\cos^2(\beta t) - (-\beta^2 s(t) \cdot s(t)) \\ &= \alpha^2\beta^2\cos^2(\beta t) + \alpha^2\beta^2\sin^2(\beta t) \\ &= \alpha^2\beta^2 \end{aligned}$$

**(ii)** We have

$$\frac{da}{ds} = \frac{da}{dt} \div \frac{ds}{dt} = \frac{-\beta^2 v(t)}{v(t)} = -\beta^2$$

**(iii)** We have

$$\begin{aligned} s\frac{da}{dt} &= s(t)(-\beta^2 v(t)) \\ &= v(t)(-\beta^2 s(t)) \\ &= v(t)a(t) \end{aligned}$$

# EXAMPLES

Examples pose a specific problem using concepts already presented and then work through the solution. These serve to enhance the students' understanding of the subject matter.

**Example 3.6**

Consider Fig. 3.4. $OAB$ is a right-angled triangle, right angled at $A$. Suppose $P$ is a moving point on $AB$ with uniform velocity $v$. Find the angular velocity of $P$ with respect to $O$.

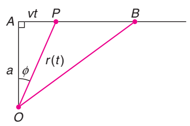



**FIGURE 3.4** Example 3.6.

***Solution:*** Since the point $P$ is moving with uniform velocity $v$ we have $AP = vt$ (since $ds = vdt$ and $v$ is constant, $s = vt$). Let $\theta(t)$ be the angle made by $OP$ with $OA$ at time $t$. Let $OA = a$, $OP = rt$ and $\angle AOP = \theta(t)$. Then

$$a = r(t)\cos(\theta(t))$$

and $$[r(t)]^2 = OP^2 = OA^2 + AP^2 = a^2 + v^2t^2 \qquad (3.1)$$

Differentiating both sides of Eq. (3.1) with respect to $t$, we get

$$2r(t)\frac{dr}{dt} = 2v^2t \qquad (3.2)$$

Differentiating both sides of the equation $a = r(t)\cos(\theta(t))$, w.r.t. $t$ we have

$$0 = \frac{dr}{dt}\cos(\theta(t)) + r(t)(-\sin(\theta(t))\frac{d\theta}{dt}$$
$$= \frac{dr}{dt}\cdot\frac{a}{r(t)} - r(t)\left(\frac{vt}{r(t)}\right)\frac{d\theta}{dt}$$
$$= \frac{dr}{dt}\cdot\frac{a}{r(t)} - (vt)\frac{d\theta}{dt}$$
$$= \frac{a}{\ }\cdot\frac{v\ t}{\ } - (vt)\frac{d\theta}{\ } \quad [\text{from Eq. (3.2)}]$$

Therefore

$$\frac{d\theta}{dt} = \frac{a}{[r(t)]^2}\left(\frac{v^2t}{vt}\right)$$
$$= \frac{av}{(r(t))^2}$$
$$= \frac{av}{(OP)^2}$$

Thus the angular velocity of $P$ w.r.t. $O$ is $av/(OP)^2$.

# THEOREMS

Relevant theorems are provided along with proofs to emphasize conceptual understanding rather than rote learning.

**Theorem 1.1 (Associative Law for Composition of Function)** Suppose $A$, $B$, $C$ and $D$ are non-empty subsets of $\mathbb{R}$ and $f: A \to B$, $g: B \to C$ and $h: C \to D$ are functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$.

**Proof** First of all, observe that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are functions from $A$ to $D$. Further, for $x \in A$,

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x))$$
$$= h(g(f(x))$$
$$= h((g \circ f)(x))$$
$$= (h \circ (g \circ f))(x)$$

■

***Note:*** Theorem 1.1 is given as Theorem 1.27 (*Try it out*) in Chapter 1 (Vol. 1) on p. 41. The proof also runs on the same lines as above.

# QUICK LOOK

Some important formulae and concepts that do not require exhaustive explanation, but their mention is important, are presented in this section. These are marked with a **magnifying glass.**

**QUICK LOOK 7**

If $f:[a,b] \to \mathbb{R}$ has absolute minimum and absolute maximum, then $f$ is bounded on $[a, b]$.

## TRY IT OUT

Within each chapter the students would find problems to reinforce and check their understanding. This would help build confidence as one progresses in the chapter. These are marked with a **pointed finger**.

**Try it out** If

$$f(x+h) = f(x) + \frac{h}{\lfloor 1} f'(x) + \frac{h^2}{\lfloor 2} f''(x+\theta h)$$

where $0 < \theta < 1$, then for

$$f(x) = ax^3 + bx^2 + cx + d$$

and $a \neq 0$, the value of $\theta$ is 1/3.

**23.** Consider the function $f(x) = 4x^3 - 12x, x \in [-1, 3]$. Then

(A) $f$ has local maximum at $x = -1$
(B) $f$ has local minimum at $x = 1$
(C) the image of the interval $[-1, 3]$ under the function is $[-8, 72]$
(D) $f$ has no extremum value in $[-1, 3]$

**Solution:** Differentiating the given function we get

$$f'(x) = 12x^2 - 12$$

Now

$$f'(x) = 0 \Leftrightarrow x = -1, 1$$

$$f'(x) = \begin{cases} 12x^2 + 2x - 2 & \text{for } 0 \le x \le 2 \\ 12x^2 - 2x + 2 & \text{for } 2 < x \le 3 \end{cases}$$

So

$$f'(2-0) = 48 + 4 - 2 = 50$$

and $$f'(2+0) = 48 - 4 + 2 = 46$$

Therefore, $f$ is not differentiable at $x = 2$. Hence 2 is a critical point. Now $0 \le x \le 2$ and $f'(x) = 0$ implies

$$6x^2 + x - 1 = 0$$
$$\Rightarrow (3x-1)(2x+1) = 0$$
$$\Rightarrow x = 1/3, -1$$

So $x = 1/3$ is a critical point. Now $2 \le x \le 3$ and $f'(x) = 0$ implies

$$6x^2 - x + 1 = 0$$

But $6x^2 - x + 1 = 0$ has no real roots. Thus, the only critical points are 1/3, 2.

**(i)** Also, $x < 1/3 \Rightarrow$ the sign of $f'(x)$ is $(-)(+) < 0$ and $x > 1/3 \Rightarrow f'(x) > 0$. Hence $f$ has local minimum at $x = 1/3$.
**(ii)** $f$ is decreasing for $x < 1/3$ and increasing for $x > 1/3$.

## SUMMARY

At the end of every chapter, a summary is presented that organizes the key formulae and theorems in an easy to use layout. The related topics are indicated so that one can quickly summarize a chapter.

### SUMMARY

**2.1** ***Derivative:*** Suppose $f: [a, b] \to IR$ is a function and $a < c < b$. If $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists, then we say that $f$ is differentiable at "$c$" and this limit is denoted by $f'(c)$ derivative or differential coefficient at $c$. If we write $y = f(x)$, then $f'(c)$ is also denoted by $(dy/dx)_{x=c}$. If $f$ is differentiable at each point of $(a, b)$ then we say that $f$ is differentiable in $(a, b)$ and $f'(x)$ or $(dy/dx)$ is called derivative or derived function of $f(x)$.

**QUICK LOOK**

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists and is equivalent to

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

**2.** The converse of Theorem 2.3 is not true; for example, take $f(x) = |x|$ at $c = 0$.

**3.** The function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(3^n x)$$

is continuous for all real $x$, but not differentiable at any $x \in \mathbb{R}$.

**2.4** Suppose $f$ and $g$ are differentiable at $c$, and $\lambda$ and $\mu$ be any two real numbers. Then

**(i)** $\lambda f + \mu g$ is differentiable at $c$ and

$$(\lambda f + \mu g)'(c) = \lambda f'(c) + \mu g'(c)$$

**(ii)** $fg$ is differentiable at $c$ and

$$(fg)'(c) = f'(c)g(c) + g'(c)f(c)$$

**(iii)** If $g(c) \neq 0$, then $f/g$ is differentiable at $c$ and

## B. WORKED-OUT PROBLEMS AND ASSESSMENT – AS PER IIT-JEE PATTERN

Mere theory is not enough. It is also important to practice and test what has been proved theoretically. The worked-out problems and exercise at the end of each chapter are in resonance with the IIT-JEE paper pattern. Keeping the IIT-JEE pattern in mind, the worked-out problems and exercises have been divided into:

1. **Single Correct Choice Type Questions**
2. **Multiple Correct Choice Type Questions**
3. **Matrix-Match Type Questions**
4. **Comprehension-Type Questions**
5. **Assertion–Reasoning Type Questions**
6. **Integer Answer Type Questions**

## WORKED-OUT PROBLEMS

In-depth solutions are provided to all worked-out problems for students to understand the logic behind and formula used.

### WORKED-OUT PROBLEMS

*Single Correct Choice Type Questions*

1. $\displaystyle\int \frac{x^m - 1}{x^{m+1}\sqrt{1-2x^m+mx^{2m}}}dx = \frac{f(x)}{mx^m}+c$ where $f(x)$ is equal to

(A) $\sqrt{1-2x^m+mx^{2m}}$ (B) $\sqrt{1-x^m+x^{2m}}$

(C) $\sqrt{x^m+\dfrac{1}{x^m}}$ (D) $\sqrt{x^{2m}+\dfrac{1}{x^{2m}}}$

**Solution:** Let

$$I = \int \frac{x^m-1}{x^{m+1}\sqrt{1-2x^m+mx^{2m}}}dx$$

Dividing numerator and denominator with $x^{2m+1}$ we get

$$I = \int \frac{\dfrac{1}{x^{m+1}} - \dfrac{1}{x^{2m+1}}}{\sqrt{\dfrac{1}{x^{2m}} - \dfrac{2}{x^m} + m}}dx$$

**Solution:** Let

$$I = \int \frac{dx}{(x^2-4)\sqrt{x+1}}$$

Put $\sqrt{x+1} = t$ so that $dx = 2t\,dt$. Therefore

$$I = \int \frac{1}{\left[(t^2-1)^2-4\right]t}(2t)\,dt$$

$$= \int \frac{2}{(t^2-1-2)(t^2-1+2)}dt$$

$$= 2\int \frac{dt}{(t^2-3)(t^2+1)}$$

$$= \frac{1}{2}\int\left(\frac{1}{t^2-3} - \frac{1}{t^2+1}\right)dt$$

$$= \frac{1}{2}\int\left(\frac{dt}{t^2-3}\right) - \frac{1}{2}\int\frac{dt}{t^2+1}$$

## SINGLE CORRECT CHOICE TYPE QUESTIONS

These are the regular multiple choice questions with four choices provided. Only one among the four choices will be the correct answer.

## MULTIPLE CORRECT CHOICE TYPE QUESTIONS

Multiple correct choice type questions have four choices provided, but one or more of the choices provided may be correct.

*Multiple Correct Choice Type Questions*

1. Let

$$f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$$

where [ ] denotes the greatest integer function. Then

(A) domain of $f$ is $\mathbb{R}-[-1,0)$ (B) $\lim\limits_{x\to 0+0} f(x) = 0$

(C) $f$ is continuous on $[0, 1)$ (D) $\lim\limits_{x\to 1+0} f(x) = 1$

**Solution:**

(A) $f$ is not defined for all those values of $x$ such that

$$[x+1] = 0 \Leftrightarrow 0 \le x+1 < 1$$
$$\Leftrightarrow -1 \le x < 0$$

Therefore domain of $f$ is $\mathbb{R}-[1,0)$. This implies (A) is true.

(B) We have

$$\lim_{x\to 0+0} f(x) = \lim_{\substack{h\to 0\\ h>0}}(f(0+h))$$

$$= \lim_{\substack{h\to 0\\ h>0}}[0+h]\sin\left(\frac{\pi}{[0+h]+1}\right)$$

(A) $\lim\limits_{x\to 0} f(x)$ does not exist (B) $f$ is continuous at $x = 0$

(C) $\lim\limits_{x\to\infty} f(x) = 0$ (D) $\lim\limits_{x\to\infty} f(x) = 1$

**Solution:** Since $x \to 0$ and $\sin 1/x$ is a bounded function, by Corollary 1.4

$$x\sin\frac{1}{x} \to 0 \text{ as } x \to 0$$

Therefore

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

So (B) is true. Now put $x = 1/y$ so that $y \to 0$ as $x \to \infty$. Therefore

$$f(x) = \frac{1}{y}\sin(y) = \frac{\sin y}{y} \to 1 \text{ as } y \to 0$$

So

$$\lim_{x\to\infty} f(x) = 1$$

Therefore (D) is also true.

# MATRIX-MATCH TYPE QUESTIONS

These questions are the regular "Match the Following" variety. Two columns each containing 4 subdivisions or first column with four subdivisions and second column with more subdivisions are given and the student should match elements of column I to that of column II. There can be one or more matches.

### *Matrix-Match Type Questions*

**1.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\lim_{x\to 0} \dfrac{2^x - 1}{\sqrt{1+x}-1}$ is | (p) 2 log 2 |
| (B) $f(x) = \dfrac{\sqrt{1+2x}-\sqrt{3x}}{\sqrt{3+x}-2\sqrt{x}}$. Then $\lim_{x\to 1} f(x)$ equals | (q) 1 |
| (C) $\lim_{x\to 0}\left(\dfrac{2\sin x - \sin 2x}{x^3}\right)$ is | (r) 2 |
| (D) $x_1 = 1$ and $x_{n+1} = \sqrt{2+x_n}$. Define $y_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ Then $\lim_{n\to\infty} y_n$ is | (s) $\dfrac{4}{3}(\sqrt{3}-\sqrt{2})$ |
| | (t) $\dfrac{2}{3\sqrt{3}}$ |

**Solution:**

(A) $$\lim_{x\to 0}\left(\frac{2^x-1}{\sqrt{1+x}-1}\right) = \lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\left(\frac{x}{\sqrt{1+x}-1}\right)$$

(C) $$\lim_{x\to 0}\frac{2\sin x - \sin 2x}{x^3} = \lim_{x\to 0}\frac{2\sin x}{x}\left(\frac{1-\cos x}{x^2}\right)$$
$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{4\sin^2\frac{x}{2}}{x^2}\right)$$
$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$
$$= 1\times 1 = 1$$

**Answer: (C) → (q)**

(D) $x_1 = 1$, $x_2 = \sqrt{3} \Rightarrow x_1 < x_2$. Assume that $x_n < x_{n+1}$. Therefore

$$x_{n+2} = \sqrt{2+x_{n+1}}$$
$$\Rightarrow x_{n+2} - x_{n+1} = \sqrt{2+x_{n+1}} - \sqrt{2+x_n}$$

which is positive, because $x_n < x_{n+1}$. So $\{x_n\}$ is an increasing sequence and bounded above by 2. By Theorem 1.42, $\{x_n\}$ converges to a finite limit, say $L$. So

$$L = \lim (x_{n+1}) = \lim \sqrt{2+x_n} = \sqrt{2+\text{L}}$$

# COMPREHENSION-TYPE QUESTIONS

Comprehension-type questions consist of a small passage, followed by three multiple choice questions. The questions are of single correct answer type.

### *Comprehension-Type Questions*

**1. Passage:** If $f$ is continuous on closed $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$. Answer the following three questions.

**(i)** If in the passage, $f(a) = f(b) = 0$, then the equation $f'(x) + \lambda f(x) = 0$ has

(A) solutions for all real $\lambda$
(B) no solution for any real $\lambda$
(C) exactly one solution for all real $\lambda$
(D) solution only for $\lambda = 1$

**(ii)** If

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are reals, then the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

has a root in

(A) $(-\infty, -1)$ (B) $(-1, 0)$
(C) $(0, 1)$ (D) $(1, \infty)$

**(iii)** The number of values of $c$ in Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ in the interval $[-\sqrt{2}, \sqrt{2}]$ is

(A) 0 (B) 1
(C) 2 (D) 3

**Solution:**

**(i)** Let $\phi(x) = e^{\lambda x} f(x)$ so that

(a) $\phi$ is continuous on $[a, b]$,
(b) differentiable in $(a, b)$,
(c) $\phi(a) = \phi(b)$ ($\because f(a) = f(b) = 0$)

Therefore by Rolle's theorem $\phi'(c) = 0$ for some $c \in (a, b)$. Then

$$e^{\lambda c}[\lambda f(c) + f'(c)] = 0$$
$$\Rightarrow f'(c) + \lambda f(c) = 0$$

## ASSERTION–REASONING TYPE QUESTIONS

These questions check the analytical and reasoning skills of the students. Two statements are provided – Statement I and Statement II. The student is expected to verify if (a) both statements are true and if both are true, verify if statement I follows from statement II; (b) both statements are true and if both are true, verify if statement II is not the correct reasoning for statement I; (c), (d) which of the statements is untrue.

### *Assertion–Reasoning Type Questions*

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation Statement I.

(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.

(C) Statement I is true and Statement II is false.

(D) Statement I is false and Statement II is true.

**1. Statement I:** $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x}$ is a bounded function.

**Statement II:** Every continuous function defined on a closed interval is bounded.

**2. Statement I:** If $-1 < x^1$, than $\lim\limits_{x\to 0} x^n = 0$.

**Statement II:** For $n \geq 3$, $-x^2 \leq x^n \leq x^2$ where $-1 < x < 1$.

**3. Statement I:** Let $f(x)=\begin{cases} x\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Then $f$ is continuous at $x=0$.

**Statement II:** Let $a\in\mathbb{R}$. In $a$ neighbourhood of $a$ two functions $f$ and $g$ are defined such that $\lim\limits_{x\to a} f(x)=0$ and $g(x)$ is bounded. Then $\lim\limits_{x\to a} f(x)g(x)=0$.

**Hint:** See Corollary 1.3.

**4. Statement I:** If $n$ is a positive integer, then $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$.

**Statement II:**

$$\sin\theta+\sin 2\theta+\sin 3\theta+\cdots+\sin n\theta=\frac{\sin\left(\dfrac{n+1}{2}\right)\theta\sin\dfrac{n\theta}{2}}{\sin(\theta/2)}$$

## INTEGER-TYPE QUESTIONS

The questions in this section are numerical problems for which no choices are provided. The students are required to find the exact answers to numerical problems and enter the same in OMR sheets. Answers can be one-digit or two-digit numerals.

### *Integer Answer Type Questions*

**1.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $0<\alpha<1$. If

$$|f(x)-f(y)|\leq\alpha|x-y|\ \forall x,y\in\mathbb{R}$$

Then the number of points of $x\in\mathbb{R}$ such that the graph of $y=f(x)$ intersects the line $y=x$ is _______.

**Solution:** Let $x_0\in\mathbb{R}$. Now define

$$x_1=f(x_0),\ x_2=f(x_1),\ldots,x_{n+1}=f(x_n)\ldots$$

Consider the sequence $\{x_n\}$. By the construction of the sequence $\{x_n\}$ and the property that $f$ is satisfying, we get that

$$|x_n-x_{n+1}|\leq\alpha^n|x_0-x_1| \qquad (1.82)$$

Therefore for $m>n$, we have

$$|x_n-x_m|=|x_n-x_{n+1}|+|x_{n+1}-x_{n+2}| + \cdots + |x_{n+m-(n+1)}-x_{n+m-n}|$$
$$\leq\alpha^n|x_0-x_1|+\alpha^{n+1}|x_0-x_1|$$

$\Rightarrow x=y \quad (\because 1-\alpha>0 \text{ and } |x-y|\geq 0)$

**Answer: 1**

**2.** Let

$$f(x)=\begin{cases}\dfrac{3x+4\tan x}{x} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases}$$

If $f$ is continuous at $x=0$, then the value of $k$ is _______.

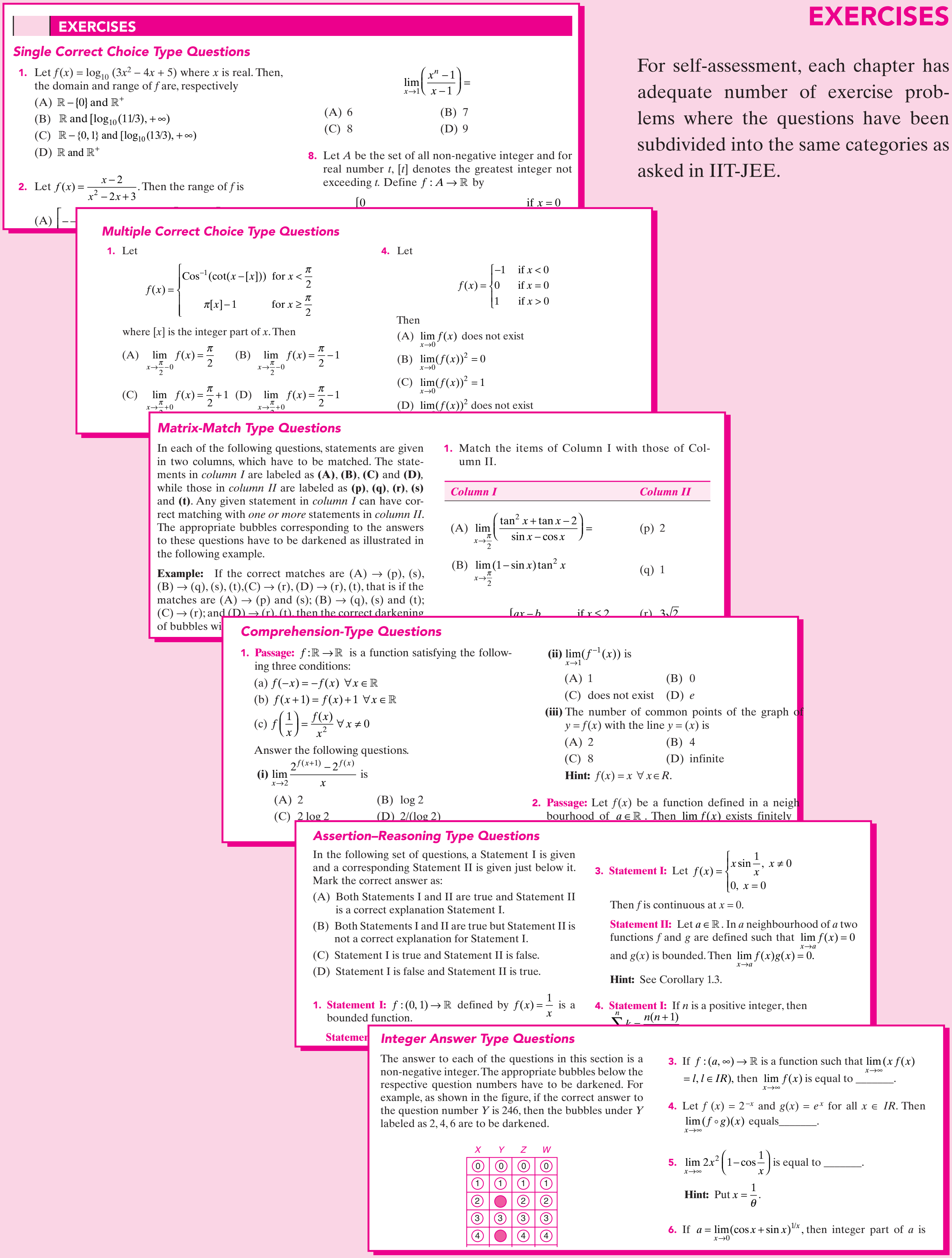
**Solution:** We have

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(3+\frac{4\tan x}{x}\right) = 3+4=7$$

Therefore $k=7$.

**Answer: 7**

**3.** Let $f:(-1,1)\to(-1,1)$ be continuous and $f(x)=f(x^2)$ for all $x\in(-1,1)$ and $f(0)=\frac{1}{2}$, then the value of



EXERCISES

Single Correct Choice Type Questions

1. Let $f(x) = \log_{10}(3x^2 - 4x + 5)$ where $x$ is real. Then, the domain and range of $f$ are, respectively
(A) $\mathbb{R} - \{0\}$ and $\mathbb{R}^+$
(B) $\mathbb{R}$ and $[\log_{10}(11/3), +\infty)$
(C) $\mathbb{R} - \{0, 1\}$ and $[\log_{10}(13/3), +\infty)$
(D) $\mathbb{R}$ and $\mathbb{R}^+$

2. Let $f(x) = \dfrac{x-2}{x^2-2x+3}$. Then the range of $f$ is

$\lim_{x \to 1}\left(\dfrac{x^n - 1}{x-1}\right) =$
(A) 6 (B) 7
(C) 8 (D) 9

8. Let $A$ be the set of all non-negative integer and for real number $t$, $[t]$ denotes the greatest integer not exceeding $t$. Define $f: A \to \mathbb{R}$ by

Multiple Correct Choice Type Questions

1. Let
$$f(x) = \begin{cases} \mathrm{Cos}^{-1}(\cot(x-[x])) & \text{for } x < \dfrac{\pi}{2} \\ \pi[x] - 1 & \text{for } x \geq \dfrac{\pi}{2} \end{cases}$$
where $[x]$ is the integer part of $x$. Then
(A) $\lim_{x \to \frac{\pi}{2}-0} f(x) = \dfrac{\pi}{2}$ (B) $\lim_{x \to \frac{\pi}{2}-0} f(x) = \dfrac{\pi}{2} - 1$
(C) $\lim_{x \to \frac{\pi}{2}+0} f(x) = \dfrac{\pi}{2} + 1$ (D) $\lim_{x \to \frac{\pi}{2}+0} f(x) = \dfrac{\pi}{2} - 1$

4. Let
$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$
Then
(A) $\lim_{x \to 0} f(x)$ does not exist
(B) $\lim_{x \to 0}(f(x))^2 = 0$
(C) $\lim_{x \to 0}(f(x))^2 = 1$
(D) $\lim(f(x))^2$ does not exist

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

**Example:** If the correct matches are (A) → (p), (s), (B) → (q), (s), (t), (C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t), then the correct darkening of bubbles wi

1. Match the items of Column I with those of Column II.

| Column I | Column II |
|---|---|
| (A) $\lim_{x \to \frac{\pi}{2}}\left(\dfrac{\tan^2 x + \tan x - 2}{\sin x - \cos x}\right) =$ | (p) 2 |
| (B) $\lim_{x \to \frac{\pi}{2}}(1 - \sin x)\tan^2 x$ | (q) 1 |
| | (r) $3\sqrt{2}$ |

Comprehension-Type Questions

1. **Passage:** $f: \mathbb{R} \to \mathbb{R}$ is a function satisfying the following three conditions:
(a) $f(-x) = -f(x)\ \forall x \in \mathbb{R}$
(b) $f(x+1) = f(x) + 1\ \forall x \in \mathbb{R}$
(c) $f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{x^2}\ \forall x \neq 0$
Answer the following questions.
**(i)** $\lim_{x \to 2} \dfrac{2^{f(x+1)} - 2^{f(x)}}{x}$ is
(A) 2 (B) log 2
(C) 2 log 2 (D) 2/(log 2)
**(ii)** $\lim_{x \to 1}(f^{-1}(x))$ is
(A) 1 (B) 0
(C) does not exist (D) $e$
**(iii)** The number of common points of the graph of $y = f(x)$ with the line $y = (x)$ is
(A) 2 (B) 4
(C) 8 (D) infinite
**Hint:** $f(x) = x\ \forall x \in R$.

2. **Passage:** Let $f(x)$ be a function defined in a neighbourhood of $a \in \mathbb{R}$. Then $\lim f(x)$ exists finitely

Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:
(A) Both Statements I and II are true and Statement II is a correct explanation Statement I.
(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.
(C) Statement I is true and Statement II is false.
(D) Statement I is false and Statement II is true.

1. **Statement I:** $f: (0, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is a bounded function.

3. **Statement I:** Let $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
Then $f$ is continuous at $x = 0$.
**Statement II:** Let $a \in \mathbb{R}$. In a neighbourhood of $a$ two functions $f$ and $g$ are defined such that $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded. Then $\lim_{x \to a} f(x)g(x) = 0$.
**Hint:** See Corollary 1.3.

4. **Statement I:** If $n$ is a positive integer, then

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number $Y$ is 246, then the bubbles under $Y$ labeled as 2, 4, 6 are to be darkened.

3. If $f: (a, \infty) \to \mathbb{R}$ is a function such that $\lim_{x \to \infty}(x f(x) = l, l \in IR)$, then $\lim_{x \to \infty} f(x)$ is equal to ______.

4. Let $f(x) = 2^{-x}$ and $g(x) = e^x$ for all $x \in IR$. Then $\lim_{x \to \infty}(f \circ g)(x)$ equals______.

5. $\lim_{x \to \infty} 2x^2\left(1 - \cos\dfrac{1}{x}\right)$ is equal to ______.
**Hint:** Put $x = \dfrac{1}{\theta}$.

6. If $a = \lim_{x \to 0}(\cos x + \sin x)^{1/x}$, then integer part of $a$ is

# EXERCISES

For self-assessment, each chapter has adequate number of exercise problems where the questions have been subdivided into the same categories as asked in IIT-JEE.

## SUBJECTIVE QUESTIONS

Since Calculus requires a lot of practice, some chapters in addition to providing numerous solved examples in IIT-JEE pattern, also give subjective questions as exercises.

### EXERCISES

To have a grip over integration, the student has to practice problems on various methods. That is why we are supplying the student with a number of problems for evaluation. Hence, the exercise contains only subjective problems.

**1.** Evaluate $\int \frac{xe^x}{(1+x)^2}dx$.

**2.** Evaluate $\int \frac{xe^x}{\sqrt{1+e^x}}dx$

**Hint:** Take $u = x, dv = \frac{e^x}{\sqrt{1+e^x}}$ so that $v = 2\sqrt{1+e^x}$ and use integration by parts.

**3.** Show that

$$\int \frac{dx}{\sqrt{1+e^x+e^{2x}}} = \log_e\left(\frac{-1+e^x+\sqrt{1+e^x+e^{2x}}}{1+e^x+\sqrt{1+e^x+e^{2x}}}\right)+c$$

**4.** Show that

**9.** Show that

$$\int x^3 \mathrm{Tan}^{-1}x\,dx = \frac{x^4-1}{4}\mathrm{Tan}^{-1}x - \frac{x^3}{12}+\frac{x}{4}+c$$

**Hint:** Use integration by parts.

**10.** Evaluate $\int \frac{\mathrm{Sin}^{-1}x}{\sqrt{1+x}}dx$.

**11.** Show that

$$\int \frac{\sqrt{x^2+1}}{x^4}[\log_e(x^2+1)-2\log_e x]dx = \frac{(x^2+1)^{3/2}}{9x^3}\left[2-3\log_e\left(1+\frac{1}{x^2}\right)\right]+c$$

**Hint:** Put $1+\frac{1}{x^2}=t$.

**12.** Compute $\int \sin x \log_e \tan x dx$.

## ANSWERS

The Answer key at the end of each chapter contains answers to all exercise problems.

### ANSWERS

#### *Single Correct Choice Type Questions*

**1.** (B)
**2.** (D)
**3.** (D)
**4.** (A)
**5.** (D)
**6.** (C)
**7.** (B)
**8.** (A)
**9.** (A)
**10.** (C)
**11.** (B)
**12.** (D)
**13.** (B)
**14.** (A)
**15.** (C)
**16.** (D)
**17.** (B)
**18.** (B)
**19.** (A)
**20.** (A)
**21.** (D)
**22.** (A)
**23.** (A)
**24.** (B)
**25.** (A)
**26.** (C)
**27.** (D)
**28.** (B)
**29.** (C)
**30.** (B)
**31.** (A)
**32.** (B)
**33.** (B)
**34.** (A)
**35.** (A)
**36.** (B)
**37.** (D)
**38.** (A)
**39.** (C)
**40.** (A)
**41.** (A)
**42.** (C)
**43.** (B)
**44.** (D)
**45.** (C)
**46.** (B)
**47.** (A)
**48.** (D)
**49.** (C)
**50.** (B)
**51.** (A)
**52.** (C)
**53.** (C)
**54.** (A)
**55.** (B)
**56.** (B)
**57.** (B)
**58.** (C)
**59.** (B)
**60.** (B)
**61.** (D)
**62.** (A)
**63.** (A)
**64.** (B)
**65.** (D)

#### *Multiple Correct Choice Type Questions*

**1.** (A), (D)
**2.** (B), (C), (D)
**3.** (A), (D)
**4.** (A), (C)
**5.** (A), (C), (D)
**6.** (B), (C), (D)
**7.** (A), (B), (C)
**8.** (A), (B), (C), (D)
**9.** (A), (B), (D)
**10.** (A), (B), (C), (D)

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## BOOK FEEDBACK FORM

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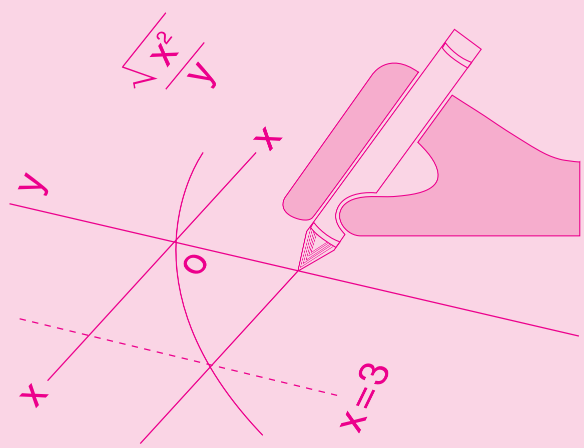
# Contents

# Pre-Requisites

# 0




Pre-Requisites

## Contents



This chapter summarizes the concepts and important formulae required for understanding "Calculus".

## 0.1 | Sets

Even though we discussed *set theory* completely in Chapter 1 of Vol. 1, in this section, we recall some of the concepts, symbols and theorems (for the convenience of the reader) which are going to be used throughout this volume. First, we begin with some notations.

### 0.1.1 Notation

Suppose $S$ is a set (i.e., a collection of objects enjoying a certain property).

1. The symbol $\in$ stands for "belongs to" or "is a member of." $a \in S$ means $a$ belongs to $S$ or is a member of $S$.
2. The symbol $\notin$ stands for "does not belong to." $a \notin S$ means $a$ is not a member of $S$.
3. The symbols $\exists$ and $\ni$ stand for "there exists" and "such that," respectively. Generally these two symbols go together, for example, $\exists$ real number $x \ni x^2 = 2$.
4. The symbol "$\forall$" stands for "for all" or "for every." For example, "$x^2$ is a positive integer $\forall$ non-zero integer $x$."
5. If $S$ is the set of all objects satisfying a property $P$, then $S$ is represented as

$$S = \{x \mid x \text{ has property } P\}$$

6. The set having no objects is called the *empty set* or *null set* and is denoted by "$\phi$".
7. If a set has only a finite number of members $x_1, x_2, \ldots, x_n$, then we write

$$S = \{x_1, x_2, \ldots, x_n\}$$

### 0.1.2 Union and Intersection

Suppose $A$ and $B$ are sets. Then

1. The collection of all objects which either belong to $A$ or belong to $B$ is denoted by $A \cup B$ and is called *union* of $A$ and $B$. That is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

2. The collection of all objects which belong to both $A$ and $B$ is denoted by $A \cap B$ and is called the *intersection* of $A$ and $B$. That is

$$A \cap B = \{x \mid x \in A \text{ or } x \in B\}$$

### 0.1.3 Disjoint Sets

Two sets $A$ and $B$ are called disjoint sets if $A \cap B = \phi$.

### 0.1.4 Indexed Family of Sets

A family $\mathscr{F}$ of sets is called *indexed family of sets* if there exists a set $I$ such that for each element $i \in I$, there exists unique member $A_i$ in $\mathscr{F}$ associated with $i$ and $\mathscr{F} = \{A_i : i \in I\}$ or $\mathscr{F} = \{A_i\}_{i \in I}$. For example, if $h$ is a house in Delhi and $A_h$ is the set of all persons belonging to the house $h$, then $\mathscr{F} = \{A_h \mid h$ is a house in Delhi$\}$ is an indexed family of sets, the index set being the set of all houses in Delhi.

### 0.1.5 Subset

We say that set $A$ is a subset of set $B$ and we write $A \subset B$ or $B \supset A$, if every member of $A$ is also a member of $B$. If $A \subset B$ and $A \neq B$, then $A$ is called *proper subset* of $B$.

### 0.1.6 Power Set

If $A$ is a set, then the set of all subsets of $A$ is called the *power set* of $A$ and is denoted by $P(A)$.

**QUICK LOOK 1**

$\phi$ and $A$ belong to $P(A)$.

### 0.1.7 Set Difference

Let $A$ and $B$ be two sets. Then the set $A - B$ is defined as

$$\{x \in A \mid x \notin B\}$$

### 0.1.8 Universal Set

If $\{A_i\}_{i\in I}$ is a class of sets then the set $X = \bigcup_{i\in I} A_i$ is called universal set of this family of sets $\{A_i\}_{i\in I}$.

### 0.1.9 Complement

If $X$ is universal set and $A \subset X$, then $X - A$ is called complement of $A$ and is denoted by $A'$ or $\overline{A}$ or $A^c$.

**QUICK LOOK 2**

If $A$ and $B$ are any two subsets of universal set $X$, then

$$A - B = A \cap B' = A \cap \overline{B} = A \cap B^c$$

The following theorems are easy to prove.

**THEOREM 0.1** Let $A$, $B$, $C$ be sets. Then

1. $(A \cup B) \cup C = A \cup (B \cup C)$
2. $(A \cap B) \cap C = A \cap (B \cap C)$
3. $A \cup \phi = A$
4. $A \cap \phi = \phi$
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**THEOREM 0.2 (DE MORGAN'S LAWS)** Let $A$, $B$ be subsets of a universal set $X$. Then

1. $(A \cup B)' = A' \cap B'$
2. $(A \cap B)' = A' \cup B'$

**Theorem 0.3 (Generalized De Morgan's Laws)** Let $\{A_i\}_{i\in I}$ be an indexed family of subsets of universal set $X$. Then,

**1.** $\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i'$

**2.** $\left(\bigcap_{i\in I} A_i\right)' = \bigcup_{i\in I} A_i'$

## 0.2 | Real Numbers

Real analysis is considered to be the Brain of Mathematics. Real analysis makes us to think, analyze and solve problems in mathematics. Knowingly or unknowingly we learned about real analysis and used various tools to solve problems in real analysis including certain basic properties. As a process of recalling them from our acquired earlier knowledge, let us begin with the binary operations in the real number system and list out some of their properties.

### 0.2.1 Addition and Multiplication

$\mathbb{R}$ stands for the real number set (system). There are two binary relations (operations), namely, addition (+) and multiplication ($\cdot$) in $\mathbb{R}$. These relations have the following properties:

$A_1 : a, b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$ (Closure property of +)

$A_2 : a, b \in \mathbb{R} \Rightarrow a+b = b+a$ (Commutative property of +)

$A_3 : (a+b)+c = a+(b+c)\ \forall\, a, b, c \in \mathbb{R}$ (Associative property of +)

$A_4 : a+0 = a\ \forall a \in \mathbb{R}$ (Existence of additive identity for + and 0 is the real number zero)

$A_5 : a+(-a) = 0\ \forall\, a \in \mathbb{R}$ (Existence of additive inverse, $-a$ being the negative of $a$)

**QUICK LOOK 3**

We observe that 0 is the only real number satisfying $A_4$ (i.e., if $\alpha \in \mathbb{R}$ and $a+\alpha = a\ \forall\, a \in \mathbb{R}$, then $\alpha = 0$) and given $a \in \mathbb{R}$, then $(-a)$ is the only real number satisfying $A_5$ (i.e., if $a \in \mathbb{R}, \alpha \in \mathbb{R}$ and $a+\alpha = 0$, then $\alpha = -a$).

With regards to multiplication, we have the following properties:

$M_1 : a, b \in \mathbb{R} \Rightarrow a\cdot b \in \mathbb{R}$ (Closure property of multiplication)

$M_2 : a, b \in \mathbb{R} \Rightarrow a\cdot b = b\cdot a$ (Commutative property of multiplication)

$M_3 : (a\cdot b)\cdot c = a\cdot(b\cdot c)$ (Associativity property of multiplication)

$M_4 : a\cdot 1 = a\ \forall\, a \in \mathbb{R}$ (Existence of multiplicative identity, where 1 is the usual one)

$M_5 : a\cdot\dfrac{1}{a} = 1\ \forall\, a \in \mathbb{R}, a \neq 0$ (Existence of multiplicative inverse)

$M_6$ : If $a, b, c \in \mathbb{R}$, then $a\cdot(b+c) = a\cdot b + a\cdot c$ (Distributive law)

**QUICK LOOK 4**

We observe that 1 is the only real number satisfying $M_4$ (i.e., if $\alpha \in \mathbb{R}$ and $a \cdot \alpha = a\ \forall\, a \in \mathbb{R}$, then $\alpha = 1$) and for $a \neq 0$, $1/a$ is the only real number satisfying $M_5$ (i.e., if $a \in \mathbb{R}, a \neq 0, \alpha \in \mathbb{R}$ and $a \cdot \alpha = 1$, then $\alpha = 1/a$).

**CONVENTION:** Here afterwards we will write $ab$ for $a \cdot b$.

***Note:*** In view of $A_1$ to $A_5$ and $M_1$ to $M_6$, we can call $\mathbb{R}$ a field with respect to the two binary operations + and · we write $(\mathbb{R}, +, \cdot)$ is a field.

In addition to the properties of + and ·, $\mathbb{R}$ has another property called *ordering* which has typical properties. First, let us start with positive real numbers. The set of all positive real numbers is denoted by $\mathbb{R}^+$, that is $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a > 0\}$.

### 0.2.2 Properties of $\mathbb{R}^+$

**$R_1$:** If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$.

**$R_2$:** If $x \in \mathbb{R}$ and $x \neq 0$, then either $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$ but not both.

**$R_1$:** $0 \notin \mathbb{R}^+$.

### 0.2.3 Properties of $(\mathbb{R}, +, \cdot, <)$

$(\mathbb{R}, +, \cdot, <)$ is an ordered field having the following properties:

1. $x, y \in \mathbb{R}$, then $x < y$ if and only if $y - x \in \mathbb{R}^+$.
2. $x < y \Rightarrow y > x$.
3. $x \leq y$ means either $x = y$ or $x < y$.
4. If $x, y \in \mathbb{R}$, then exactly one of $x < y, x > y, x = y$ holds.
5. If $x, y, z \in \mathbb{R}, x < y$, and $y < z$, then $x < z$.
6. Also $1 > 0$.

### 0.2.4 Inductive Set

$\mathbb{N}$ as well as $\mathbb{Z}^+$ denote the set of positive integers (positive integers are also called natural numbers). $\mathbb{N}$ has the following properties:

1. $1 \in \mathbb{N}$.
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Suppose $S$ is a set of real numbers such that

1. $1 \in S$.
2. $s \in S$ implies $s + 1 \in S$.

Then $S$ is called an *inductive set*.

**QUICK LOOK 5**

1. $\mathbb{N}$ is an inductive set.
2. If $S$ is an inductive set, then $\mathbb{N} \subset S$ and hence $\mathbb{N}$ is the smallest inductive set.

**Notation:** $\mathbb{Z}$ denotes the set of all integers. That is

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$$

where $-\mathbb{N} = \{-n \mid n \in \mathbb{N}\}$.

## 0.3 | Bounded Set, Least Upper Bound and Greatest Lower Bound

In this section, we introduce the concept of a bounded set of real numbers, least upper bound, and greatest lower bound of a set $S$. Let us begin with the following definition

**DEFINITION 0.1** Let $S$ be a non-empty subset of $\mathbb{R}$. A real number $x$ is said to be a *lower bound* of $S$ if $x \le y$ for every $y \in S$. If $S$ has at least one lower bound, then we say that $S$ is *bounded below*. If $x$ is a lower bound of $S$ and $x \in S$, then we say that $x$ is the least or minimum member (element) of $S$.

### Examples

1. If $S = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$, then 0 (zero) is a lower bound and it is the least element of $S$.
2. For the set $\mathbb{N}, 1$ is the least element of $\mathbb{N}$.
3. If $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$, then 0 is a lower bound but it is not least element of $T$ (since $0 \notin T$).

**DEFINITION 0.2** Let $S$ be a set of real numbers. A real number $z$ is called an *upper bound* of $S$, if $s \le z$ for all $s \in S$. If $S$ has at least one upper bound, then we say that $S$ is *bounded above*. If $z$ is an upper bound of $S$ and $z \in S$, then $z$ is called greatest (maximum) element of $S$.

### Examples

1. $S = \{x \in \mathbb{R} \mid x < 3\}$ is bounded above without greatest element (since $3 \notin S$).
2. $T = \{1/n \mid n \in \mathbb{N}\}$ is bounded above, 1 is the greatest element and $T$ has no least element.

**DEFINITION 0.3** **Bounded Set** A set $S$ of real numbers is said to be *bounded*, if $S$ is both bounded below and bounded above.

**DEFINITION 0.4** **Greatest Lower Bound, g.l.b.** Suppose that $S$ is bounded below and $x$ is a lower bound of $S$. If $y$ is a lower bound of $S$ which implies $y \le x$, then $x$ is called greatest lower bound of $S$ and $x$ is denoted by g.l.b. $S$.

**QUICK LOOK 6**

A real number $x$ is a g.l.b. $S$ if and only if $x$ is a lower bound of $S$ and any real number $y > x$ cannot be a lower bound of $S$.

**DEFINITION 0.5** **Least Upper Bound, l.u.b.** Suppose $S$ is bounded above set. A real number $z$ is called least upper bound of $S$ (l.u.b. $S$) if $z$ is an upper bound of $S$ and $y$ is also an upper bound of $S$ then $y \ge z$.

**QUICK LOOK 7**

A real number $u$ is l.u.b of a set $S$ if and only if $u$ is an upper bound of $S$ and any number less than $u$ cannot be an upper bound of $S$.

**DEFINITION 0.6** **Unbounded Below and Unbounded Above** A set $S$ of real numbers is said to be unbounded below if it is not bounded below. $S$ is said to be unbounded above, if $S$ is not bounded above.

**DEFINITION 0.7** **Unbounded Set** A set $S$ of real numbers is said to be unbounded set if either $S$ is unbounded below or unbounded above.

## 0.4 | Completeness Property of $\mathbb{R}$ and Archimedes' Principle

In this section we state the so-called completeness property of $\mathbb{R}$ and the Archimedes' principle and discuss how well ordering principle in $\mathbb{R}$ leads to the concept of the integral part of a real number. We will also mention some of its properties. Let us begin with the completeness of $\mathbb{R}$.

**Theorem 0.4 (Completeness Property of $\mathbb{R}$)** Every non-empty set of real numbers which is bounded above has a least upper bound (l.u.b) in $\mathbb{R}$.

Generally l.u.b of $S$ is called supremum of $S$ and the completeness property of $\mathbb{R}$ is called *supremum property* of $\mathbb{R}$. From the completeness property (or supremum property) of $\mathbb{R}$, we can conclude the following.

**Theorem 0.5**

1. Every non-empty set $S$ of $\mathbb{R}$ which is bounded below has g.l.b. (also called *infimum*).
2. $S$ is bounded below if and only if $-S = \{-x \mid x \in S\}$ is bounded above and g.l.b. $S = -$l.u.b. $(-S)$.
3. If $S$ and $T$ are non-empty subsets of $\mathbb{R}$ and $x \in S, y \in T \Rightarrow x \le y$, then $S$ is bounded above, $T$ is bounded below and l.u.b. $S \le$ g.l.b. $T$.

### 0.4.1 Archimedes' Principle

*Statement:* If $x$ is a real number and $y$ is a positive real number, then there exists a positive integer $n$ such that $x < ny$. As a consequence of Archimedes' principle we have the following theorem.

**Theorem 0.6**

1. If $x$, $y$ are integers, then $x + y$ and $xy$ are also integers.
2. If $x$ is an integer, then there is no integer between $x$ and $x + 1$. That is, if $x < y < x + 1$, then $y$ cannot be an integer.
3. If $S$ is a non-empty subset of $\mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers) which is bounded above, then l.u.b. $S$ is in $S$. Similarly, if $S$ is bounded below, then $S$ has minimum (least) element.

From part (3) of Theorem 0.6, we have the following property called well ordering principle.

**Theorem 0.7 (Well Ordering Principle)** Every non-empty subset of the natural number set $\mathbb{N}$ has minimum (or least) element.

From the well ordering principle, we have the following theorem which is the base for the concept of *integral part* of a real number.

**Theorem 0.8** If $x \in \mathbb{R}$, then there exists a unique integer $n$ such that $n \le x < n+1$.

**DEFINITION 0.8** Let $x \in \mathbb{R}$. Then the unique integer $n$ (guaranteed by Theorem 0.8) such that $n \le x < n+1$ is called the *integral part* of $x$ and is denoted by $[x]$. In other words, $[x]$ is the largest integer not exceeding $x$.

**DEFINITION 0.9** **Fractional Part** If $x \in \mathbb{R}$, then $x-[x]$ is called *fractional part* of $x$ and is denoted by $\{x\}$.

**QUICK LOOK 8**

$$0 \le \{x\} < 1 \text{ for all } x \in \mathbb{R} \text{ and } x = [x] + \{x\}$$

In the following theorem, we list out some properties of integral part (for proofs, refer the authors Vol. 1 pp. 46–48) for quick reference whenever necessary.

**THEOREM 0.9** The following hold for any real number $x$:

1. $[x] \le x < [x]+1$
2. $x-1 < x \le [x]$
3. $[x] = \sum_{1 \le i \le x} 1$, if $x > 1$
4. $[x] = x$ if and only if $x \in \mathbb{Z} \Leftrightarrow \{x\} = 0$
5. $\{x\} = x$ if and only if $[x] = 0$
6. $[x]+[-x] = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ -1 & \text{if } x \notin \mathbb{Z} \end{cases}$

**THEOREM 0.10** The following hold for any real numbers $x$ and $y$:

1. $[x+y] = \begin{cases} [x]+[y] & \text{if } \{x\}+\{y\} < 1 \\ [x]+[y]+1 & \text{if } \{x\}+\{y\} \ge 1 \end{cases}$
2. $[x+y] \ge [x]+[y]$ and equality holds if and only if $\{x\}+\{y\} < 1$.
3. If $x$ or $y$ is an integer, then $[x+y] = [x]+[y]$.

**THEOREM 0.11**

1. If $x$ is any real number and $m$ is any non-zero integer, then
$$\left[\frac{x}{m}\right] = \left[\frac{[x]}{m}\right]$$
2. If $n$ and $k$ are positive integers and $k > 1$, then
$$\left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] \le \left[\frac{2n}{k}\right]$$
3. Also
$$[2x] - 2[x] = \begin{cases} 0 & \text{if } [2x] \text{ is even} \\ 1 & \text{if } [2x] \text{ is odd} \end{cases}$$

**THEOREM 0.12** If $n$ is a positive integer and $p$ is any prime number, then the highest power $p$ dividing $n!$ (factorial $n$) is

$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

**QUICK LOOK 9**

Since $n$ is a finite number, on and after a stage $p^k$ exceeds $n$ so that the infinite sum is actually a finite sum.

## 0.5 | Rational Numbers, Irrational Numbers and Density Property of Rational Numbers

**DEFINITION 0.10** **Rational Number** A real number $x$ is called rational number, if there exists $n \in \mathbb{N}$ (i.e., $n$ is a positive integer) such that $nx \in \mathbb{Z}$. The set of all rational numbers is denoted by $\mathbb{Q}$.

**QUICK LOOK 10**

1. $\mathbb{Z} \subset \mathbb{Q}$
2. There are rational numbers which are not integers. For example 1/2, 4/3 are rational numbers but are not integers.
3. $\mathbb{Z}$ is a proper subset of $\mathbb{Q}$.

The following theorems enlist some properties of the set $Q$ of rational numbers, especially the density property of $Q$.

**THEOREM 0.13** If $x,\ y \in \mathbb{Q}$, then $-x \in \mathbb{Q}$, $x + y \in \mathbb{Q}$, $xy \in \mathbb{Q}$ and $x / y \in \mathbb{Q}$ whenever $y \neq 0$.

**THEOREM 0.14 (DENSITY PROPERTY OF $\mathbb{Q}$)** Between two distinct real numbers, there lies a rational number. That is, if $x, y$ are real numbers and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.

**DEFINITION 0.11** **Even Integer** $n \in \mathbb{N}$ is said to be *even*, if there exits $m \in \mathbb{N}$ such that $m + m = n$. Otherwise we say that $n$ is *odd*.

**THEOREM 0.15** There exists a unique real number $x > 0$ such that $x^2 = 2$ and this unique $x$ is denoted by $\sqrt{2}$.

**DEFINITION 0.12** **Irrational Number** A real number which is not rational is called *irrational number*.

**THEOREM 0.16** $\sqrt{2}$ is irrational number.

Finally we conclude this section with the following theorem.

**THEOREM 0.17**
1. $\mathbb{Q}$ is a proper set of $\mathbb{R}$.
2. Between any two distinct real numbers there lies an irrational number.

## 0.6 | Intervals

In this section, we define an interval, its length, bounded and unbounded intervals, and g.l.b., l.u.b. of intervals (considering interval as a set). For this section we have $a,\ b \in \mathbb{R}$ and $a < b$.

**DEFINITION 0.13** **Open Interval** The set $\{x \in \mathbb{R} \mid a < x < b\}$ is called an *open interval.* It is denoted by $(a, b)$ with initial end point $a$ and terminal end point $b$.

**DEFINITION 0.14** **Closed Interval** The set $\{x \in \mathbb{R} \mid a \le x \le b\}$ is called *closed interval*. It is denoted by $[a, b]$ with $a$ and $b$ as initial and terminal end points, respectively.

**DEFINITION 0.15** **Half Open and Half Closed Intervals**

1. The set $\{x \in \mathbb{R} \mid a \le b < x\}$ is denoted by $[a, b)$.
2. The set $\{x \in \mathbb{R} \mid a < x \le b\}$ is denoted by $(a, b]$.

**DEFINITION 0.16** **Length of the Intervals** $b - a$ is called the length of all the four intervals $(a, b)$, $[a, b]$, $[a, b)$ and $(a, b]$.

**QUICK LOOK 11**

1. If $a = b$, then $[a, b] = \{a\}$ (singleton set) so that $(a, b) = [a, b) = (a, b] = \phi$ (empty set).
2. If $a \le b$, then g.l.b. $[a, b] = a$ and l.u.b. $[a, b] = b$.
3. If $a < b$, then g.l.b. $(a, b) =$ g.l.b. $[a, b) =$ g.l.b. $(a, b] = a$ and l.u.b. $(a, b) =$ l.u.b. $[a, b) =$ l.u.b. $(a, b] = b$.

**DEFINITION 0.17** All the four intervals $(a, b)$, $[a, b]$, $[a, b)$ and $(a, b]$ are called *bounded intervals*. In the next definition we list out unbounded intervals.

**DEFINITION 0.18** **Unbounded Intervals** Let $a \in \mathbb{R}$. Then

1. $\{x \in \mathbb{R} \mid x > a\}$ is denoted by $(a, \infty)$.
2. $\{x \in \mathbb{R} \mid x < a\}$ is denoted by $(-\infty, a)$.
3. $\{x \in \mathbb{R} \mid x \ge a\}$ is denoted by $[a, \infty)$.
4. $\{x \in \mathbb{R} \mid x \le a\}$ is denoted by $(-\infty, a]$.
5. $\mathbb{R}$ is also denoted by $(-\infty, \infty)$.

***Note:***

**(i)** $\infty, -\infty$ are only symbols but not real numbers.

**(ii)** The intervals $(a, \infty)$, $[a, \infty)$ are bounded below with $a$ as their g.l.b, but these intervals are not bounded above. Similarly the intervals $(-\infty, a)$ and $(-\infty, a]$ are bounded above with $a$ as l.u.b. and these intervals are not bounded below.

**(iii)** $(-\infty, \infty)$ is neither bounded below nor bounded above.

## 0.6.1 Interesting Features of Intervals

Let $I$ denote any one of the intervals defined above. Then $I$ has the following basic property:

$x, y \in I, x < y \Rightarrow [x, y] \subset I$ and conversely, if $A \subset \mathbb{R}$ has the property that $x, y \in A, x < y \Rightarrow [x, y] \subset A$, then $A$ is an interval.

**DEFINITION 0.19** **Neighbourhood** Let $a \in \mathbb{R}$ and $\in > 0$. Then the open interval $(a - \in, a + \in)$ is called $\in$-neighbourhood of $a$ and $(a - \in, a + \in) - \{a\}$ is called the deleted $\in$-neighbourhood of $a$ where $\{a\}$ is the singleton set containing the element $a$.

## 0.7 | Absolute Value of a Real Number

In this section, we introduce the concept of *absolute value* or *modulus value* of a real number and list some of its properties which are useful. Let us begin with the following definition.

**DEFINITION 0.20** Let $x \in \mathbb{R}$. We write

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|x|$ is called the absolute values of $x$ or modulus value of $x$.

The following theorem enlists the important properties of $|x|$.

**THEOREM 0.18**

1. $|0| = 0$ and $|x| = |-x| \; \forall \; x \in \mathbb{R}$.
2. For $x, \; y \in \mathbb{R}$, we have
$$-|x| \leq x \leq |x|$$
and whenever $y \neq 0$
   **(a)** $|xy| = |x||y|$
   **(b)** $|x + y| \leq |x| + |y|$
   **(c)** $||x| - |y|| \leq |x - y|$
   **(d)** $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$
3. For $a \in \mathbb{R}$ and $\in > 0$, we have
$$(a - \in, a + \in) = \{x \in \mathbb{R} \,|\, |x - a| < \in\}$$
4. If $x \in (a - \in, a + \in) \; \forall \in > 0$, then $x = a$.
5. $\text{Max}\{x, -x\} = |x|$.
6. **(a)** $\text{Max}\{a, b\} = \dfrac{1}{2}(a + b + |a - b|)$
   **(b)** $\text{Min}\{a, b\} = \dfrac{1}{2}(a + b - |a - b|)$

**QUICK LOOK 12**

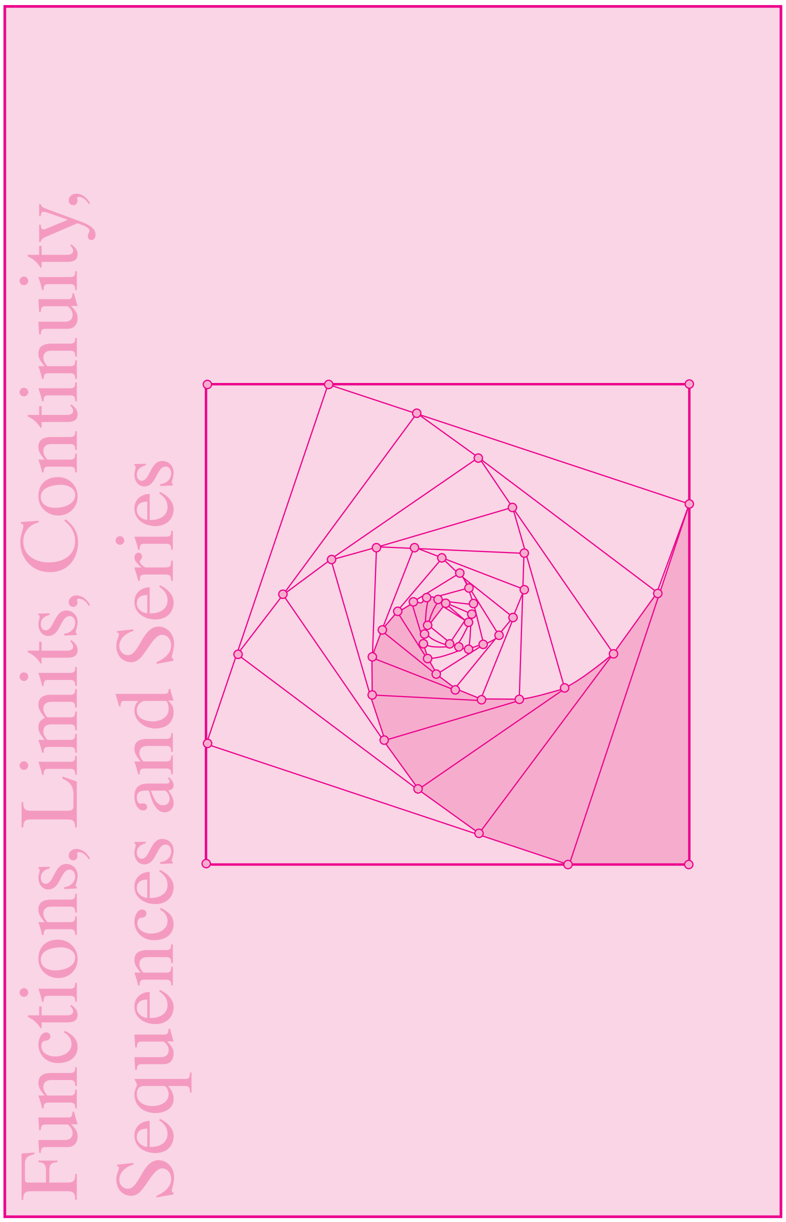
$|x + y| = |x| + |y|$ if and only if both $x$ and $y$ are of same sign.

Finally we conclude this chapter with the following property of closed intervals.

**THEOREM 0.19** Let $\{x_n \,|\, n = 1, 2, 3, \ldots\}$ and $\{y_n \,|\, n = 1, 2, 3, \ldots\}$ be subsets of $\mathbb{R}$ such that $[x_{n+1}, y_{n+1}] \subset [x_n, y_n]$ for $n = 1, 2, 3\ldots.$ Then $\bigcap_{n \in \mathbb{N}} [x_n, y_n]$ is non-empty. If further $\underset{n \in \mathbb{N}}{\text{g.l.b.}} (y_n - x_n) = 0$, then $\bigcap_{n \in \mathbb{N}} [x_n, y_n]$ is a singleton set.

# Functions, Limits, Continuity, Sequences and Series

# 1




*Contents*



A **function** associates one quantity – *argument* or *input* of the function – with another quantity – the *value* or *output* of the function. **Limit** is used to describe the value that a function or sequence "approaches" as the input approaches some value. A **sequence** is a list of objects/events ordered in a sequential fashion such that each member either comes before, or after, every other member. A **series** is a sum of a sequence of terms.

Calculus is a part of Mathematical Analysis which was developed by Issac Newton (1642–1727), Gottfried Leibniz (1646–1716), Leonhard Euler (1707–1783), Augustin–Louis Cauchy (1789–1857) and Karl Weierstrass (1815–1897). The world of Mathematics came to know that Calculus was the brain child of both Newton and Leibniz who developed it simultaneously without knowing each other and was improved by Euler, Cauchy and Weierstrass. It was Weierstrass who gave the precise definition of limit and more teeth to continuity and differentiability. It was he who constructed a function which is continuous for all real values of $x$ but not differentiable at any real value of $x$.

## 1.1 | Functions: Varieties

Even though the general notion of a function, real-valued functions and operations among real-valued functions were exhaustively studied in Chapter 1 of Vol. 1 (Algebra), for completeness sake, we recall some of the basic definitions and results (possibly without proofs) here. This process is undertaken to facilitate the reader to have immediate reference. Let us begin with the definition of a function.

**DEFINITION 1.1** Suppose $A$ and $B$ are two non-empty sets $A \times B = \{(a, b) \mid a \in A, b \in B\}$ is the set of ordered pairs of $A$ and $B$ and is called the ***Cartesian product*** of $A$ and $B$.

**DEFINITION 1.2** Any non-empty subset $S$ of $A \times B$ is called a ***function*** if

**(i)** $a \in A \Rightarrow \exists\, b \in B$ such that $(a, b) \in S$
**(ii)** $(a, b_1) \in S, (a, b_2) \in S \Rightarrow b_1 = b_2$

We also say that $S$ is a map or mapping. If $S \subset A \times B$ is a function, then

**(i)** $A$ is called the ***domain*** of $S$ and $B$ is called ***codomain*** of $S$.
**(ii)** $\{b \in B \mid (a, b) \in S\}$ is called the ***range*** of $S$.

If $(a, b) \in S$, it is customary to denote $b$ by $S(a)$. We observe that $S(a)$ is unique for a given $a \in A$ and $S = \{(a, S(a)) \mid a \in A\}$. $S(a)$ is called the ***value or image*** of $S$ at $a$. If $S(a) = b$, then $a$ is called the ***preimage*** of $b$. Generally, we write $S: A \to B$ and say that $S$ is a function defined on $A$ with values in $B$.

**DEFINITION 1.3** **Equal Functions** If $S: A \to B$ and $T: A \to B$ are functions such that $S(a) = T(a)$ for every $a \in A$, then we say that $S$ and $T$ are equal and we write $S = T$.

For convenience sake, we use the symbol $f$ for a function. Thus, $f: A \to B$ is taken as a function. For our purpose, in general, $A$ is a subset of $\mathbb{R}$ and $B = \mathbb{R}$. When this is the case, that is $f: A \to \mathbb{R}$ where $A \subset \mathbb{R}$, then $f$ is a real-valued function from the subset $A$ of $\mathbb{R}$.

### Examples

1. $A$ is the set of positive integers, $f: A \to \mathbb{R}$ where $f(n) = 2n$ for $n = 1, 2, 3, \ldots$. Thus $f(1) = 2, f(2) = 4, f(3) = 6, \ldots$.
2. $A = [0, 1]$, $f: A \to \mathbb{R}$ where $f(x) = x^2$ for $x \in A$. Thus

$$f(1) = 1^2 = 1$$
$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$f\left(\frac{\sqrt{2}}{3}\right) = \left(\frac{\sqrt{2}}{3}\right)^2 = \frac{2}{9} \cdots$$

3. $A = \mathbb{R}$ and $f: A \to \mathbb{R}$ is defined by $f(x) = x^2 + x + 1$ for $x \in A$. Thus

$$f(0) = 0^2 + 0 + 1 = 1$$
$$f(1) = 1^2 + 1 + 1 = 3$$
$$f(-1) = (-1)^2 + (-1) + 1 = 1$$
$$f(\sqrt{2}) = (\sqrt{2})^2 + \sqrt{2} + 1 = 3 + \sqrt{2}$$

4. $A$ = The set of all non-negative real numbers, and $f: A \to \mathbb{R}$ is defined by $f(x) = \sqrt{x}$. Thus

$$f(0) = 0,\ f(1) = 1,\ f(2) = \sqrt{2},\ f\left(\frac{1}{4}\right) = \frac{1}{2}$$

[Here $-1/2$ should not be taken (why?)]. $f(-1)$ is not defined because $-1 \notin A$.

5. $A = \mathbb{R}$ and $f: A \to \mathbb{R}$ is defined by $f(x) = |x|$. Thus

$$f(0) = 0,\ f(1) = 1,\ f(-1) = |-1| = 1$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2},\ f\left(-\frac{1}{2}\right) = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

**DEFINITION 1.4** **Constant Function** If $A \subset \mathbb{R}$ and $k \in \mathbb{R}$ then $f: A \to \mathbb{R}$ defined by $f(x) = k$ for every $x \in A$ is called *constant function*.

In the following definition we introduce operations on the set of real-valued functions defined on a common domain, a subset of real numbers.

**DEFINITION 1.5** Suppose $A \subset \mathbb{R}$, $f$ and $g$ are two real-valued functions defined on $A$ with values in $\mathbb{R}$. Then we define the sum $f+g$, the product $f \cdot g$, the quotient $f/g$, $|f|$ and $[f]$ ($[\cdot]$ is integral part) as follows:

1. $(f+g)(x) = f(x) + g(x)\ \forall x \in A$.
2. $(f \cdot g)(x) = f(x) \cdot g(x)\ \forall x \in A$.
3. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ if $g(x) \neq 0\ \forall x \in A$.
4. $|f|(x) = |f(x)|\ \forall x \in A$.
5. $[f](x) = [f(x)]\ \forall x \in A$.

Also for any positive integer $n$, we have

6. $f^n(x) = (f(x))^n\ \forall x \in A$.
7. If $\alpha \in \mathbb{R}$ and $f(x) > 0\ \forall x \in A$, then $f^{\alpha}(x) = (f(x))^{\alpha}\ \forall x \in A$.

**QUICK LOOK 1**

1. If $k$ is a constant, then $(kf)(x) = kf(x)\ \forall x \in A$.
2. $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{f(x)}\ \forall x \in A$, if $f(x) \neq 0\ \forall x \in A$.

**DEFINITION 1.6** **Composite Function** Suppose $A$, $B$ and $C$ are non-empty subsets of $\mathbb{R}$, $f: A \to B$ and $g: B \to C$ are functions, then we define the function $g \circ f: A \to C$ as

$$(g \circ f)(x) = g(f(x))\ \forall x \in A$$

$g \circ f$ is called *composite function* of $g$ and $f$. Note that $g \circ f$ may be defined but $f \circ g$ may not be defined.

**QUICK LOOK 2**

If $f: A \to A$ and $g: A \to A$, then both $g \circ f$ and $f \circ g$ are defined, but still $f \circ g$ and $g \circ f$ may not be equal functions.

## Examples

Let $A = \mathbb{R}$ and $f(x) = x^2$ and $g(x) = 2x$ for all $x \in A$. Then

1. $(f+g)(x) = f(x) + g(x) = x^2 + 2x$
2. $(f \cdot g)(x) = f(x)g(x) = x^2(2x) = 2x^3$
3. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x}{2}\ \ \forall x \in \mathbb{R}$ and $x \neq 0$. Here we take $A = \mathbb{R} - \{0\}$.

**4.** $|f|(x) = |f(x)| = |x^2| = x^2$

**5.** $[f](x) = [f(x)]$. We can observe that

$$[f](1/2) = [f(1/2)] = [1/4] = 0$$

In fact

**(a)** $[f](x) = 0 \ \forall x \in (-1, 1)$

**(b)** $[f](2.1) = [f(2.1)] = [(2.1)^2] = 4$.

**6.** **(a)** $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$

**(b)** $(f \circ g)(x) = f(g(x)) = f(2x) = 4x^2$

We observe that $f \circ g \neq g \circ f$.

***Note:*** Suppose $A \subset \mathbb{R}$ and $f: A \to \mathbb{R}$ is a function. Then the functions $|f|, [f], f^n$ and $f^\alpha$ can be realized as composite functions by taking $g: \mathbb{R} \to \mathbb{R}$ as $g(x) = |x|$, $g(x) = [x]$, $g(x) = x^n$ (where $n$ is a positive integer) and $g(x) = x^\alpha$ ($\alpha$ is real), respectively. That is

**1.** If $g(x) = |x|$, then

$$(g \circ f)(x) = g(f(x)) = |f(x)| = |f|(x)$$

**2.** If $g(x) = [x]$, then

$$(g \circ f)(x) = g(f(x)) = [f(x)] = [f](x)$$

**3.** If $g(x) = x^n$, then

$$(g \circ f)(x) = g(f(x)) = (f(x))^n = f^n(x)$$

**4.** If $g(x) = x^\alpha$, then

$$(g \circ f)(x) = (f(x))^\alpha = f^\alpha(x)$$

**DEFINITION 1.7** **Identity Function** Let $A \subset \mathbb{R}$ and $A$ be non-empty. The function $f: A \to A$ defined by $f(x) = x \ \forall x \in A$ is called the *identity function* on $A$. Sometimes, it is denoted by $I_A$. That is, $I_A(x) = x \ \forall x \in A$.

**THEOREM 1.1 (ASSOCIATIVE LAW FOR COMPOSITION OF FUNCTION)** Suppose $A, B, C$ and $D$ are non-empty subsets of $\mathbb{R}$ and $f: A \to B$, $g: B \to C$ and $h: C \to D$ are functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$.

**PROOF** First of all, observe that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are functions from $A$ to $D$. Further, for $x \in A$,

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= h(g(f(x)) \\ &= h((g \circ f)(x)) \\ &= (h \circ (g \circ f))(x) \end{aligned}$$

■

***Note:*** Theorem 1.1 is given as Theorem 1.27 (*Try it out*) in Chapter 1 (Vol. 1) on p. 41. The proof also runs on the same lines as above.

## 1.2 | Functions and Their Inverse

In this section, we once again recall from Vol. 1 (Algebra) the notions of one-one, onto and bijective functions. Further we introduce the concept of the inverse of a bijective function and their properties.

**DEFINITION 1.8** Let $A$ and $B$ be two non-empty subsets of $\mathbb{R}$. Let $f: A \to B$ be a function. Then

**1.** $f$ is said to be ***one-one*** function if $x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$.

2. $f$ is said to be ***onto*** if $y \in B \Rightarrow \exists\, x \in A \ni f(x) = y$.
3. If $f$ is both one-one and onto, then we say that $f$ is a ***bijection***.

## Examples

1. Let $A = B = [0, \infty)$ (i.e., the set of all non-negative reals). The function $f$ defined by
$$f(x) = \frac{x}{x+1}$$
for every $x \in A$ is one-one but not onto [since $f(x) = f(y) \Rightarrow x = y$ and $1 \in B$, but there is no $x \in A$ such that $f(x) = 1$].
2. Let $A = \mathbb{R}$ and $B = [0, \infty)$. Then the function $f$ defined by $f(x) = |x|$ is onto but not one-one, since
$$y \in B \Rightarrow y = |y| = f(y)$$
and $$f(x) = f(-x) = |y| \;\; \forall\, x \in A$$
3. Let $A = B = \mathbb{R}$. Then the function $f$ defined by $f(x) = 2x$ is a bijection.
4. Let $A = \mathbb{R}$ and $B = \mathbb{R}$. Define $f: A \to B$ by
$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$
Then $f$ is neither one-one nor onto.

### QUICK LOOK 3

If $f: A \to B$ is a function and range of $f = C$ such that $C \subset B$, then $f: A \to C$ is onto.

**DEFINITION 1.9** **Inverse of a Function** Suppose $f: A \to B$ is a bijection. Define $g: B \to A$ by $g(b) = a$, where $f(a) = b$. Then $g$ is called the *inverse function* of $f$ and is denoted by $f^{-1}$.

## Example

Let $A = B = \mathbb{R}$. Define $f: A \to B$ by
$$f(x) = x + 1$$
Then clearly $f$ is a bijection and also $f^{-1}(y) = y - 1$ is the inverse function of $f$ since
$$f^{-1}(x+1) = (x+1) - 1 = x \;\; \forall\, x \in A$$

### QUICK LOOK 4

1. If $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is also a bijection and $f^{-1} \circ f = I_A$, $f \circ f^{-1} = I_B$.
2. If $f: A \to A$ is a bijection, then $f \circ f^{-1} = f^{-1} \circ f = I_A$.

**DEFINITION 1.10** **Restriction of a Function** Let $f: A \to B$ be a function and $C$ be a non-empty subset of $A$. Define $g: C \to B$ by $g(c) = f(c)$ for all $x \in C$. Then $g$ is called *restriction* of $f$ to $C$ and is denoted by $f|_C$.

## Examples

1. Let $f: \mathbb{R} \to [0, \infty)$ be defined by $f(x) = |x| \; \forall\, x \in \mathbb{R}$ and let $C = [0, \infty)$. Then clearly $C \subset \mathbb{R}$ and $f|_C : C \to [0, \infty)$ is given by
$$(f|_C)(x) = f(x) = |x| = x \;\; \forall\, x \in C \quad (\because x \geq 0)$$
2. Let $f: \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$. Take $C = [0, \infty)$. Then
$$(f|_C)(x) = f(x) = x^2 \;\; \forall\, x \in C$$
Notice that $f|_C$ is a bijection where $f$ is not one-one so that $f$ is not a bijection. Also if $C = (-\infty, 0]$, then $f|_C$ is a bijection.
3. Let $f: \mathbb{R} \to [0, \infty)$ be defined by $f(x) = \sqrt{|x|} \;\; \forall\, x \in \mathbb{R}$. If $C = [0, \infty)$ and $D = (-\infty, 0]$, then

$$(f|_C)(x) = \sqrt{|x|} = \sqrt{x} \ \ \forall\, x \in C$$

and $$(f|_D)(x) = \sqrt{-x} \ \ \forall\, x \in D$$

Observe that $f|_C$ and $f|_D$ are bijections but $f$ is not.

***Note:*** Even though a function $f: A \to B$ may not be a bijection, sometimes $f$ is restricted to a suitable subset of $A$ so that $f$ restricted to that subset has inverse function from the range of $f$ to the restricted domain. The following is an illustration of this.

### Examples

1. Let $f: \mathbb{R} \to B = [0, \infty)$ be a mapping defined by $f(x) = |x|$. Clearly $f$ is not a bijection. If $A = B = [0, \infty)$ then $f|_A$ is a bijection and the inverse of $f|_A$ is given by
$$(f|_A)^{-1}(x) = f^{-1}(x) = x \ \ \forall\, x \in A$$
2. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sin x$ so that range of $f$ is $[-1, 1]$. If $A = [-\pi/2, \pi/2]$, then $f|_A$ is a bijection from $A$ to $[-1, 1]$ and $f|_A$ has inverse function from $[-1, 1]$ to $A$. This inverse function is denoted by $\text{Sin}^{-1}$ (see Chapter 2, Vol. 2).
3. Let $f: \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$. Taking $A = [0, \infty)$, we have $f|_A$ is a bijection on $A$ and
$$(f|_A)^{-1}(x) = f^{-1}(x) = \sqrt{x} \ \ \forall\, x \in A$$

## 1.3 | Even and Odd Functions, Periodic Functions

Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, then we can observe $f(-x) = f(x)\ \forall\, x \in \mathbb{R}$. Functions satisfying this property are called ***even functions***. Functions satisfying the property $f(-x) = -f(x)$, for example $f(x) = x^3$, are called ***odd functions***. Functions satisfying the property $f(x + T) = f(x)$ whenever $x$ and $x + T$ are in the domain of $f$ and $T > 0$ are called ***periodic functions***. In this section, we give precise definitions of these and state some of their properties. Let us begin with the following.

**DEFINITION 1.11** **Symmetric Set** Let $X$ be a non-empty subset of $\mathbb{R}$. Then $X$ is called symmetric set if
$$x \in X \Leftrightarrow -x \in X$$

### Examples

1. $\mathbb{R}$ is a symmetric set.
2. $[-1, 1]$ is a symmetric set.
3. The set $\mathbb{Q}$ of rational numbers and the set $\mathbb{Z}$ of all integers are symmetric sets.
4. $[0, 1]$ is not a symmetric set.
5. $\{-1, 0, 1\}$ is a symmetric set.

***Note:*** The real number 0 need not belong to a symmetric set. For example, the set $\{-1, 1\}$ is a symmetric set without 0 and $[-1, 0) \cup (0, 1]$ is a symmetric set not containing the element 0.

**DEFINITION 1.12** **Even Function** Let $X$ be a symmetric set and $f: X \to \mathbb{R}$ be a function. If $f(-x) = f(x)\ \forall\, x \in X$, then $f$ is called an *even function*.

### Examples

1. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is an even function because $|x| = |-x|\ \forall\, x \in \mathbb{R}$.
2. $f(x) = x^2$ is an even function from $\mathbb{R}$ to $\mathbb{R}$ because $(-x)^2 = x^2\ \forall\, x \in \mathbb{R}$.
3. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$ is an even function, because $\cos(-x) = \cos x$.

**DEFINITION 1.13** **Odd Function** If $X$ is a symmetric set, then $f: X \to \mathbb{R}$ is called an *odd function* if $f(-x) = -f(x)$ for all $x \in X$.

## Examples

1. $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = x^3$ is an odd function because $(-x)^3 = -x^3\ \forall\ x \in \mathbb{R}$.
2. $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = \sin x$ is an odd function because $\sin(-x) = -\sin x$.
3. Let $X = [-2,-1] \cup [1,2]$ and define $f : X \to \mathbb{R}$ as $f(x) = x^3$ so that $f$ is an odd function.

***Note:*** If $X$ is a symmetric set having the element 0 and $f:X\to\mathbb{R}$ is an odd function, then $f(0)$ is necessarily zero, because

$$f(0) = f(-0) = -f(0)$$

In part (3) of the example above, the set $X = [-2,-1] \cup [1,2]$ does not contain 0, so $f(0)$ cannot exist.

The statements stated in the following theorem can be easily verified [see pp. 56, 57 of Chapter 1, Vol. 1 (Algebra)].

**THEOREM 1.2** Let $X$ be a symmetric set and $f$ and $g$ be functions from $X$ into $\mathbb{R}$. Then

1. If both $f$ and $g$ are even (both are odd), then so is $f \pm g$.
2. If both $f$ and $g$ are even (both are odd), then $fg$ is even.
3. If one of $f$ and $g$ is even and the other is odd, then $fg$ is odd.
4. $f$ is even if and only if $\lambda f$ is even for any non-zero real $\lambda$.
5. $f$ is odd if and only if $\lambda f$ is odd for any non-zero real $\lambda$.
6. $f$ is even (odd) if and only if $-f$ is even (odd).
7. If $f:X\to\mathbb{R}$ is any function, then the function

$$g(x) = \frac{f(x)+f(-x)}{2} \text{ is even}$$

$$h(x) = \frac{f(x)-f(-x)}{2} \text{ is odd}$$

and
$$f(x) = g(x) + h(x)$$

Here $g(x)$ is called *even part* of $f$ and $h(x)$ is called *odd part* of $f(x)$. Further $g(x)$ and $h(x)$ are *unique*.

**DEFINITION 1.14** **Periodic Function** Let $A$ be a non-empty subset of $\mathbb{R}$. If there exists a positive number $p$ such that $f(x+p) = f(x)$ whenever $x$ and $x+p$ belong to $A$, then $f$ is called a *periodic function* and $p$ is called a period of $f$. The smallest period (if it exists) is called the ***period of f***.

***Note:***

1. A function can be periodic without smallest period. For example, $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = k\ \forall\ x \in \mathbb{R}$ is a periodic function, but it has no smallest period.
2. If $p$ is a period of a function $f$, then $np$ is also a period of $f$ where $n$ is any positive integer.

## Examples

1. $\sin x$ and $\cos x$ are periodic functions with $2\pi$ as the smallest period. The functions $\tan x$ and $\cot x$ are periodic functions with smallest period $\pi$. For proofs refer to Chapter 1 of Vol. 2.
2. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = x - [x]$. Then

$$\begin{aligned} f(x+1) &= (x+1) - [x+1] \\ &= (x+1-([x]+1) \\ &= x - [x] = f(x)\ \forall\ x \in \mathbb{R} \end{aligned}$$

[See (3) of Theorem 0.10] so that 1 is a period of $f$ and 1 is the smallest period of $f$ (verify).

**3.** Define $f:\mathbb{N}\to\mathbb{N}$ by

$$f(n)=\begin{cases}1 & \text{if } n \text{ is odd}\\ 2 & \text{if } n \text{ is even}\end{cases}$$

If $n$ is even, then $n+2$ is also even so that $f(n+2)=1=f(n)$. If $n$ is odd, then $n+2$ is also odd and therefore $f(n+2)=2=f(n)$. Hence, in either case $f(n+2)=f(n)$ for all $n\in\mathbb{N}$. Therefore, 2 is the period of $f$.

**THEOREM 1.3 (EUCLID'S ALGORITHM)** Let $n>1$ be an integer. If $m$ is any integer, there exists an integer $K$ and a non-negative integer $r$ such that $m=Kn+r, r\in\{0,1,2,\ldots,(n-1)\}$.

One can observe that when $m$ is divided by $n$, $K$ is the quotient and $r$ is the remainder. The following is the *finest* example of a periodic function.

**DEFINITION 1.15** **Congruence Function** Let $n>1$ be an integer. Define $f:\mathbb{Z}\to\mathbb{Z}$ by $f(m)=r$ for $m\in\mathbb{Z}$ and $r$ is the remainder when $m$ is divided by $n$. Then for any $m\in\mathbb{Z}$,

$$\begin{aligned}f(m+n)&=f((Kn+r)+n)\\&=f((K+1)n+r)\\&=r\quad(\because 0\le r\le n-1)\\&=f(m)\end{aligned}$$

Thus, $f(m+n)=f(m)$. Therefore, *$n$ is the period of $f$*. The function $f$ is denoted by $\mathbb{Z}^{(n)}$ so that $\mathbb{Z}^{(n)}(m)=r$, where $r$ is the remainder when $m$ is divided by $n$. We call this function as ***congruence function with index n***. For example

$$\mathbb{Z}^{(2)}_{(m)}=\begin{cases}0 & \text{if } m \text{ is even}\\ 1 & \text{if } m \text{ is odd}\end{cases}$$

is a congruence function with index 2.

**Try it out** If $p_1, p_2$ are periods of $f_1(x)$ and $f_2(x)$, respectively, and there exist positive integers $m$ and $n$ such that $mp_1=np_2=p$, then $p$ is a period of $af_1(x)+bf_2(x)$ for all non-zero constants $a$ and $b$.

**DEFINITION 1.16** **Step Function** Suppose $f:[a,b]\to\mathbb{R}$ is a function. Further suppose that $a=a_0<a_1<a_2<\cdots<a_{n-1}<a_n=b$. If there exist constants $k_1, k_2, \ldots, k_n$ such that $k_{i-1}<k_i$ for $i=2, 3, \ldots, n$ and $f(x)=k_i$ for $x\in[a_{i-1}, a_i]$ for $i=1,2,\ldots,n$, then $f$ is called the *step function* on $[a,b]$ with steps $k_1, k_2, \ldots, k_n$.

We observe that a step function assumes only finite number of values, that is range of a step function is a finite set.

## Examples

**1.** Suppose $f:[0,1]\to\mathbb{R}$ is such that

$$f(x)=\begin{cases}1/2 & \text{if } 0\le x<1/2\\ 1 & \text{if } 1/2\le x<1\\ 0 & \text{if } x=1\end{cases}$$

Here $a=0$, $b=1$ and $a_0=0$, $a_1=1/2$, $a_2=1$, $k_1=1/2$, $k_2=1$.

**2.** Suppose $f:[-1,1]\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}1 & \text{if } x\in[-1,-1/2)\\ 2 & \text{if } x\in[-1/2,0)\\ 0 & \text{if } x\in[0,3/4)\\ -1 & \text{if } x\in[3/4,1)\\ 3 & \text{if } x=1\end{cases}$$

Then $f$ is a step function with steps 1, 2, 0, –1. Here $a=-1, b=1, a_0=-1, a_1=1/2, a_2=0, a_3=3/4, a_4=1$ with $k_1=1, k_2=2, k_3=0, k_4=-1$.

**3.** Define $f:[-2,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases}1 & \text{if } x\in[-1,0)\\ 0 & \text{otherwise}\end{cases}$$

Then $f$ is a step function with steps 0, 1, 0 and $a=-2$, $b=-1$, $a_0=-2$, $a_1=-1$, $a_2=0$, $a_3=1$ and $k_1=0$, $k_2=1$, $k_3=0$. This is so because explicitly

$$f(x)=\begin{cases}0 & \text{if } x\in[-2,-1)\\ 1 & \text{if } x\in[-1,0)\\ 0 & \text{if } x\in[0,1]\end{cases}$$

**Try it out** If $a<b$ are integers, then the function $f:[a,b]\to\mathbb{R}$ defined by

$$f(x)=[x]$$

where $[x]$ denotes integral part of $x$, is a step function.

***Note:*** The function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

is not a step function, even though $f$ assumes finite number of values. There are many functions that are not step functions.

## 1.4 | Graphs of Functions

The concept of function is introduced where the domains and codomains may be any sets. However, for calculus we consider those functions (real-valued functions) where the domain and codomain both are non-empty subsets of $\mathbb{R}$. In the category of real-valued functions, in this section we introduce the idea of graph of a function which is helpful in studying some vital clues of the function. In this pursuit, we consider a plane in which we set up $x$-axis and $y$-axis; this plane is called the $xy$-plane.

**DEFINITION 1.17** **Graph of a Function** Let $S$ be a non-empty subset of $\mathbb{R}$ and $f:S\to\mathbb{R}$ a function. Then $f=\{(x,f(x))\mid x\in S\}$ can be regarded as a set of points on the $xy$-plane, with $x$ as $x$-coordinate and $f(x)$ as $y$-coordinate. Keeping this in mind, in general, it is possible to plot the set of points $\{(x,f(x))\mid x\in S\}$ on the $xy$-plane. The resultant figure on the $xy$-plane is called the *graph of f*.

For almost all functions we have discussed, we can plot their graphs.

### Example 1.1

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x\ \forall\, x\in\mathbb{R}$. Draw its graph.

***Solution:*** The graph of $f=\{(x,x)\mid x\in\mathbb{R}\}$ is shown in Fig. 1.1.

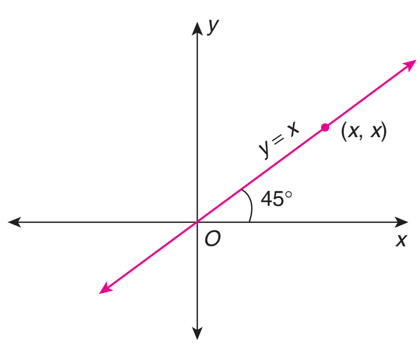



**FIGURE 1.1** Example 1.1.

## Example 1.2

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=|x|$. Draw its graph.

***Solution:*** The graph of $f=\{(x,|x|)\,|\,x\in\mathbb{R}\}$ is shown in Fig 1.2.

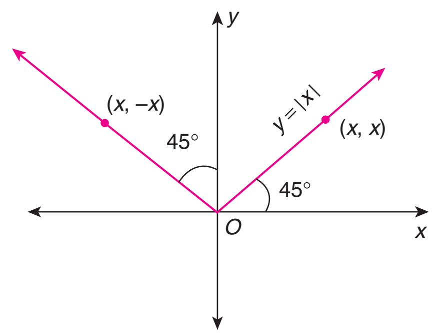



**FIGURE 1.2** Example 1.2.

## Example 1.3

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=2x\ \forall\, x\in\mathbb{R}$. Draw its graph.

***Solution:*** The graph is shown in Fig. 1.3.

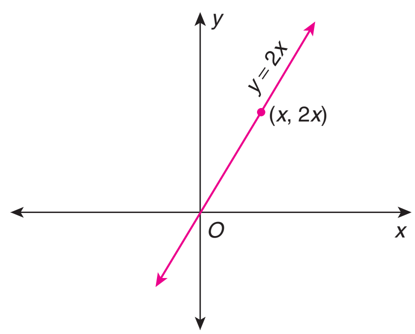



**FIGURE 1.3** Example 1.3.

## Example 1.4

Define $f:[-2,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases}0 & \text{if } x\in[-2,-1)\\ 1 & \text{if } x\in[-1,0)\\ 0 & \text{if } x\in[0,1]\end{cases}$$

Draw its graph.

***Solution:*** The graph is shown in Fig. 1.4 by colored line segments.

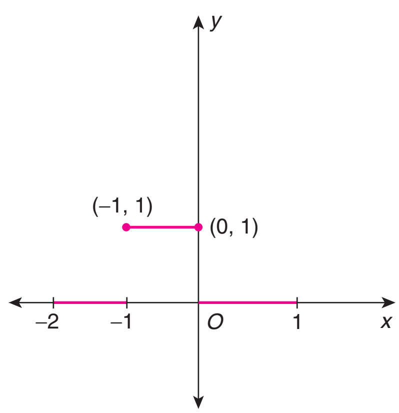



**FIGURE 1.4** Example 1.4.

It may be observed that there are functions for which drawing a graph is not possible. The following example shows such a function.

## Example 1.5

Let the function $f:[0,1] \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

Draw its graph.

***Solution:*** We know that in between any two real numbers, there exist rational numbers as well as irrational numbers. This property is called density property (see Theorems 0.14 and 0.17). The graph of $f$, if forcibly drawn, will look like two straight line segments as shown in Fig. 1.5.

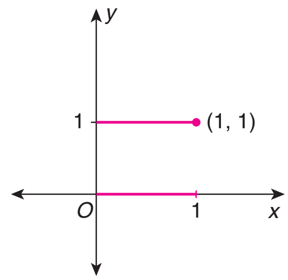



**FIGURE 1.5** Example 1.5.

***Note:***

1. Observe that a graph may not give rise to a function. For example, look at the graph of Fig. 1.6 which shows that $(1/2, a)$ and $(1/2, b)$ are points on the graph. Since $a \neq b$, 1/2 has two images.

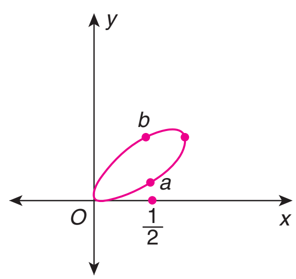



**FIGURE 1.6**

2. If $f:[a,b] \to \mathbb{R}$ is a function, then for any $x \in [a,b]$, the line through $x$, drawn parallel to $y$-axis, meets the graph at exactly one point as shown in Fig. 1.7.

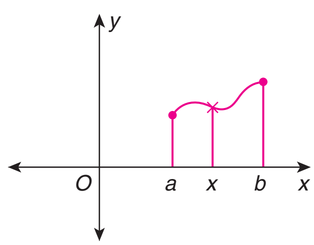



**FIGURE 1.7**

## 1.5 | Construction of Graphs and Transforming Theorem

In this section, we illustrate the process of drawing graphs of functions of the form $y = \lambda f(ax+b)+\mu$, where $\lambda$, $\mu$, $a$ and $b$ are some real numbers from the graph of the given function $y = f(x)$. Suppose $f:[a,b] \to \mathbb{R}$ is a function. Then the set of points on the graph of $f$ are given by

$$\{(x, f(x)) \mid x \in [a, b]\}$$

In other words, if we write $y = f(x)$, then the graph of $f$ is the set of points

$$\{(x, y) \mid y = f(x), x \in [a, b]\}$$

### 1.5.1 Transformation of y-Coordinate by a Constant

Suppose $k \in \mathbb{R}$ and the function $f$ is changed or transformed to $f(x)+k$. That is, now we have a new function $g:[a, b] \to \mathbb{R}$ defined by

$$g(x) = f(x) + k \ \forall x \in [a, b]$$

By this transformation, the graph of $f$ is transformed to the set of points

$$\begin{aligned}\{(x, g(x)) \mid x \in [a, b]\} &= \{(x, f(x)+k) \mid x \in [a, b]\} \\ &= \{(x, y+k) \mid y = f(x),\ x \in [a, b]\}\end{aligned}$$

In other words, the point $(x, y)$ on the graph of $y = f(x)$ is changed to $(x, y + k)$ [see Fig. 1.8(a)]. Hence the graph of $f$ and the transformed graph of $f$ (i.e., the graph of $g$) look like as shown in Fig. 1.8(b). This is a simple transformation where the $y$-coordinate $y$ is changed to $y + k$.

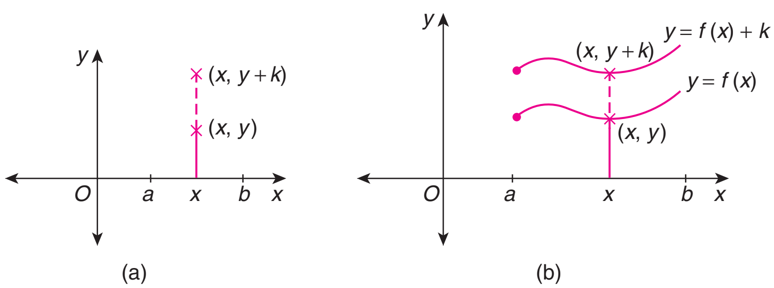



**FIGURE 1.8**

## Example 1.6

For the function $f(x) = x$ for $0 \le x \le 4$ draw the graph and the transformed graph. Assume $k = 3$.

***Solution:*** The graph of $y = f(x) = x$ and the transformed graph of $f$ are shown in Fig. 1.9.

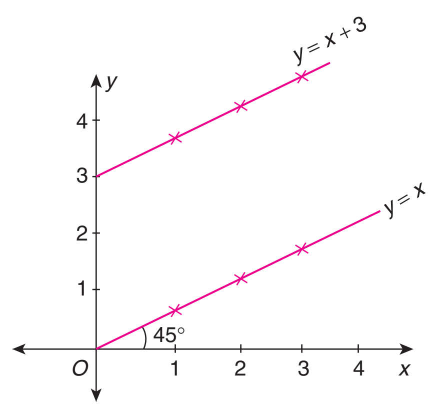



**FIGURE 1.9** Example 1.6.

## Example 1.7

Let $y = 2^x$, $x \in \mathbb{R}$. Define $y = 2^x + 1$. Draw the graph and the transformed graph.

***Solution:*** Shift the graph of $y = 2^x$ by 1 unit parallel to the existing graph. This yields the graph of $y = 2^x + 1$ (see Fig. 1.10).

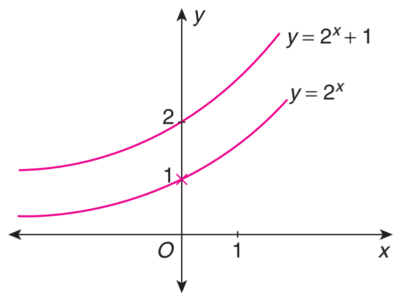



**FIGURE 1.10** Example 1.7.

### 1.5.2 Transformation of the x-Coordinate by a Constant

In this subsection we will discuss how to draw the graph of $y = f(x+k)$ from the graph of $y = f(x)$. Here the $x$-coordinate is transformed by a constant $k$. That is the point $(x, f(x))$ is transformed to the point $(x, f(x+k))$. See the graphs of $y = f(x)$ and $y = f(x+k)$ in general [Fig. 1.11(a)] and in particular $y = x^2$ and $y = (x+1)^2$ [Fig. 1.11(b)].

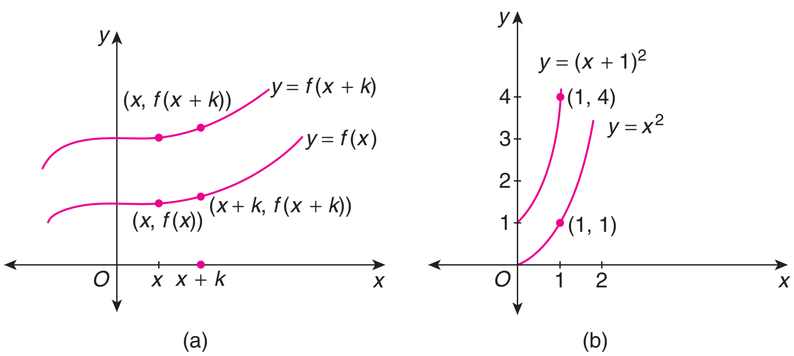



**FIGURE 1.11** (a) Graph of $y = f(x)$ and $y = f(x+k)$; (b) graph of $y = x^2$ and $y = (x+1)^2$.

### 1.5.3 Construction of y = kf(x) from y = f(x) where k ≠ 0 is a Constant

In this we have $k > 1$, $0 < k < 1$ and $k < 0$. The point $(x, f(x))$ will be transformed to $(x, kf(x))$.

1. If $k > 1$, the graph of $y = kf(x)$ is an expansion of $y = f(x)$ [Fig. 1.12(a)].
2. If $0 < k < 1$, then the graph is a contraction [Fig. 1.12(b)].
3. If $k < 0$, then the graph of $y = kf(x)$ is the reflection of $y = -kf(x)$ [Fig. 1.12(c)].

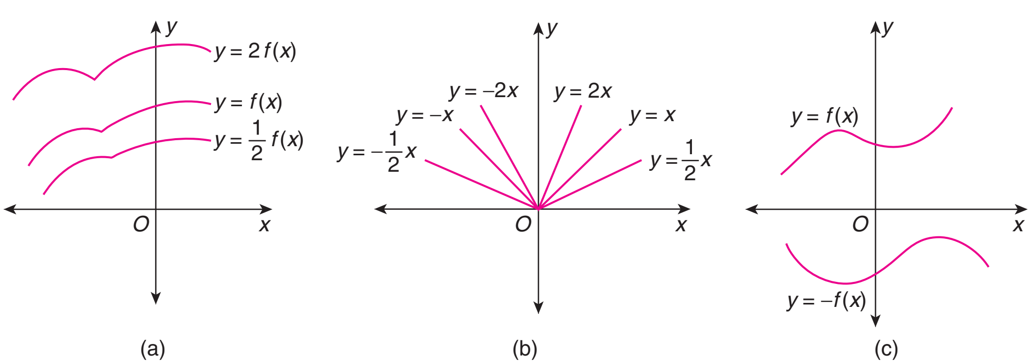



**FIGURE 1.12** (a), (b), (c) illustrate points (1), (2) and (3) above.

## Example 1.8

Draw the graph of $y = f(kx)$.

***Solution:*** The graph of $y = f(kx)$ can be obtained from $y = f(x)$ by contraction towards the $y$-axis when $k > 0$ and expansion from the $y$-axis when $0 < k < 1$. That is, $k$-fold contraction ($k > 1$) and $(1/k)$-fold expansion $(0 < k < 1)$ (see Fig. 1.13).

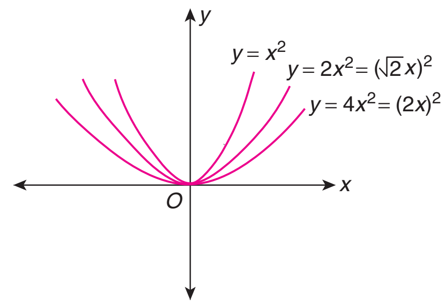



**FIGURE 1.13** Example 1.8.

***Note:*** The essence of the transformation of the graph is the point $(x, f(x))$ which is transformed to the point $(x, g(x))$ where $g(x)$ is a transformation of $f(x)$ such as

$$g(x) = f(x) + k$$

$$g(x) = f(x + k)$$

$$g(x) = kf(x)$$

and

$$g(x) = f(kx)$$

## 1.6 | Limit of a Function

In this section, we introduce the concept of a function which is the root for the entire development of calculus (in general, for Mathematical Analysis), the concepts of left and right limits of a function, graphical meaning of the limit and discuss limits of some standard functions. Further, we assume the limits of some functions whose proofs are beyond the scope of this book. First, let us begin with the concept of an interior point of an interval and neighbourhood of a point on the real line $\mathbb{R}$.

**DEFINITION 1.18** **Interior Point** If $[a, b]$ is a closed interval and $a < c < b$, then $c$ is called an *interior point* of the interval $[a, b]$.

**DEFINITION 1.19** **Neighbourhood** Let $c$ be an interior point of a closed interval $[a, b]$ and $\varepsilon = \min\{c - a, b - c\}$ so that $(c - \varepsilon, c + \varepsilon) \subset (a, b)$. If $0 < \delta < \varepsilon$, then the interval $(c - \delta, c + \delta)$ is called $\delta$-neighbourhood of $c$. The intervals $(c - \delta, c)$ and $(c, c + \delta)$ are called left and right $\delta$-neighbourhoods, respectively, of $c$.

**DEFINITION 1.20** **Limit** Let $c$ be an interior point of $[a, b]$ and $f$ a real-valued function defined on $[a, b]$ (possibly except at $c$). Let $l$ be a real number. If to each $\varepsilon > 0$, there corresponds a small positive number $\delta$ such that $(c - \delta, c + \delta) \subset (a, b)$ and $|f(x) - l| < \varepsilon$ (equivalently $l - \varepsilon < f(x) < l + \varepsilon$) for all $x \in (c - \delta, c + \delta)$ and $x \neq c$ then we say that $f(x)$ tends to $l$ as $x$ tends $c$. In this case we write

$$f(x) \to l \text{ as } x \to c \quad \text{or} \quad \lim_{x \to c} f(x) = l \quad \text{or} \quad \underset{x \to c}{\text{Lt}}\, f(x) = l$$

**DEFINITION 1.21** **Right Limit** Suppose $a \le c < b$ and $f$ is a real-valued function defined on $(a, b)$ (possibly except at $c$). Suppose there exists a real number $l$ such that to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $(c, c + \delta) \subset (a, b)$ and $|f(x) - l| < \varepsilon$ for all $x \in (c, c + \delta)$. Then we say that limit of $f$ as $x$ approaches $c$ from the right side is $l$. In this case we write

$$\lim_{x \to c+0} f(x) = l \quad \text{or} \quad \underset{x \to c+0}{\text{Lt}}\, f(x) = l$$

**DEFINITION 1.22** **Left Limit** Suppose $a < c \leq b$ and $f$ is a real-valued function defined on $(a, b)$ (possibly except at $c$). Suppose there exists a real number $l$ such that to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $(c - \delta, c) \subset (a, b)$ and $|f(x) - l| < \varepsilon \ \forall x \in (c - \delta, c)$. Then we say that $f$ approaches $l$ from the left of $c$ and we write

$$\lim_{x \to c-0} f(x) = l \quad \text{or} \quad \underset{x \to c-0}{\text{Lt}} f(x) = l$$

The concept of limit can be explained lucidly using the graph of the function (see Fig. 1.14). Given a neighbourhood $N_\varepsilon(l) = (l - \varepsilon, l + \varepsilon)$ of $l$, there exists a neighbourhood $N_\delta(c) = (c - \delta, c + \delta)$ such that the rectangle formed by the two vertical lines at $c - \delta$, $c + \delta$ and the horizontal lines drawn at $l - \varepsilon$, $l + \varepsilon$ contain some part of the graph of $y = f(x)$.

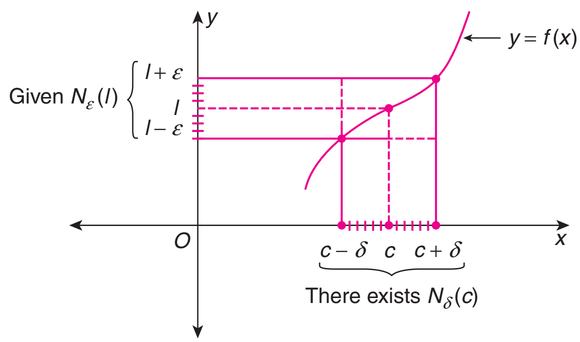



**FIGURE 1.14**

## Example 1.9

Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x \ \forall x \in [0, 1]$. Take $c = 1/2$ and $l = 1/2$. Calculate its limit.

***Solution:*** See Fig. 1.15. Let $\varepsilon > 0$. Choose $\delta = \text{Min}\{\varepsilon, 1/2\}$ so that

$$\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right) \subset (0, 1)$$

Further,

$$x \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right) \quad \text{and} \quad x \neq \frac{1}{2}$$

$$\Rightarrow \left|f(x) - \frac{1}{2}\right| = \left|x - \frac{1}{2}\right| < \delta \leq \delta \text{ (by the choice of } \delta)$$

$$\Rightarrow \left|f(x) - \frac{1}{2}\right| < \varepsilon$$

Therefore

$$f(x) \to \frac{1}{2} \quad \text{as } x \to \frac{1}{2}$$

or

$$\lim_{x \to 1/2} f(x) = \frac{1}{2}$$

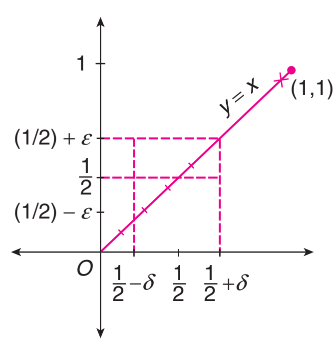



**FIGURE 1.15** Example 1.9.

## Example 1.10

Suppose

$$f(x) = \begin{cases} 2 & \forall x \in [-1, 1] \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Calculate its limit at $O$.

***Solution:*** See Fig. 1.16. Suppose $\varepsilon > 0$ and $\delta = \text{Min}\{\varepsilon, 1\}$. Then $(0 - \delta, 0 + \delta) \subset (-1, 1)$. Take $l = 2$ and $c = 0$. Now,

$x \in (0-\delta,\ 0+\delta)$ and $x \neq 0$ implies

$$\begin{aligned} |f(x)-2| &= |2-2| \\ &= 0 \\ &< \varepsilon \end{aligned}$$

Hence $f(x) \to 2$ as $x \to 0$.

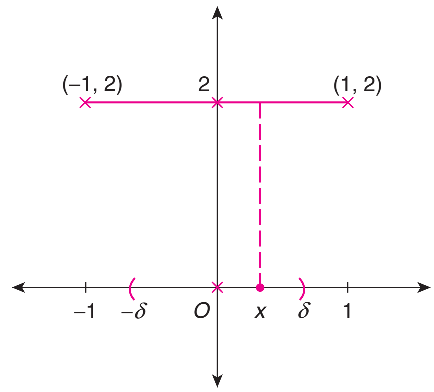



**FIGURE 1.16** Example 1.10.

## Example 1.11

Obtain the limit for $f(x) = k\ \forall x \in [a, b]$.

***Solution:*** Let $a < c < b$. Let $\varepsilon > 0$ and take $\delta = \text{Min}\{\varepsilon, c-a, b-c\}$. Then

$$(c-\delta, c+\delta) \subset (a, b)$$

and $|f(x)-k| = |k-k| = 0 < \varepsilon\ \forall x \in (c-\delta, c+\delta)$

Hence,

$$\lim_{x\to c} f(x) = k$$

## Example 1.12

Obtain the limit for $f(x) = x^2\ \forall x \in [-1, 1]$ as $x \to 0$ and $x \neq 0$.

***Solution:*** Take $c = 0$ and $l = 0$. Suppose $\varepsilon > 0$. Let

$$\delta = \text{Min}\{\sqrt{\varepsilon}, 1/2\}$$

so that $0 < \delta \le 1/2$ and $0 < \delta \le \sqrt{\varepsilon}$. Then $x \in (0-\delta,\ 0+\delta)$ implies

$$|f(x)-0| = |x^2| = x^2 < \delta^2 \le (\sqrt{\varepsilon})^2 = \varepsilon$$

Therefore $\lim_{x\to 0} f(x) = 0$ or equivalently $x^2 \to 0$ as $x \to 0$.

## Example 1.13

Let $f:[-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Obtain the left and right limits.

***Solution:*** See Fig. 1.17.

1. Take $c = 0$ and $l = 0$. Let $0 < \varepsilon < 1/2$ and $\delta = \varepsilon$. Then $(c-\delta, 0) \subset (-1, 1)$ (since $-1 < -1/2 < 0$) and $x \in (-\delta, 0)$. This implies

$$|f(x)-l| = |x-0| = |x| = -x < \delta = \varepsilon$$

Therefore, the left limit of $f$ at 0 is 0 or $\lim_{x\to 0-0} f(x) = 0$.

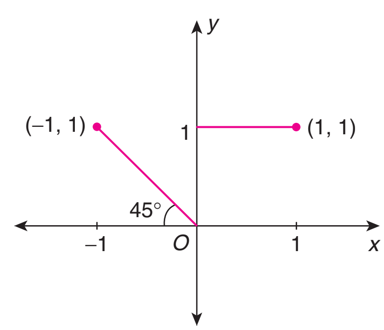



**FIGURE 1.17** Example 1.13.

**2.** Take $c = 0$ and $l = 1$. Let $0 < \varepsilon < 1/2$ and $\delta = \varepsilon$. Then

$$(0, c+\delta) = (0, \delta) \subset (-1, 1)$$

and $$x \in (0, c+\delta)$$

This implies

$$|f(x) - l| = |1 - 1| = 0 < \varepsilon$$

[since $f(x) = 1$]. Therefore the right limit of $f$ at 0 is 1 or $\lim_{x \to 0+0} f(x) = 1$.

## Example 1.14

Using the function given in Example 1.13, show that $\lim_{x \to 0} f(x)$ does not exist.

***Solution:*** Suppose the limit exists and $\lim_{x \to 0} f(x) = L$, say. Then by definition, to $\varepsilon = 1/2$, there corresponds $\delta > 0$ such that

$$x \in (0-\delta, 0+\delta) \Rightarrow |f(x) - L| < \varepsilon = 1/2$$

Hence

$$\begin{aligned} x \in (-\delta, 0) &\Rightarrow |f(x) - L| = |-x - L| = |x + L| < 1/2 \\ &\Rightarrow -1/2 < x + L < 1/2 \\ &\Rightarrow x < 1/2 - L \\ &\Rightarrow 0 = \sup\{x \mid x \in (-\delta, 0)\} \\ &\le \frac{1}{2} - L \Rightarrow L \le \frac{1}{2} \end{aligned} \tag{1.1}$$

Again

$$\begin{aligned} y \in (0, \delta) &\Rightarrow |f(y) - L| < 1/2 \\ &\Rightarrow |1 - L| < \frac{1}{2} \quad [\because f(y) = 1] \\ &\Rightarrow -\frac{1}{2} < 1 - L < \frac{1}{2} \\ &\Rightarrow \frac{1}{2} < L \end{aligned} \tag{1.2}$$

From Eqs. (1.1) and (1.2) we have $1/2 < L \le 1/2$ which is a contradiction. This contradiction arose due to our supposition that $\lim_{x \to 0} f(x)$ exists. Hence $\lim_{x \to 0} f(x)$ does not exist.

The following example illustrates that for a function *neither of the left and right limit* exist nor the limits exist at a point.

## Example 1.15

Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that the limit does not exist.

***Solution:*** If possible, assume that, $\lim_{x \to 0+0} f(x)$ exists and is equal to $L$. Then to $\varepsilon = 1/4$, there corresponds $\delta > 0$ such that $(0, \delta) \subset [0, 1]$ and

$$x \in (0, \delta) \Rightarrow |f(x) - L| < \frac{1}{4}$$

According to Theorems 0.14 and 0.17 [part (2)] both rational and irrational numbers exist in the interval $(0, \delta)$. Hence, for rational number $x \in (0, \delta)$ we have

$$\begin{aligned} |f(x) - L| &< \frac{1}{4} \\ \Rightarrow |1 - L| &< \frac{1}{4} \\ \Rightarrow -\frac{1}{4} < 1 - L &< \frac{1}{4} \\ \Rightarrow \frac{3}{4} &< L \end{aligned} \tag{1.3}$$

Again, irrational number $x \in (0, \delta)$ implies

$$\begin{aligned} |f(x) - L| &< \frac{1}{4} \\ \Rightarrow |0 - L| &< \frac{1}{4} \quad [\because f(x) = 0] \\ \Rightarrow -\frac{1}{4} < L &< \frac{1}{4} \\ \Rightarrow L &< \frac{1}{4} \end{aligned} \tag{1.4}$$

From Eqs. (1.3) and (1.4) we have $3/4 < L < 1/4$ which is a contradiction. Therefore, $\lim_{x \to 0+0} f(x)$ does not exist.

In a similar way as shown in Example 1.15 we can prove that $\lim_{x\to 0-0} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. In fact, $\lim_{x\to c-0} f(x)$, $\lim_{x\to c+0} f(x)$ and $\lim_{x\to c} f(x)$ do not exist for any $c \in (0,1)$.

**Theorem 1.4 (Uniqueness of the Limit)** Let $c \in (a, b)$ and $f$ be a real-valued function defined on $[a, b]$ (possibly except at $c$). Suppose $\lim_{x\to c} f(x) = l$ and $\lim_{x\to c} f(x) = L$. Then $l = L$. In other words, at the point $c \in (a,b)$, $\lim_{x\to c} f(x)$ is unique, if it exists.

**Proof** Let $\varepsilon > 0$. Then by definition, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $(c - \delta_1, c + \delta_1) \subset (a, b)$ and $(c - \delta_2, c + \delta_2) \subset (a, b)$. Therefore

$$x \in (c-\delta_1, c+\delta_1) \Rightarrow |f(x)-l| < \frac{\varepsilon}{2}$$

and

$$x \in (c-\delta_2, c+\delta_2) \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}$$

Now, let $\delta = \text{Min}\{\delta_1, \delta_2\}$ so that $x \in (c - \delta, c + \delta)$. This implies $x \in$ both $(c - \delta_1, c + \delta_1)$ and $(c - \delta_2, c + \delta_2)$. Therefore $x \in (c - \delta, c + \delta)$ implies

$$\begin{aligned}|l-L| &= |f(x)-L-f(x)+l| \\ &\le |f(x)-l| + |f(x)-L| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

That is, $|l-L| < \varepsilon$ for all $\varepsilon > 0$. Therefore $|l-L| = 0$ or $l = L$. ■

***Note:*** In a similar way we can show that

1. $\lim_{x\to c+0} f(x)$ is unique (if it exists) where $a \le c < b$.
2. $\lim_{x\to c-0} f(x)$ is unique (if it exists) where $a < c \le b$.

The following theorem gives the relation between left limit, right limit and limit of a function at a point.

**Theorem 1.5** Suppose $a < c < b$ and $f : [a, b] \to \mathbb{R}$ is defined except possibly $c$. Then,

1. If $\lim_{x\to c} f(x)$ exists and is equal to $l$, then both $\lim_{x\to c+0} f(x)$ and $\lim_{x\to c-0} f(x)$ exist and are equal to $l$.
2. Conversely, if both $\lim_{x\to c+0} f(x)$ and $\lim_{x\to c} f(x)$ exist and are equal to $l$ say, then $\lim_{x\to c} f(x)$ exists and is equal to $l$.

**Proof**

1. Suppose $\lim_{x\to c} f(x)$ exists and is equal to $l$. Then, to $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(c - \delta, c + \delta) \subset (a, b)$$

and

$$|f(x)-l| < \varepsilon \ \forall\, x \in (c-\delta, c+\delta),\ x \ne c$$

Hence $(c - \delta, c) \subset (a, b)$ and $x \in (c - \delta, c)$ implies

$$|f(x)-l| < \varepsilon \tag{1.5}$$

and $(c, c + \delta) \subset (a, b)$ and $x \in (c, c + \delta)$ implies

$$|f(x)-l| < \varepsilon \tag{1.6}$$

Equations (1.5) and (1.6) show that $\lim_{x\to c-0} f(x)$ exists and is equal to $l$ and also that $\lim_{x\to c+0} f(x)$ exists and is equal to $l$.

Conversely, suppose $\lim_{x\to c-0} f(x)$ exists and is equal to $l$ and $\lim_{x\to c+0} f(x)$ exists and is equal to $l$. Let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$(c-\delta_1, c) \subset (a,b) \Rightarrow |f(x)-l| < \varepsilon \;\; \forall\, x \in (c-\delta_1, c)$$

and $$(c, c+\delta_2) \subset (a,b) \Rightarrow |f(x)-l| < \varepsilon \;\; \forall\, x \in (c, c+\delta_2)$$

Let $\delta = \text{Min}\{\delta_1, \delta_2\}$. Hence

$$c \neq x \in (c-\delta, c+\delta) \subset (a,b)$$
$$\Rightarrow |f(x)-l| < \varepsilon \;\; \forall\, x \in (c-\delta, c+\delta)$$

Therefore $\lim_{x\to c} f(x) = l$. ■

**Corollary 1.1** Suppose $a < c < b$ and $f$ is a real-valued function defined on $[a, b]$ (except possibly at $c$). If $\lim_{x\to c} f(x)$ exists then $f$ is bounded on some neighbourhood of $c$.

**Proof** Suppose $\lim_{x\to c} f(x)$ exists and is equal to $l$, say. Therefore to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and $|f(x)-l| < \varepsilon$ for all $x \in (c-\delta, c+\delta)$ and $x \neq c$. In particular, to $\varepsilon = 1$, there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta)$ and $x \neq c$ implies

$$|f(x)-l| < 1 \;\forall\, x \in (c-\delta, c+\delta)$$

possibly except at $x = c$. If $f(c)$ is not defined we take $M = (|l|+1)$ and if $f(c)$ is defined, then we take

$$M = \text{Max}\{|f(c)|, |l|+1\}$$

In any case $x \in (c - \delta, c + \delta)$ implies

$$|f(x)| - |l| \leq |f(x)-l| < 1$$
$$\Rightarrow |f(x)| \leq |l|+1 \leq M$$

Therefore, according to Definition 0.3, $f$ is bounded on $(c - \delta, c + \delta)$. ■

***Note:*** In Corollary 1.1, $[a, b]$ can replaced by any non-empty subset $A$ of $\mathbb{R}$.

In the following theorems, we discuss about the limits of sum, product and quotient of real-valued functions having limits at a point.

**Theorem 1.6** Suppose $a < c < b$ and $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ are defined (except possibly at $c$). Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then $\lim_{x\to c}(f+g)(x)$ exists and is equal to $L + M$. Converse need not be true.

**Proof** Let $\varepsilon > 0$. Then by hypothesis, $\exists\, \delta_1 > 0 \ni (c - \delta_1, c + \delta_1) \subset (a, b)$, $x \in (c - \delta, c + \delta_1)$ and $x \neq c$. This implies

$$|f(x)-L| < \frac{\varepsilon}{2\lambda} \tag{1.7}$$

where $\lambda = |L| + |M| + 1$. Also $\exists\, \delta_2 > 0 \ni (c - \delta_2, c + \delta_2) \subset (a, b)$ and $x \in (c - \delta_2, c + \delta_2)$, $x \neq c$. This implies

$$|f(x)-M| < \frac{\varepsilon}{2\lambda} \tag{1.8}$$

Let $\delta = \text{Min}\{\delta_1, \delta_2\}$. Then from Eqs. (1.7) and (1.8), $(c-\delta, c+\delta) \subset (a, b)$, $x \in (c-\delta, c+\delta)$ and $x \neq c$. These imply

$$\begin{aligned} |(f+g)(x)-(L+M)| &= |f(x)+g(x)-L-M| \\ &\leq |f(x)-L|+|g(x)-M| \\ &< \frac{\varepsilon}{2\lambda}+\frac{\varepsilon}{2\lambda}=\frac{\varepsilon}{\lambda}\leq \varepsilon \end{aligned}$$

Therefore

$$\lim_{x\to c}(f+g)(x) = L+M$$

We now show that the converse is not true. Consider the functions $f, g:[0,1]\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x) = \begin{cases} -1 & \text{if } x \text{ rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

so that $(f+g)(x)=0\ \forall x\in[0,1]$ and hence for $c\in(0,1)$,

$$\lim_{x\to c}(f+g)(x)=0$$

However, neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exist. ■

***Note:*** Replacing $g$ by $-g$, we have that $\lim_{x\to c}(f-g)(x) = L-M$.

**THEOREM 1.7** Suppose $a<c<b$ and $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ are defined except possibly at $c$. If $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, then $\lim_{x\to c}(f\cdot g)(x)$ exists and is equal to $LM$.

**PROOF** Let $\varepsilon>0$. Then there correspond $\delta_1>0$ and $\delta_2>0$ such that $(c-\delta_1, c+\delta_1)\subset(a,b)$, $x\in(c-\delta_1, c+\delta_1)$ and $x\neq c$. This implies

$$|f(x)-L| < \frac{\varepsilon}{2\lambda} \tag{1.9}$$

where $\lambda = |L|+|M|+1$. Also $x\in(c-\delta_2, c+\delta_2)$ and $x\neq c$ implies

$$|g(x)-M| < \frac{\varepsilon}{2\lambda} \tag{1.10}$$

Now $x\in(c-\delta, c+\delta)$ and $x\neq c$ implies [using Eqs. (1.9) and (1.10)]

$$\begin{aligned} |(f\cdot g)(x)-LM| &= |f(x)g(x)-LM| \\ &= |(f(x)-L)g(x)+L(g(x)-M)| \\ &\leq |f(x)-L||g(x)|+|L||g(x)-M| \\ &\leq \frac{\varepsilon}{2\lambda}|g(x)|+|L|\frac{\varepsilon}{2\lambda} \end{aligned} \tag{1.11}$$

Therefore, for $x\in(c-\delta, c+\delta)$, $x\neq c$ and from Eq. (1.10) we have

$$|g(x)|-|M| \leq |g(x)-M| < \frac{\varepsilon}{2\lambda}$$

so that

$$|g(x)| < |M| + \frac{\varepsilon}{2\lambda} < |M| + 1$$

Now, from Eq. (1.11) we have $x \in (c-\delta, c+\delta)$, $x \neq c$ which implies

$$\begin{aligned} |(f \cdot g)(x) - LM| &\leq \frac{\varepsilon}{2\lambda}|g(x)| + |L|\frac{\varepsilon}{2\lambda} \\ &\leq \frac{\varepsilon}{2\lambda}(|M| + 1 + |L|) \\ &= \frac{\varepsilon}{2} < \varepsilon \quad (\because \lambda = |L| + |M| + 1) \end{aligned}$$

Thus, given a positive number $\varepsilon$, there exists $\delta > 0$ such that $x \in (c-\delta, c+\delta) \subset (a, b)$ and $x \in (c-\delta, c+\delta)$, $x \neq c$ which implies

$$|(f \cdot g)(x) - LM| < \varepsilon$$

Hence, $\lim\limits_{x \to c}(f \cdot g)(x)$ exists and is equal to $LM$. ■

**THEOREM 1.8** Suppose $a < c < b$ and $g : [a, b] \to \mathbb{R}$ is defined except possibly at $c$. Let $\lim\limits_{x \to c} g(x)$ exists and be non-zero. Then, there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset (a, b)$ and

$$x \in (c-\delta, c+\delta), x \neq c \Rightarrow g(x) \neq 0$$

**PROOF** Let $M = \lim\limits_{x \to c} g(x)$ so that by hypothesis $M \neq 0$. Without loss of generality, we may suppose that $M > 0$. Let $0 < \varepsilon < M$ (if $M < 0$, then we consider $|M|$). Then by hypothesis, there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset (a, b)$ and $x \in (c-\delta, c+\delta)$, $x \neq c$ which implies

$$\begin{aligned} &|g(x) - M| < \varepsilon \\ &\Rightarrow -\varepsilon < g(x) - M < \varepsilon \\ &\Rightarrow M - \varepsilon < g(x) < M + \varepsilon \\ &\Rightarrow 0 < M - \varepsilon < g(x) \quad (\because \varepsilon < M) \\ &\Rightarrow 0 < g(x) \\ &\Rightarrow g(x) \neq 0 \ \forall x \in (c-\delta, c+\delta) \end{aligned}$$

■

**THEOREM 1.9** Suppose $a < c < b$ and $g : [a, b] \to \mathbb{R}$ is defined except possibly at $c$. If $\lim\limits_{x \to c} g(x)$ exists and is not equal to zero, then $\lim\limits_{x \to c}(1/g)(x)$ exists and is equal to $(\lim\limits_{x \to c} g(x))^{-1}$.

**PROOF** Let $\lim\limits_{x \to c} g(x) = M \neq 0$ (by hypothesis). We may suppose that $M > 0$. Let $\varepsilon > 0$. Choose $\eta > 0$ such that

$$0 < \eta < \mathrm{Min}\left\{\varepsilon, \frac{M^2\varepsilon}{1 + M\varepsilon}\right\}$$

Corresponding to $\eta > 0$, there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset (a, b)$ and

$$x \in (c-\delta, c+\delta), x \neq c \Rightarrow |g(x) - M| < \eta \tag{1.12}$$

$$\Rightarrow 0 < M - \eta < g(x) < M + \eta \tag{1.13}$$

Now, $x \in (c-\delta, c+\delta)$ implies [using Eqs. (1.12) and (1.13)]

$$\begin{aligned}\left|\left(\frac{1}{g}\right)(x) - \frac{1}{M}\right| &= \left|\frac{1}{g(x)} - \frac{1}{M}\right| \\ &= \frac{|M - g(x)|}{M\,|g(x)|} \\ &< \frac{\eta}{M(M-\eta)} \end{aligned} \tag{1.14}$$

By the choice of $\eta$, we have

$$\begin{aligned}\eta < \frac{M^2\varepsilon}{1+M\varepsilon} &\Rightarrow \eta(1+M\varepsilon) < M^2\varepsilon \\ &\Rightarrow \eta < M^2\varepsilon - \eta M\varepsilon = M(M-\eta)\,\varepsilon \\ &\Rightarrow \frac{\eta}{M(M-\varepsilon)} < \varepsilon \end{aligned} \tag{1.15}$$

From Eqs. (1.14) and (1.15), we have

$$x \in (c-\delta, c+\delta) \Rightarrow \left|\left(\frac{1}{g}\right)(x) - \frac{1}{M}\right| < \varepsilon$$

Thus

$$\lim_{x\to c}\left(\frac{1}{g}\right)(x) = \frac{1}{M} = \left(\lim_{x\to c} g(x)\right)^{-1}$$

■

**THEOREM 1.10** Suppose $a < c < b$, and $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ are defined except possibly at $c$ and $g(x)\neq 0\ \forall\, x\in[a,b]$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M \neq 0$, then

$$\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$$

**PROOF** Define $h:[a,b]\to\mathbb{R}$ by

$$h(x) = \left(\frac{1}{g}\right)(x) = \frac{1}{g(x)}\quad \forall\, x\in[a,b]$$

By Theorem 1.9,

$$\lim_{x\to c} h(x) = \lim_{x\to c}\left(\frac{1}{g}\right)(x) = \frac{1}{M}$$

Now by Theorem 1.7, we have

$$\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \lim_{x\to c}(f\cdot h)(x) = L\left(\frac{1}{M}\right) = \frac{L}{M}$$

■

**COROLLARY 1.2** Suppose $a<c<b$ and $f:[a,b]\to\mathbb{R}$ is defined except possibly at $c$. If $\lim_{x\to c} f(x) = L$ and $\lambda\in\mathbb{R}$, thus $\lim_{x\to c}(\lambda f)(x)$ exists and is equal to $\lambda L$.

**PROOF** Write $g(x) = \lambda\ \forall\, x\in[a,b]$ so that by Example 1.11 we have $\lim_{x\to c} g(x) = \lambda$. Now, by Theorem 1.7, we have

$$\lim_{x\to c}(f\cdot g)(x) = \lambda L$$

■

**Corollary 1.3** Suppose $a < c < b$, $f:[a,b] \to \mathbb{R}$, $g:[a,b] \to \mathbb{R}$ are defined except possibly at $c$ and $g(x) \neq 0$ $\forall\, x \in [a,b]$. If $\lim_{x\to c} f(x)/g(x)$ exists finitely and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} f(x) = 0$.

**Proof** Since

$$f(x) = \frac{f(x)}{g(x)} \cdot g(x) = \left(\frac{f}{g}\right)(x) \cdot g(x)$$

for all $x \in [a,b]$, by Theorem 1.7 we have

$$\begin{aligned}\lim_{x\to c} f(x) &= \left(\lim_{x\to c}\left(\frac{f}{g}\right)(x)\right)\left(\lim_{x\to c} g(x)\right)\\ &= (\text{Finite number})(0)\\ &= 0\end{aligned}$$

■

**Corollary 1.4** Suppose $a < c < b$, $f:[a,b] \to \mathbb{R}$, $g:[a,b] \to \mathbb{R}$ are defined except possibly at $c$ and $g(x)$ is not a zero function. If $\lim_{x\to c} f(x) = 0$ and $g(x)$ is bounded on $[a,b]$, then $\lim_{x\to c}(f \cdot g)(x) = 0$.

**Proof** Since $g(x)$ is bounded, suppose $|g(x)| \le M$ $\forall\, x \in [a,b]$. Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = 0$, there corresponds $\delta > 0$ such that for all $x \in (c-\delta, c+\delta)$, $x \neq c$,

$$\begin{aligned}(c-\delta, c+\delta) \subset (a,b) \Rightarrow |f(x)| &< \frac{\varepsilon}{M+1}\\ \Rightarrow |(f\cdot g)(x)| &= |f(x)g(x)|\\ &= |f(x)||g(x)|\\ &< \left(\frac{\varepsilon}{M+1}\right)M < \varepsilon\end{aligned}$$

for all $x \in (c-\delta, c+\delta)$, $x \neq c$. Thus, $\lim_{x\to c}(f\cdot g)(x) = 0$. ■

***Note:***

1. In practice, we write "$\lim_{x\to c} f(x) = l$" or "$f(x) \to l$ as $x \to c$", $a < c < b$ for some $a, b \in \mathbb{R}$ to mean that $f$ is defined on $[a,b]$ except possibly at $c$ and $\lim_{x\to c} f(x) = l$.
2. In the definition of $\lim_{x\to c} f(x) = l$, the value of $f$ at $c$ is not playing any role; in fact, $f$ may not be defined at $c$.

The above results stated for limits are equally valid for left limits as well as right limits, with obvious modifications. We state them briefly for right limits as follows.

**Theorem 1.11** Suppose $a \le c < b$ and $f$ and $g$ are two real-valued functions defined on $(a,b)$ except possibly at $c$. Suppose $\lim_{x\to c+0} f(x) = L$ and $\lim_{x\to c+0} g(x) = M$. Then

1. $\lim_{x\to c+0} f(x)$ is unique when exists.
2. $\lim_{x\to c+0} (f+g)(x)$ exists and is equal to $\lim_{x\to c+0} f(x) + \lim_{x\to c+0} g(x) = L + M$.
3. $\lim_{x\to c+0} (f\cdot g)(x)$ exists and is equal to $(\lim_{x\to c+0} f(x))(\lim_{x\to c+0} g(x)) = L\cdot M$.
4. If $\lim_{x\to c+0} g(x) = M \neq 0$, then there exists $\delta > 0$ such that $g(x) \neq 0$ in $(c, c+\delta)$ and $\lim_{x\to c+0} (1/g)(x)$ exits and is equal to $1/M$.
5. If $\lim_{x\to c+0} g(x) = M \neq 0$, then $\lim_{x\to c+0} (f/g)(x)$ exists and is equal to

$$\frac{\lim\limits_{x\to c+0} f(x)}{\lim\limits_{x\to c+0} g(x)} = \frac{L}{M}$$

**6.** If $\lim\limits_{x\to c+0} f(x) = L$ and $K \in \mathbb{R}$, then $\lim\limits_{x\to c+0} (Kf)(x)$ exists and is equal to $K \cdot L$.

**7.** If $\lim\limits_{x\to c+0} f(x) = L$ and $\lim\limits_{x\to c+0} g(x) = M$, then $\lim\limits_{x\to c+0} (f-g)(x)$ exists and is equal to $L - M$.

**THEOREM 1.12** Suppose $a < c < b, f_1, f_2, \ldots, f_n$ are real-valued functions defined on $[a, b]$ except positively at $c$ and $\lim\limits_{x\to c} f_i(x)$ exists for $i = 1,2,3, \ldots, n$. Then

**1.** $\lim\limits_{x\to c}(f_1 + f_2 + \cdots + f_n)(x)$ exists and is equal to $\lim\limits_{x\to c} f_1(x) + \lim\limits_{x\to c} f_2(x) + \cdots + \lim\limits_{x\to c} f_n(x)$.

**2.** $\lim\limits_{x\to c}(f_1 \cdot f_2 \cdot f_3 \cdots f_n)(x)$ exists and is equal to $\left(\lim\limits_{x\to c} f_1(x)\right)\left(\lim\limits_{x\to c} f_2(x)\right)\ldots\left(\lim\limits_{x\to c} f_n(x)\right)$.

**PROOF**

**1.** Follows from Theorem 1.6 by induction.

**2.** Follows from Theorem 1.7 by induction. ■

***Note:*** Theorem 1.12 is valid when limit is replaced by left limit or right limit.

**THEOREM 1.13** Suppose $a < c < b$ and $f(x) \ge 0 \;\forall x \in [a, b], x \ne c$. If $\lim\limits_{x\to c} f(x)$ exists and is equal to $L$, then $L \ge 0$.

**PROOF** Suppose $L < 0$. Take $\varepsilon = -L/2 > 0$. Since $\lim\limits_{x\to c} f(x) = L$ corresponds to $\varepsilon = -L/2$, there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \;\forall x \in (c-\delta, c+\delta),\; x \ne c$$

That is

$$L - \varepsilon < f(x) < L + \varepsilon = L - \frac{L}{2} = \frac{L}{2} < 0 \;\forall x\; \varepsilon (c-\delta, c+\delta),\; x \ne c$$

which is a contradiction. Hence $L \ge 0$. ■

**The following theorem has useful applications in evaluation of limits.**

**THEOREM 1.14 (SQUEEZING THEOREM OR SANDWICH THEOREM)** Suppose

**1.** $a < c < b$.

**2.** $f, g, h$ are real-valued functions defined on $[a, b]$ except possibly at $c$.

**3.** $f(x) \le g(x) \le h(x)$ for all $x \in [a, b], x \ne c$.

**4.** $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} h(x)$ exist and are equal.

Then $\lim\limits_{x\to c} g(x)$ exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \lim_{x\to c} h(x)$$

**PROOF** Write $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} h(x) = L$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$L - \varepsilon < f(x) < L + \varepsilon$$

and

$$L - \varepsilon < h(x) < L + \varepsilon$$

for all $x \in (c - \delta, c + \delta), x \ne c$. Now, by hypotheses

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$$

and hence

$$L - \varepsilon < g(x) < L + \varepsilon$$

for all $x \in (c - \delta, c + \delta), x \neq c$. Thus

$$|g(x) - L| < \varepsilon \;\; \forall\, x \in (c - \delta, c + \delta), x \neq c$$

Hence $\lim_{x \to c} g(x)$ exists and is equal to $L$. ■

**THEOREM 1.15** Suppose

1. $f(x) \leq g(x) \;\; \forall\, x \in [a, b], x \neq c$.
2. $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

Then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

**PROOF** Write $h(x) = f(x) - g(x) \;\; \forall\, x \in [a, b], x \neq c$. Then $h(x) \leq 0 \;\; \forall\, x \in [a, b], x \neq c$ and hence by Theorem 1.13, $\lim_{x \to c} h(x) \leq 0$. Therefore

$$\lim_{x \to c} f(x) - \lim_{x \to c} g(x) = \lim_{x \to c} (f - g)(x) = \lim_{x \to c} h(x) \leq 0$$

This gives

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$ ■

***Note:*** Theorems 1.13–1.15 are also valid when the limit is replaced by left limit or right limit with relevant modifications. The following examples will enhance the understanding of the concepts and results discussed so far.

## Example 1.16

Let $c \in \mathbb{R}$, $n$ be a positive integer and $f(x) = x^n$ for all $x \in \mathbb{R}$, $x \neq c$. Then $\lim_{x \to c} x^n = c^n$.

***Solution:*** If $n = 1$, then $f(x) = x$. Given $\varepsilon > 0$, take $\delta = \varepsilon$ so that $x \in (c - \delta, c + \delta), x \neq c$. This implies

$$|f(x) - c| = |x - c| < \delta = \varepsilon$$

Hence $f(x) \to c$ as $x \to c$. Assume that the result is true for $n = K \geq 1$. Then $x^K \to c^K$ as $x \to c$. Now (by Theorem 1.7) as $x \to c$

$$x^{K+1} = x^K \cdot x \to c^K \cdot c$$

That is, $x^{K+1} \to c^{K+1}$ as $x \to c$. Thus, the result is true for $K + 1$. Hence by induction, the result is true for every positive integer $n$.

## Example 1.17

Let $c > 0$ and $n$ be a positive integer. Then $x^{1/n} \to c^{1/n}$ as $x \to c$.

***Solution:*** Let $\varepsilon > 0$. Choose $0 < \delta < \min\{c, c^{(n-1)/n}, \varepsilon\}$. Suppose $x\, \varepsilon\, (c - \delta, c + \delta), x \neq c$. Then

$$\begin{aligned} x - c &= (x^{1/n})^n - (c^{1/n})^n \\ &= (x^{1/n} - c^{1/n})\,[x^{(n-1)/n} + x^{(n-2)/n} \cdot c^{1/n} + x^{(n-3)/n} \cdot c^{2/n} \\ &\quad + \cdots + x^{1/n} \cdot cn^{(n-2)/n} + c^{(n-1)/n}] \end{aligned}$$

We know that

$$y^n - a^n = (y - a)(y^{n-1} + y^{n-2}a + y^{n-3} \cdot a^2 + \cdots + a^{n-1})$$

where $n$ is a positive integer. Take $y = x^{1/n}$ and $a = c^{1/n}$. Therefore

$$\begin{aligned} |x - c| &= |x^{1/n} - c^{1/n}|(x^{(n-1)/n} + x^{(n-2)/n} \cdot c^{1/n} + \cdots + c^{(n-1)/n}) \\ &\geq |x^{1/n} - c^{1/n}| \cdot c^{(n-1)/n} \end{aligned}$$

so that

$$|x^{1/n} - c^{1/n}| \leq \frac{|x - c|}{c^{(n-1)/n}} < \frac{\delta}{c^{(n-1)/n}} < \varepsilon$$

(by the choice of $\delta$). Consequently $x^{1/n} \to c^{1/n}$ as $x \to c$.

## Example 1.18

Let $c \in \mathbb{R}$ and $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial where $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_n \neq 0$. Then $\lim_{x\to c} p(x) = p(c)$.

***Solution:*** By Example 1.16, $x^i \to c^i$ as $x \to c$. Hence by Corollary 1.3 and Theorem 1.12, we get

$$\begin{aligned}\lim_{x\to c}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) &= a_0 + a_1c + a_2c^2 + \cdots + a_nc^n \\ &= p(c)\end{aligned}$$

## Example 1.19

Find the limit of $1/(\sqrt{x} - 2)$ as $x \to 1$.

***Solution:*** It is known that $\sqrt{x} = x^{1/2} \to 1$ as $x \to 1$. Therefore

$$(\sqrt{x} - 2) \to 1 - 2 = -1 \quad \text{as } x \to 1$$

Hence by Theorem 1.9,

$$\frac{1}{\sqrt{x} - 2} \to \frac{1}{-1} = -1 \quad \text{as } x \to 1$$

## Example 1.20

Find the limit of $x^{-1/2} + 1$ as $x \to 1$.

***Solution:*** We have

$$x^{-1/2} + 1 = \frac{1}{x^{1/2}} + 1 = \frac{1 + x^{1/2}}{x^{1/2}} = \frac{f(x)}{g(x)}$$

where

$$f(x) = 1 + x^{1/2} \quad \text{and} \quad g(x) = x^{1/2}$$

Also

$$\lim_{x\to 1} f(x) = 1 + 1 = 2$$

and

$$\lim_{x\to 1} g(x) = 1^{1/2} = 1 \neq 0$$

Therefore, by Theorem 1.10

$$\lim_{x\to 1}(x^{-1/2} + 1) = \lim_{x\to 1}\frac{f(x)}{g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)} = \frac{2}{1} = 2$$

## Example 1.21

Find $\lim_{x\to c} x^{m/n}$ where $m$ and $n$ are positive integers and $c$ is a positive real number.

***Solution:*** Write $f(x) = x^{1/n}$. Then

$$x^{m/n} = f(x)\cdot f(x)\cdots f(x)\ (m \text{ times}) = (f(x))^m$$

Then by part (2) of Theorem 1.12,

$$\begin{aligned}\lim_{x\to c}(x^{m/n}) &= (\lim_{x\to c} x^{1/n})^m \\ &= (\lim_{x\to c} f(x))^m \\ &= (c^{1/n})^m = c^{m/n}\end{aligned}$$

## Example 1.22

Find $\lim_{x\to 0+0}(x^{1/n})$ for $x > 0$.

***Solution:*** Let $f(x) = x^{1/n}$. We have to find $\lim f(x)$ as $x \to 0 + 0$. Let $\varepsilon > 0$. Take $\delta = \varepsilon^n$. Then for $x \in (0, \delta)$,

$$0 < x < \delta \Rightarrow 0 < x^{1/n} < \delta^{1/n} = (\varepsilon^n)^{1/n} = \varepsilon$$

Therefore

$$\begin{aligned}0 < x < \delta &\Rightarrow |x^{1/n} - 0| < \varepsilon \\ &\Rightarrow \lim_{x\to 0+0} x^{1/n} = 0\end{aligned}$$

## Example 1.23

Evaluate $\lim_{x\to 4-0}(4-x)^{3/2}$.

***Solution:*** Write

$$f(x) = (4-x)^{3/2} \text{ for } x < 4$$

If $x > 4$, then $4 - x < 0$ and consequently $(4 - x)^{3/2}$ is not defined. Now, let

$$g(x) = (4-x)^{1/2} \text{ for } x < 4$$

Here $c = 4 > 0$. Therefore

$$\lim_{x\to 4-0} g(x) = (4-c)^{1/2} = (4-4)^{1/2} = 0$$

Hence

$$\lim_{x\to 4-0} f(x) = \lim_{x\to 4-0}(g(x))^3 = (\lim_{x\to 4-0} g(x))^3 = 0^3 = 0$$

## Example 1.24

Find the limit of

1. $(2+x-x^2)^{1/2}$ as $x \to 2-0$.
2. $(2+x-x^2)^{1/2}$ as $x \to (-1)+0$.
3. $(2+x-x^2)^{1/2}$ as $x \to 0$.

***Solution:*** Write

$$f(x) = 2+x-x^2$$

We observe that

$$f(x) = 2+x-x^2 = (x+1)(2-x)$$

so that $f(x) < 0$ for $x < -1$ or $x > 2$ and $f(x) > 0$ if $x \in (-1, 2)$.

1. Now

$$\lim_{x\to 2-0} f(x) = \lim_{x\to 2-0}(2+x-x^2) = 0$$

so that

$$\lim_{x\to 2-0}(2+x-x^2)^{1/2} = \lim_{x\to 2-0}[f(x)]^{1/2} = 0^{1/2} = 0$$

2. Similarly, $\lim_{x\to -1+0} f(x) = 0$ so that

$$\lim_{x\to -1+0}(2+x-x^2)^{1/2} = \lim_{x\to -1+0}[f(x)]^{1/2} = 0^{1/2} = 0$$

3. Again

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0}(2+x-x^2)\\ &= \lim_{x\to 0}(2) + \lim_{x\to 0} x + \lim_{x\to 0}(x^2)\\ &= 2\end{aligned}$$

Hence

$$\lim_{x\to 0}(2+x-x^2)^{1/2} = [\lim_{x\to 0} f(x)]^{1/2} = 2^{1/2} = \sqrt{2}$$

## Example 1.25

Suppose $f(x) \to l$ as $x \to c$, $l > 0$ and $n$ is a positive integer. Show that $(f(x))^{1/n} \to l^{1/n}$ as $x \to c$.

***Solution:*** Let $0 < \varepsilon < l$ and $0 < \eta < \varepsilon \cdot l^{(n-1)/n}$. Then there exists $\delta > 0$ such that $x\ \varepsilon\ (c-\delta, c+\delta)$, $x \neq c$ which implies

$$0 < l - \eta < f(x) < l + \eta$$

$$\Rightarrow f(x) > 0 \text{ and } (l-\eta)^n < [f(x)]^{1/n} < (l+\eta)^{1/n}$$

$$\Rightarrow \left|[f(x)]^{1/n} - l^{1/n}\right| \le |f(x) - l| \cdot \frac{1}{l^{(n-1)/n}} \text{ (as in Example 1.15)}$$

$$< \frac{\eta}{l^{(n-1)/n}} < \varepsilon \text{ (by the choice of } \eta)$$

Therefore

$$[f(x)]^{1/n} \to l^{1/n} \quad \text{as } x \to c$$

## Example 1.26

If $n$ is a positive integer and $a \in \mathbb{R}$, then find

$$\lim_{x\to a}\frac{x^n - a^n}{x-a}$$

***Solution:*** Write

$$f(x) = \frac{x^n - a^n}{x-a}$$

We observe that $f$ is not defined at $x = a$ and in fact it need not be defined at all. Now, for $x \neq a$,

$$f(x)(x-a) = x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3} \cdot a^2 + \cdots + a^{n-1})$$

so that

$$f(x) = \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}$$

Hence using Example 1.18, we have

$$\begin{aligned}\lim_{x\to a} f(x) &= a^{n-1} + a^{n-2} \cdot a + a^{n-3} \cdot a^2 + \cdots + a^{n-1} \\ &= a^{n-1} + a^{n-1} + \cdots + \text{upto } n \text{ times} \\ &= na^{n-1}\end{aligned}$$

## 1.7 | Some Useful Inequalities

Students at the 10 + 2 level need the limit of $(x^n - a^n)/(x - a)$ as $x \to a$ where $0 < a$ and $n$ is a rational number. In fact this limit is $na^{n-1}$. For completeness sake, here we want to prove that $(x^n - a^n)/(x - a) \to na^{n-1}$ as $x \to a$ for any real number $n$. For this, we have to use certain basic inequalities which we discuss in this section.

**THEOREM 1.16** Suppose $k$ and $n$ are positive integers, $n \geq 1$ and $0 < c \neq 1$. Then

$$(n+k)(c^n - 1) < n(c^{n+k} - 1)$$

**PROOF** **Case I:** Suppose $c > 1$. Then

$$k \leq c^{k-1} + c^{k-2} + \cdots + c + 1$$

and

$$c^{n-1} + c^{n-2} + \cdots + c + 1 < \underbrace{c^n + c^n + \cdots + c^n}_{(n \text{ times})} = nc^{n-1}$$

Therefore

$$k(c^{n-1} + c^{n-2} + \cdots + c + 1) < (c^{k-1} + c^{k-2} + \cdots + c + 1)nc^n$$

On multiplying both sides with $c - 1$, we get

$$k(c^n - 1) < (c^k - 1)nc^n$$

Adding $n(c^n - 1)$ to both sides we get

$$(n+k)(c^n - 1) < (c^k - 1)nc^n + n(c^n - 1) = n(c^{n+k} - 1)$$

Therefore

$$(n+k)(c^n - 1) < n(c^{n+k} - 1) \tag{1.16}$$

**Case II:** Let $0 < c < 1$. Then

$$c^{k-1} + c^{k-2} + \cdots + c + 1 \leq k$$

and

$$c^{n-1} + c^{n-2} + \cdots + c + 1 > nc^n$$

Therefore

$$nc^n(c^{k-1} + c^{k-2} + \cdots + c + 1) < k(c^{n-1} + c^{n-2} + \cdots + c + 1)$$

Multiplying both sides with $(1 - c) > 0$, we get that

$$nc^n(1 - c^k) < k(1 - c^n)$$

or

$$nc^n(c^k - 1) > k(c^n - 1)$$

Now, adding $n(c^n - 1)$ to both sides we get

$$(n+k)(c^n - 1) < n(c^{n+k} - 1) \tag{1.17}$$

From Eqs. (1.16) and (1.17), we obtain the desired inequality. ■

***Note:*** From Theorem 1.16, we have the following theorem.

**THEOREM 1.17** Suppose $m, n$ are positive integers, $1 \le n < m$ and $c > 0$. Then

$$\frac{c^n - 1}{n} \le \frac{c^m - 1}{m}$$

with equality holding if and only if $c = 1$.

**PROOF** In Theorem 1.16, take $k = m - n$ so that we have

$$m(c^n - 1) < n(c^m - 1)$$

and hence

$$\frac{c^n - 1}{n} < \frac{c^m - 1}{m}$$

Obviously equality holds if and only if $c = 1$. ■

**COROLLARY 1.5** Suppose $0 < p < 1$ is a rational number and $0 < a \ne 1$. Then $a^p - 1 < p(a - 1)$.

**PROOF** Write $p = n/m$ where $m, n$ are positive integers such that $1 \le n < m$. Take $b = a^{1/m}$ so that

$$0 < b \ne 1,\ b^n = a^{n/m} = a^p > 0 \quad \text{and} \quad b^m = a$$

From Theorem 1.17, we get that

$$\frac{b^n - 1}{n} < \frac{b^m - 1}{m}$$

and hence

$$b^n - 1 < (n/m)(b^m - 1) = p(b^m - 1)$$

Therefore

$$a^p - 1 < p(a - 1) \quad (\because b^n = a^p \text{ and } b^m = a)$$ ■

**COROLLARY 1.6** Suppose $\alpha$ is a real number, $0 < \alpha < 1$ and $0 < a \ne 1$. Then $a^\alpha - 1 < \alpha(a - 1)$.

**PROOF** **Case I:** Suppose $a > 1$. Take any rational number $p$ such that $\alpha < p < 1$ (this choice is possible by Theorem 0.14). Therefore, by Corollary 1.5

$$a^\alpha - 1 < a^p - 1 < p(a - 1)$$

Hence

$$\frac{a^\alpha - 1}{a - 1} < p$$

This is true for all rational numbers $p$ such that $\alpha < p < 1$. Hence

$$\frac{a^\alpha - 1}{a - 1} \le \inf \{p \mid p \text{ is rational and } \alpha < p < 1\} = \alpha$$

Therefore

$$a^\alpha - 1 \le \alpha(a - 1) \tag{1.18}$$

**Case II:** Suppose $0 < a < 1$. Take any rational number $p$ such that $0 < p < \alpha$. Then, since $0 < \alpha < 1$ and $p < \alpha$,

$$a^\alpha - 1 < a^p - 1 < p(a - 1) \text{ (by Corollary 1.5)}$$

Therefore

$$\frac{a^\alpha - 1}{a-1} > p \quad (\because a-1<0)$$

Since this is true for all rational numbers $p$ such that $0 < p < \alpha$, it follows that

$$\frac{a^\alpha - 1}{a-1} \geq \sup\{p \mid p \text{ is rational and } 0 < p < \alpha\} = \alpha$$

Therefore

$$a^\alpha - 1 < \alpha(a-1) \quad (\because a-1<0) \tag{1.19}$$

From Eqs. (1.18) and (1.19) we have $a^\alpha - 1 < \alpha(a-1)$. ■

**THEOREM 1.18** If $0 < a \neq 1$ and $\alpha$ is a positive real number, then

$$a^\alpha - 1 < \alpha(a-1) \text{ if } \alpha < 1$$

$$a^\alpha - 1 > \alpha(a-1) \text{ if } \alpha > 1$$

**PROOF** If $0 < \alpha < 1$, then the result follows from Corollary 1.6. Suppose $\alpha > 1$ so that $0 < 1/\alpha < 1$. Write $b = a^\alpha$ so that $b^{1/\alpha} = a$. Again by Corollary 1.6,

$$b^{1/\alpha} - 1 < \frac{1}{\alpha}(b-1)$$

$$\Rightarrow a - 1 < \frac{1}{\alpha}(a^\alpha - 1)$$

$$\Rightarrow \alpha(a-1) < a^\alpha - 1$$

$$\Rightarrow a^\alpha - 1 > \alpha(a-1)$$

Thus, $a^\alpha - 1 > \alpha(a-1)$ when $\alpha > 1$. ■

**THEOREM 1.19** If $0 < a \neq 1$ and $\alpha$ is real, then

1. $\alpha a^{\alpha-1}(a-1) < a^\alpha - 1 < \alpha(a-1)$ if $0 < \alpha < 1$
2. $\alpha a^{\alpha-1}(a-1) > a^\alpha - 1 > \alpha(a-1)$ if $\alpha < 0$ or $\alpha > 1$

**PROOF** **1.** Suppose $0 < \alpha < 1$ and let $b = 1/a$. Then by Theorem 1.18

$$b^\alpha - 1 < \alpha(b-1)$$

$$\Rightarrow \frac{1}{a^\alpha} - 1 < \alpha\left(\frac{1}{a} - 1\right)$$

$$\Rightarrow 1 - a^\alpha < \alpha a^\alpha \frac{(1-a)}{a} = \alpha a^{\alpha-1}(1-a)$$

$$\Rightarrow \alpha a^{\alpha-1}(a-1) < a^\alpha - 1$$

**2.** Suppose $\alpha > 1$. Then $0 < 1/\alpha < 1$. Hence from (1)

$$\frac{1}{\alpha} b^{[(1/\alpha)-1]}(b-1) < b^{1/\alpha} - 1 < \frac{1}{\alpha}(b-1)$$

$$\Rightarrow \frac{1}{\alpha}\left(\frac{b^{1/\alpha}}{b}\right)(b-1) < b^{1/\alpha} - 1 < \frac{1}{\alpha}(b-1)$$

Using $b = a^{\alpha}$ we get

$$\frac{a}{b}(a^{\alpha} - 1) < \alpha(a - 1) < (a^{\alpha} - 1)$$

$$\Rightarrow a^{1-\alpha}(a^{\alpha} - 1) < \alpha(a - 1) < (a^{\alpha} - 1)$$

$$\Rightarrow \frac{a^{\alpha} - 1}{a^{\alpha - 1}} < \alpha(a - 1) < (a^{\alpha} - 1)$$

$$\Rightarrow a^{\alpha} - 1 < \alpha(a - 1)a^{\alpha - 1} < a^{\alpha - 1}(a^{\alpha} - 1)$$

$$\Rightarrow \alpha(a - 1)a^{\alpha - 1} > a^{\alpha} - 1 > \alpha(a - 1)$$

Thus (2) is proved when $\alpha > 1$.
Now, suppose $\alpha < 0$ so that $-\alpha > 0$. Write $\beta = 1 - \alpha > 1$. Now by (2) (i.e., $\alpha > 1$)

$$\beta a^{\beta - 1}(a - 1) > a^{\beta} - 1 > \beta(a - 1)$$

$$\Rightarrow (1 - \alpha)a^{-\alpha}(a - 1) > a^{1-\alpha} - 1 > (1 - \alpha)(a - 1)$$

$$\Rightarrow \frac{(1 - \alpha)(a - 1)}{a^{\alpha}} > \frac{a - a^{\alpha}}{a^{\alpha}} > (a - 1) - \alpha(a - 1)$$

$$\Rightarrow (1 - \alpha)(a - 1) > a - a^{\alpha} > a^{\alpha}[(a - 1) - \alpha(a - 1)] \tag{1.20}$$

From the inequality in Eq. (1.20) we have

$$(1 - \alpha)(a - 1) > a - a^{\alpha}$$

$$\Rightarrow (a - 1) - \alpha(a - 1) > a - a^{\alpha}$$

$$\Rightarrow a^{\alpha} - 1 > \alpha(a - 1) \tag{1.21}$$

Again from the second inequality in Eq. (1.20), we have

$$a - a^{\alpha} > a^{\alpha}[(a - 1) - \alpha(a - 1)]$$

$$\Rightarrow a - a^{\alpha} - a^{\alpha}(a - 1) > -\alpha a^{\alpha}(a - 1)$$

$$\Rightarrow a - a^{\alpha + 1} > -\alpha a^{\alpha}(a - 1)$$

$$\Rightarrow 1 - a^{\alpha} > -\alpha a^{\alpha - 1}(a - 1)$$

$$\Rightarrow a^{\alpha} - 1 < \alpha a^{\alpha - 1}(a - 1) \tag{1.22}$$

Thus (2) follows from Eqs. (1.21) and (1.22) when $\alpha < 0$. ■

**THEOREM 1.20** Suppose $a > 0$ and $\alpha$ is real. Then

**1.** $\alpha a^{\alpha - 1}(a - 1) \le a^{\alpha} - 1 \le \alpha(a - 1)$ if $0 \le \alpha \le 1$

**2.** $\alpha a^{\alpha - 1}(a - 1) \ge a^{\alpha} - 1 \ge \alpha(a - 1)$ if $\alpha \le 0$ or $\alpha > 1$

Equality holds if and only if $a = 1$ or $\alpha = 0$.

**PROOF** Suppose $0 < a \ne 1$. Then strict inequality holds in (1) and (2) as according to $0 < \alpha < 1$, $\alpha < 0$ or $\alpha > 1$, respectively (by Theorem 1.19). If $a = 1$ or $\alpha = 0$ or $\alpha = 1$, clearly equality holds in (1) and (2). ■

**THEOREM 1.21** Suppose $a$ and $x$ are distinct positive real numbers and $\alpha$ is real. Then

**1.** $\alpha a^{\alpha - 1}(a - x) < a^{\alpha} - x^{\alpha} < \alpha x^{\alpha - 1}(a - x)$ if $0 < \alpha < 1$

**2.** $\alpha a^{\alpha - 1}(a - x) > a^{\alpha} - x^{\alpha} > \alpha x^{\alpha - 1}(a - x)$ if $\alpha < 0$ or $\alpha > 1$

**PROOF** In Theorem 1.19, if we replace $a$ by $a/x$ we get the above result. ■

***Note:*** The inequalities in (1) and (2) of Theorem 1.21 become equality if $a = x$.

**THEOREM 1.22** If $a > 0$ and $\alpha$ is real, then $\lim_{x \to a} x^{\alpha} = a^{\alpha}$.

**PROOF** Suppose $\alpha > 1$. Then by part (2) of Theorem 1.21, whenever $x \in (a/2, 3a/2)$, $x \neq a$, we have

$$x < a \Rightarrow 0 < a^{\alpha} - x^{\alpha} < \alpha\, a^{\alpha-1}(a - x) < \alpha\left(\frac{3}{2}a\right)^{\alpha-1}(a - x)$$

and
$$x > a \Rightarrow 0 < x^{\alpha} - a^{\alpha} < \alpha\, x^{\alpha-1}(x - a) < \alpha\left(\frac{3}{2}a\right)^{\alpha-1}(x - a)$$

Hence, $x \in (a/2, 3a/2)$, $x \neq a$, implies

$$|x^{\alpha} - a^{\alpha}| < \alpha\left(\frac{3}{2}a\right)^{\alpha-1}|x - a| \to 0 \quad \text{as } x \to a$$

Therefore, if $\alpha > 1$, $x^{\alpha} \to a^{\alpha}$ as $x \to a$. In a similar way, using parts (1) and (2) of Theorem 1.21 we can show that $x^{\alpha} \to a^{\alpha}$ as $x \to a$, if $0 < \alpha < 1$ or $\alpha < 0$. Clearly $x^{\alpha} \to a^{\alpha}$ if $\alpha = 0$ or 1. Hence $\lim_{x \to a} x^{\alpha} = a^{\alpha}$ if $a > 0$ and $\alpha$ is real. ■

**THEOREM 1.23** Suppose $0 < a \neq 1$ and $\alpha$ is real. Define

$$f(x) = \frac{x^{\alpha} - a^{\alpha}}{x - a}$$

for all real numbers $x \neq a$. Then $\lim_{x \to a} f(x) = \alpha a^{\alpha-1}$.

**PROOF** Suppose $\alpha > 1$. Then, for $x \neq a$, we have from Theorem 1.21 that

$$\alpha a^{\alpha-1} > \frac{a^{\alpha} - x^{\alpha}}{a - x} > \alpha x^{\alpha-1} \text{ if } 0 < x < a$$

That is,

$$\alpha a^{\alpha-1} > f(x) > \alpha x^{\alpha-1} \text{ if } 0 < x < a$$

Hence,

$$\begin{aligned} \left|f(x) - \alpha a^{\alpha-1}\right| &= \alpha a^{\alpha-1} - f(x) \\ &< \alpha a^{\alpha-1} - \alpha x^{\alpha-1} \\ &= \alpha(a^{\alpha-1} - x^{\alpha-1}) \\ &= \alpha\left|x^{\alpha-1} - a^{\alpha-1}\right| \end{aligned} \tag{1.23}$$

Also

$$\alpha a^{\alpha-1} < \frac{a^{\alpha} - x^{\alpha}}{a - x} < \alpha x^{\alpha-1} \text{ if } x > a$$

That is,

$$\alpha a^{\alpha-1} < f(x) < \alpha x^{\alpha-1}$$

This implies

$$\begin{aligned} \left|f(x) - \alpha a^{\alpha-1}\right| &= f(x) - \alpha a^{\alpha-1} \\ &< \alpha x^{\alpha-1} - \alpha a^{\alpha-1} \\ &= \alpha(x^{\alpha-1} - a^{\alpha-1}) \\ &= \alpha\left|x^{\alpha-1} - a^{\alpha-1}\right| \end{aligned} \tag{1.24}$$

From Eqs. (1.23) and (1.24), we get (using Theorem 1.22), whenever $0 < x \neq a$,

$$\left|f(x) - \alpha a^{\alpha-1}\right| < \alpha\left|x^{\alpha-1} - a^{\alpha-1}\right| \to 0 \quad \text{as } x \to a$$

Hence $f(x) \to \alpha a^{\alpha-1}$ as $x \to a$ when $\alpha > 1$. Now suppose $0 < \alpha < 1$ or $\alpha < 0$. Then by parts (1) and (2) of Theorem 1.21 it follows that $f(x)$ lies between $\alpha x^{\alpha-1}$ and $\alpha a^{\alpha-1}$. Hence by Theorem 1.22, as $x \to a$

$$\left|f(x) - \alpha a^{\alpha-1}\right| \leq \left|\alpha a^{\alpha-1} - \alpha x^{\alpha-1}\right| = |\alpha|\left|x^{\alpha-1} - a^{\alpha-1}\right| \to 0$$

Here we used the fact $|p-q| < |p-r|$ where $p < q < r$. Therefore $f(x) \to \alpha a^{\alpha-1}$ as $x \to a$. ■

## 1.8 | Continuity

In this section, we give a precise definition of the most important concept of continuity of a real-valued function at a point of its domain and extend it to its domain. The notion of continuity of a function has occupied central stage in mathematical analysis and it will be used extensively in the coming sections of this volume. The term "continuous" is in practice since the time of Newton, but was not defined precisely until the 19th century.

Bernhard Bolzano in 1817 and Augustin–Louis Cauchy in 1821 came to know that continuity of a function is an important property and gave the definition. Since the concept of continuity of a function at a point is linked with the concept of limit, Karl Weierstrass carefully proposed proper definition of continuity in 1870s. In this section we will first explain the concepts of continuity at a point and continuous function and then discuss the properties of continuous functions. Linguistically, continuous means no break. Loosely speaking, a continuous function means, a function whose *graph can be drawn on a paper without lifting the hand from the plane of the paper.*

**DEFINITION 1.23** **Continuity at a Point** Let $f:[a, b] \to \mathbb{R}$ be a function and $a < c < b$.

1. We say that $f$ is *continuous* at the point $c$ if
$$\lim_{x \to c} f(x) = f(c)$$
2. $f$ is *continuous at the left end point a* if $\lim\limits_{x \to a+0} f(x) = f(a)$. In such case we say that $f$ is **right continuous** at $a$.
3. $f$ is said to be *continuous at the right end point b* if $\lim\limits_{x \to b-0} f(x) = f(b)$. In such case we say that $f$ is **left continuous** at $b$.

**DEFINITION 1.24** **Continuous Function** A function $f:[a, b] \to \mathbb{R}$ is said to be a continuous function if $f$ is continuous at every point of $[a, b]$ (including $a$ and $b$). We say that $f$ is discontinuous at $c$ if $f$ is not continuous at $c$.

**QUICK LOOK 5**

$f$ is continuous at $c$ implies

1. The function $f$ must be defined at the point $c$.
2. $\lim\limits_{x \to c} f(x)$ must exist and is equal to $f(c)$.

**THEOREM 1.24** Suppose $a < c < b$ and $f:[a, b] \to \mathbb{R}$ is a function. If $\lim\limits_{x \to c+0} f(x) = f(c)$ and $\lim\limits_{x \to c-0} f(x) = f(c)$, then $f$ is continuous at $c$. Conversely, if $f$ is continuous at $c$, then both $\lim\limits_{x \to c-0} f(x)$ and $\lim\limits_{x \to c+0} f(x)$ exist and are equal to $f(c)$.

**PROOF** Directly follows from Theorem 1.4. ■

**QUICK LOOK 6**

To prove that $f$ is continuous at $c$, it is enough if we show that

$$\lim_{\substack{h\to 0\\ h>0}} f(c+h) = \lim_{\substack{h\to 0\\ h>0}} f(c-h) = f(c)$$

## Example 1.27

If $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ is a polynomial where $a_0, a_1, \ldots, a_n$ are real, $a_0 \neq 0$ and $n$ is a positive integer, then show that $P(x)$ is continuous at every $c \in \mathbb{R}$.

**Solution:** By Example 1.16, $\lim_{x\to c} x^n = c^n$. Hence, the result follows from Corollary 1.2 and Theorem 1.6.

***Note:*** $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = x$ is continuous at every $c \in \mathbb{R}$.

## Example 1.28

Define $f:[a,b]\to\mathbb{R}$ by $f(x) = k$ (constant) for all $x \in [a,b]$. Then show that $f$ is continuous at every $c \in [a,b]$.

**Solution:** Let $a < c < b$. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k = f(c)$$

Also

$$\lim_{x\to a+0} f(x) = \lim_{x\to a+0} k = k = f(a)$$

and

$$\lim_{x\to b-0} f(x) = \lim_{x\to b-0} k = k = f(b)$$

## Example 1.29

Let $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

Then show that $f$ is discontinuous at 0.

**Solution:** We have

$$\lim_{x\to 0-0} f(x) = \lim_{x\to 0-0} 0 = 0 = f(0)$$

$$\lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} 1 = 1 \neq f(0)$$

Therefore $f$ is discontinuous at $x = 0$, but continuous at all other points (Fig. 1.18).

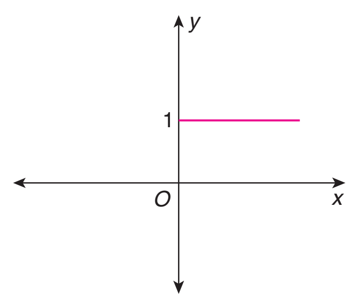



**FIGURE 1.18** Example 1.29.

## Example 1.30

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = n$ if $x \in [n, n+1]$ where $n = 0, \pm1, \pm2, \ldots$. Then show that $f$ is discontinuous at every $n = 0, \pm1, \pm2, \ldots$ and continuous at all other points.

**Solution:** See Fig. 1.19. Let $n$ be a positive integer. Then

$$\lim_{x\to n-0} f(x) = n-1 \text{ and } \lim_{x\to n+0} f(x) = n$$

Therefore

$$\lim_{x\to n-0} f(x) \neq \lim_{x\to n+0} f(x)$$

and so $f$ is not continuous at $n$. Similarly, if $n$ is a negative integer say $n = -m$ where $m$ is a positive integer, then

$$\lim_{x\to n-0} f(x) = \lim_{x\to(-m)-0} f(x) = (-m)-1 = n-1$$

and

$$\lim_{x\to n+0} f(x) = \lim_{x\to(-m)+0} f(x) = -m = n$$

Therefore, $f$ is discontinuous at $n$. Clearly $\lim_{x\to 0-0} f(x) = -1$ and $\lim_{x\to 0+0} f(x) = 0$. Hence $f$ is discontinuous at all integer values.

Now suppose $x_0$ is not an integer and let $x_0 \in [n, n+1]$ where $n$ is an integer. Then clearly $f(x_0) = n$ and

$$\lim_{x \to x_0 - 0} f(x) = \lim_{x \to x_0 + 0} f(x) = n = f(x_0)$$

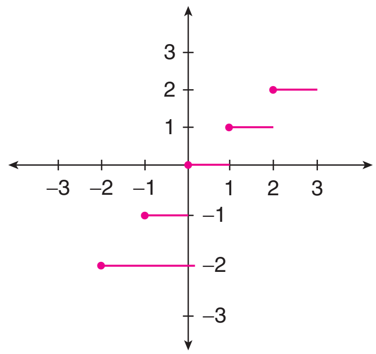



**FIGURE 1.19** Example 1.30.

## Example 1.31

Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then show that $f$ is discontinuous at all real values of $x \neq 0$.

***Solution:*** Let $c \in \mathbb{R}$. In every neighbourhood of $c$ there are infinite rationals and irrationals, and so $\lim_{x \to c} f(x)$ cannot exist.

***Note:*** The graph of this function cannot be drawn.

## Example 1.32

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then show that $f$ is continuous at $x = 0$.

***Solution:*** When $x \neq 0, f(x)$ is a product of two functions, viz., $g(x) = x$ and $h(x) = \sin 1/x$. Here $\lim_{x \to 0} g(x) = 0$ and $h(x)$ is a bounded function because $-1 \leq \sin 1/x \leq 1$ for all real $x \neq 0$. Therefore, by Corollary 1.4

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (g(x)h(x)) = 0 = f(0)$$

Therefore, $f$ is continuous at $x = 0$.

**Try it out** Show that the function

$$f(x) = \begin{cases} x \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.
*Hint:* Work on the same lines as Example 1.32.

## Example 1.33

Let $c > 0$ and $x$ be real. Define $f(x) = x^\alpha$ for $x > 0$. Then show that $f$ is continuous at $c$.

***Solution:*** By Theorem 1.22,

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^\alpha) = c^\alpha = f(c)$$

Hence $f$ is continuous at $x = c > 0$. Note that $f$ is continuous when $c = 0$ also (check).

**THEOREM 1.25** Let $c > 0$ and $\alpha$ be real. Define

$$f(x) = \begin{cases} \dfrac{x^\alpha - c^\alpha}{x - c} & \text{if } x \in \left[\dfrac{c}{2}, \dfrac{3c}{2}\right], x \neq c \\ \alpha c^{\alpha-1} & \text{if } x = c \end{cases}$$

Then $f$ is continuous at $c$.

**PROOF** By Theorem 1.23

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{x^\alpha - c^\alpha}{x - c}\right) = \alpha c^{\alpha-1} = f(c)$$

Hence $f$ is continuous at $x = c$. ■

**SPECIAL NOTE:** Generally the students at 10+2 level have to evaluate $\lim\limits_{x\to a}(x^n - a^n)/(x - a)$ when $n$ is a rational number and $a$ is positive real number for which the proof does not require all the inequality theorems proved before Theorem 1.26. As mentioned in the introduction of this section, the proof given for Theorem 1.25 is to cover when $n$ is any real number. The following theorem evaluated the same when $n$ is rational. We state and prove the result as follows.

**THEOREM 1.26 (PARTICULAR CASE)** If $n$ is a rational number, define

$$f(x) = \begin{cases} \dfrac{x^n - a^n}{x - a} & \text{if } x \neq a \\ na^{n-1} & \text{if } x = a \end{cases}$$

where $a > 0$. Then $f$ is continuous at $a$.

**PROOF** **Case I:** $n$ is a positive integer. Then

$$x^n - a^n = (x - a)[x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}]$$

Therefore

$$\begin{aligned} \lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) &= \lim_{x\to a}[x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}] \\ &= a^{n-1} + a^{n-1} + \cdots + a^{n-1} \quad (n \text{ times})(\text{by Example 1.27}) \\ &= na^{n-1} = f(a) \end{aligned}$$

**Case II:** Suppose $n$ is a negative integer say $n = -m$, where $m$ is a positive integer. Then

$$\frac{x^n - a^n}{x - a} = \frac{(1/x^m) - (1/a^m)}{x - a} = \frac{-(x^m - a^m)}{x^m a^m (x - a)}$$

Therefore

$$\begin{aligned} \lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) &= -\lim_{x\to a}\left(\frac{x^m - a^m}{x - a}\right)\lim_{x\to a}\left(\frac{1}{x^m a^m}\right) \\ &= -(ma^{m-1})\left(\frac{1}{a^m \cdot a^m}\right) \quad (\text{By Case I}) \\ &= (-m)a^{-m-1} = na^{n-1} = f(a) \end{aligned}$$

**Case III:** Suppose $n = p/q$ where $p$ and $q$ are integers and $q \neq 0, p \neq 0$. Now write $y = x^{1/q}$ and $b = a^{1/q}$ so that $x = y^q$ and $a = b^q$ and $y \to b$ as $x \to a$. Therefore

$$\begin{aligned}
\lim_{x\to}\left(\frac{x^n-a^n}{x-a}\right) &= \lim_{y\to b}\left(\frac{y^p-b^p}{y^q-b^q}\right)\\
&= \lim_{y\to b}\left(\frac{y^p-b^p}{y-b}\times\frac{y-b}{y^q-b^q}\right)\\
&= \lim_{y\to b}\left(\frac{y^p-b^p}{y-b}\right)\div\lim_{y\to b}\left(\frac{y^q-b^q}{y-b}\right)\\
&= (pb^{p-1})\div(qb^{q-1}) \quad \text{(By Cases I and II)}\\
&= \frac{p}{q}b^{p-q}\\
&= \frac{p}{q}(b^q)^{[(p/q)-1]}\\
&= na^{n-1} = f(a)
\end{aligned}$$

When $n = 0$, then clearly $f(x) = 0\ \forall x$ so that $\lim_{x\to a} f(x) = 0 = f(a)$. Hence, $\lim_{x\to a} f(x) = na^{n-1} = f(a)$, where $n$ is rational. ■

The following is another basic theorem on limit of the function $(\sin x)/x$ which is not defined when $x$ is equal to 0. We establish $\lim_{x\to 0}(\sin x)/x$ exists and is equal to 1.

**THEOREM 1.27** Define $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then $f$ is continuous at $x = 0$.

**PROOF** Suppose $0 < x < \pi/2$. Draw the circle with centre at the origin $O$ and radius equal to 1. Suppose it cuts the $x$-axis at $A$. Suppose the line "$l$" through $O$ making angle $x$ with the positive direction of the $x$-axis meets the circle in $B$ and the tangent to the circled drawn at $A$ in $C$. Draw $BD$ perpendicular to the $x$-axis (Fig. 1.20). Now

$$\text{Area of } \Delta OAB < \text{Area of the sector } OAB < \text{Area of } \Delta OAC$$

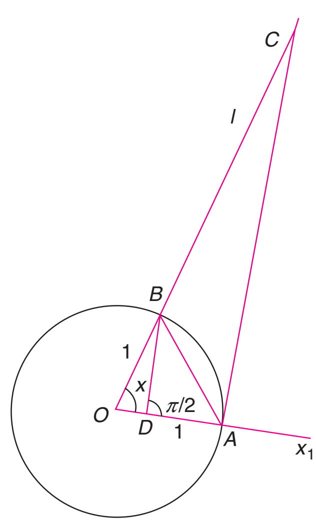



**FIGURE 1.20** Theorem 1.27.

Hence

$$OA \cdot BD < x < OA \cdot AC \tag{1.25}$$

Now

$$\left.\begin{aligned} \sin x &= \frac{BD}{OB} = BD \\ \tan x &= \frac{AC}{OA} = AC \end{aligned}\right\} \tag{1.26}$$

Therefore from Eqs. (1.25) and (1.26), $0 < \sin x < x < \tan x$, so that

$$\left.\begin{aligned} 0 < \frac{\sin x}{x} < 1 < \frac{\tan x}{x} \\ 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \end{aligned}\right\} \tag{1.27}$$

Equation (1.27) holds good when $-\pi/2 < x < 0$, since in this case

$$0 < \frac{\sin x}{x} = \frac{\sin(-x)}{-x} < 1 < \frac{\tan(-x)}{-x} = \frac{\tan x}{x}$$

Also from Eq. (1.27),

$$\cos x < \frac{\sin x}{x} < 1$$

Therefore

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\frac{\sin^2 x}{2} < \frac{x^2}{2} \to 0 \quad \text{as } x \to 0$$

By squeezing theorem (Theorem 1.14)

$$\lim_{x\to 0}\left(1 - \frac{\sin x}{x}\right) = 0 = \lim_{x\to 0}(1 - \cos x)$$

Therefore

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1 \quad \text{and} \quad \lim_{x\to 0}(\cos x) = 1$$

The graph of this function is shown in Fig. 1.21.

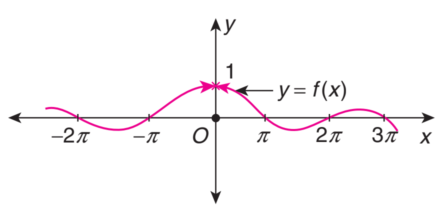



**FIGURE 1.21** Graph of Theorem 1.27. ■

***Note:*** $\lim_{x\to 0} \cos x = 1 \Rightarrow$ the function $\cos x$ is continuous at $x = 0$.

We shall now prove certain theorems on continuity of sum, product and quotient of continuous functions which are parallel to their counterparts on limits (Section 1.5).

**THEOREM 1.28** Let $a < c < b$, $f:[a,b] \to \mathbb{R}$, $g:[a,b] \to \mathbb{R}$ be continuous at $c$, then

1. $f + g$ is continuous at $c$.
2. $f \cdot g$ is continuous at $c$.
3. If $g(c) \neq 0$, then $f/g$ is continuous at $c$.
4. If $f(x)$ and $g(x)$ are polynomials and

$$f(x) \equiv (x-a)^k P(x)$$
$$g(x) \equiv (x-a)^k Q(x)$$

then

$$\lim_{x \to a}\left(\frac{f}{g}\right)(x) = \frac{P(a)}{Q(a)} \quad \text{if } Q(a) \neq 0$$

**PROOF** Since $f$ and $g$ are continuous at $c$, we have

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)$$

1. By Theorem 1.6

$$\begin{aligned}\lim_{x \to c}(f+g)(x) &= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)\\ &= f(c) + g(c)\\ &= (f+g)(c)\end{aligned}$$

Therefore $f + g$ is continuous at $c$.

2. Using Theorem 1.7

$$\begin{aligned}\lim_{x \to c}(f \cdot g)(x) &= \lim_{x \to c}(f(x) \cdot g(x))\\ &= \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)\\ &= f(c) \cdot g(c)\\ &= (f \cdot g)(c)\end{aligned}$$

Hence $f \cdot g$ is continuous.

3. Since $\lim_{x \to c} g(x) = g(c) \neq 0$, by Theorem 1.8, $g(x) \neq 0$ for all $x$ in a neighbourhood of $c$, and hence by Theorem 1.9,

$$\lim_{x \to c}\left(\frac{f}{g}\right)(x) = \lim_{x \to c}\frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)} = \left(\frac{f}{g}\right)(c)$$

Hence $f/g$ is continuous at $c$.

4. Follows from (3). ■

**COROLLARY 1.7**

1. A constant function is continuous.
2. If $\lambda \in \mathbb{R}$ and $f$ is continuous at $c$, then $\lambda f$ is continuous at $c$.
3. If $f$ and $g$ are continuous at $c$, then $f - g$ is continuous at $c$.

**PROOF**

1. Suppose $f(x) = k \; \forall x \in \mathbb{R}$. Therefore for any $c \in \mathbb{R}$ (see Example 1.11),

$$\lim_{x \to c} f(x) = \lim_{x \to c}(k) = k = f(c)$$

Hence $f$ is continuous at $c$. This being true for every $c \in \mathbb{R}$, it follows that $f$ is continuous on $\mathbb{R}$.

2. Define $g(x) = \lambda \ \forall x \in \mathbb{R}$. Since $f$ is continuous at $c$ and $g$ is continuous at $c$, by part (3) of Theorem 1.28, $f \cdot g = \lambda f$ is continuous at $c$.
3. By taking $\lambda = -1$ in (2), we get $-g = (-1)g$ is continuous at $c$ and hence $f - g = f + (-g)$ is continuous at $c$. ■

The following two theorems are about the continuity of composite function.

**THEOREM 1.29** Suppose $a < c < b$, $f : [a, b] \to \mathbb{R}$ is a function and $\lim_{x \to c} f(x) = l$. Further suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous at $l$. Then $\lim_{x \to c} (g \circ f)(x)$ exists and is equal to $g(l)$.

**PROOF** Write $p = g(l)$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $y \in (l - \delta, l + \delta) \Rightarrow |g(y) - p| < \varepsilon$ (since $g$ is continuous at $l$). Since $f(x) \to l$ as $x \to c$, corresponding to this $\delta > 0$, there exists $\eta > 0$ such that

$$\begin{aligned} x \in (c - \eta, c + \eta) &\Rightarrow |f(x) - l| < \delta \\ &\Rightarrow 1 - \delta < f(x) < 1 + \delta \\ &\Rightarrow |g(f(x)) - p| < \varepsilon \\ &\Rightarrow |(g \circ f)(x) - p| < \varepsilon \end{aligned}$$

Hence

$$\lim_{x \to c} (g \circ f)(x) = p = g(l)$$ ■

**THEOREM 1.30** Suppose $a < c < b$ and $f : [a, b] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are, respectively, continuous at $c$ and $f(c)$. Then $g \circ f : [a, b] \to \mathbb{R}$ is continuous $c$.

**PROOF** By Theorem 1.29

$$\lim_{x \to c} (g \circ f)(x) = g(f(c)) = (g \circ f)(c)$$

Hence $g \circ f$ is continuous at $c$. ■

***Note:*** If $c = a$ or $b$, in a similar way as Theorem 1.30 we can show that $g \circ f$ is continuous at $a$ or $b$.

**COROLLARY 1.8** Suppose $f : [a, b] \to \mathbb{R}$ is continuous, $f([a, b]) \subset [p, q]$ and $g : [p, q] \to \mathbb{R}$ is continuous. Then $g \circ f : [a, b] \to \mathbb{R}$ is continuous.

**PROOF** Suppose $c \in [a, b]$. Since $f$ is continuous on $[a, b]$ and $g$ is continuous on $[p, q]$ it follows that $g \circ f$ is continuous at $c$ (by Theorem 1.30 and the note). Hence $g \circ f$ is continuous on $[a, b]$. ■

**COROLLARY 1.9** Suppose $\alpha \in \mathbb{R}$, $a < c < b$, $f : [a, b] \to [0, \infty)$ and $\lim_{x \to c} f(x) = l$, then $\lim_{x \to c} f^{\alpha}(x) = l^{\alpha}$.

**PROOF** Since $f(x) \geq 0$ on $[a, b]$, by Theorem 1.12 it follows that $\lim_{x \to c} f(x) = l \geq 0$. Write $g(x) = x^{\alpha} \ \forall x \geq 0$. By Example 1.33 we get $g$ is continuous at $l$. Therefore, by Theorem 1.30 it follows that

$$\lim_{x \to c} f^{\alpha}(x) = \lim_{x \to c} (f(x))^{\alpha} = \lim_{x \to c} g(f(x)) = g(l) = l^{\alpha}$$ ■

## Example 1.34

Consider

$$f(x)=\begin{cases}\dfrac{(1+x)^{1/2}-(1-x)^{1/2}}{x} & \text{if } -1\le x\le 1, x\ne 0\\ 1 & \text{if } x=0\end{cases}$$

then $f(x)$ is continuous at $x=0$.

***Solution:*** It is enough if we show that the $\lim\limits_{x\to 0} f(x)=1$. Now

$$f(x)=2\left[\frac{(1+x)^{1/2}-(1-x)^{1/2}}{(1+x)-(1-x)}\right]$$

$$=2\left[\frac{(1+x)^{1/2}-(1-x)^{1/2}}{[(1+x)^{1/2}-(1-x)^{1/2}][(1+x)^{1/2}+(1-x)^{-1/2}]}\right]$$

$$=\frac{2}{(1+x)^{1/2}+(1-x)^{1/2}}$$

Therefore

$$\lim_{x\to 0} f(x)=\frac{2}{1+1}=1=f(0)$$

**Aliter:**

$$f(x)=\frac{(1+x)^{1/2}-1+1-(1-x)^{1/2}}{x}$$

$$=\frac{(1+x)^{1/2}-1}{x}-\frac{(1-x)^{1/2}-1}{x}$$

$$=\frac{(1+x)^{1/2}-1}{(1+x)-1}+\frac{(1-x)^{1/2}-1}{(1-x)-x}$$

$$=\frac{y-1}{y^2-1}+\frac{z-1}{z^2-1}$$

where

$$y=(1+x)^{1/2}\to 1 \quad \text{as } x\to 0$$

and $z=(1-x)^{1/2}\to 1$ as $x\to 0$ $(\because x^{\alpha}\to a^{\alpha}$ as $x\to a)$

Therefore

$$\lim_{x\to 0} f(x)=\lim_{y\to 1}\left(\frac{y-1}{y^2-1}\right)+\lim_{z\to 1}\left(\frac{z-1}{z^2-1}\right)$$

$$=\lim_{y\to 1}\left(\frac{y^2-1}{y-1}\right)^{-1}+\lim_{z\to 1}\left(\frac{z^2-1}{z-1}\right)^{-1}$$

$$=\frac{1}{2}+\frac{1}{2}\quad \text{(By Theorem 1.26)}$$

$$=1$$

## Example 1.35

Show that the function $f(x)=\sin x$ is continuous on $\mathbb{R}$.

***Solution:*** Since $|\sin x|=\sin(|x|)<|x|$ for $x\in(-\pi/2,\pi/2)$ (by Theorem 1.27), it follows that $f(x)\to 0$ as $x\to 0$. Consequently $f(x)$ is continuous at $x=0$. Now, let $a\in\mathbb{R}$. Then

$$|f(x)-f(a)|=|\sin x-\sin a|$$

$$=\left|2\cos\frac{x+a}{2}\sin\frac{x-a}{2}\right|$$

$$\le 2\sin\frac{|x-a|}{2}\to 0$$

as $x\to a$. Therefore $f(x)\to f(a)$ as $x\to a$. Thus $f(x)$ is continuous at $x=a$.

## Example 1.36

Show that $g(x)=\cos x$ is continuous on $\mathbb{R}$.

***Solution:*** Define $h(x)=(\pi/2)-x$ and $f(x)=\sin x$. Then

$$(f\circ h)(x)=f(h(x))=\sin((\pi/2)-x)=\cos x=g(x)$$

Since both $f(x)$ and $h(x)$ are continuous, it follows from Theorem 1.30 that $g(x)$ is continuous.

**Example 1.37**

Show that the function $\tan x = \sin x/\cos x$ is continuous on $\mathbb{R}$ except at odd multiples of $\pi/2$.

***Solution:*** We know that $\cos x = 0$ if and only if $x$ is an odd multiple of $\pi/2$. Hence if $\alpha$ is not an odd multiple of $\pi/2$, then $\cos \alpha \neq 0$ and $\cos x \to \cos \alpha \neq 0$ so that [by part (3) of Theorem 1.28]

$$\tan x = \frac{\sin x}{\cos x} \to \frac{\sin \alpha}{\cos \alpha} = \tan \alpha \quad \text{as } x \to \alpha$$

**Try it out** Show that $\cot x$ and $\operatorname{cosec} x$ are continuous on $\mathbb{R}$ except at multiples of $\pi$ and $\sec x$ is continuous on $\mathbb{R}$ except at odd multiples of $\pi/2$.
*Hint:* See Example 1.37.

Since $x^n \to a^n$ as $x \to a$ (see Example 1.16) and $\lim_{x\to a} (x^n - a^n)/(x-a) = na^{n-1}$ (see Theorem 1.25 or 1.26), the following limits can be easily checked.

**Try it out**

1. $\lim_{x\to 2}(x^2 + 2x + 3) = 11$
2. $\lim_{x\to 0+} (\sqrt{x} + x\sqrt{x}) = 0 \ (x > 0)$
3. $\lim_{x\to 1} \dfrac{2x-1}{3x^2 - 4x + 5} = \dfrac{1}{4}$
4. $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2} = 4$
5. $\lim_{x\to 1} \left(\dfrac{2}{x+1} - \dfrac{3}{x}\right) = -2$
6. $\lim_{x\to 8} \dfrac{x^{4/3} - 8^{4/3}}{x - 8} = \dfrac{8}{3}$

## 1.9 | Properties of Continuous Functions

In this section we will explore the properties of continuous function defined on a closed interval $[a, b]$ where $a$ and $b$ are real numbers and $a < b$.

**DEFINITION 1.25** A function $f : [a, b] \to \mathbb{R}$ is said to be bounded on $[a, b]$ if the range of $f$ in $\mathbb{R}$ is bounded subset of $\mathbb{R}$ (for bounded subset, see Definition 0.3).

Thus $f : [a, b] \to \mathbb{R}$ is bounded if the range set $f([a, b])$ is bounded in $\mathbb{R}$. That is, if there exist constants $\alpha$ and $\beta$ such that $\alpha \leq f(x) \leq \beta \ \forall x \in [a, b]$. In fact, if $f$ is bounded, then both $\underset{x\in[a,b]}{\text{l.u.b.}} f(x)$ and $\underset{x\in[a,b]}{\text{g.l.b.}} f(x)$ exist and they are finite real numbers. Also $f$ is bounded if and only if $|f|$ is bound on $[a, b]$, because

$$\begin{aligned} \alpha \leq f(x) \leq \beta &\Rightarrow -\beta \leq f(x) \leq -\alpha \\ &\Rightarrow |f(x)| \leq \text{Max}\{\beta, -\alpha\} \ \ \forall x \in [a, b] \\ &\Rightarrow |f| \text{ is bounded} \end{aligned}$$

**DEFINITION 1.26** Suppose $f:[a, b] \to \mathbb{R}$ is a function. If there exists $x^* \in [a, b]$ such that

$$f(x^*) \le f(x) \; \forall x \in [a, b]$$

then $x^*$ is called a point of **absolute minimum** of $f$ on $[a, b]$ and $f(x^*)$ is called the **absolute minimum** of $f$ on $[a, b]$. Similarly, if $x^{**} \in [a, b]$ is such that

$$f(x) \le f(x^{**}) \; \forall x \in [a, b]$$

then $x^{**}$ is called a point of **absolute maximum** of $f$ in $[a, b]$ and $f(x^{**})$ is called the **absolute maximum** of $f$ on $[a, b]$.

**QUICK LOOK 7**

If $f:[a,b] \to \mathbb{R}$ has absolute minimum and absolute maximum, then $f$ is bounded on $[a, b]$.

We now state and prove that a continuous function on a closed interval $[a, b]$ ($a$ and $b$ are real) is bounded and we show that the continuity of $f$ **on closed interval** is essential.

**THEOREM 1.31** Suppose $a$ and $b$ are real numbers and $a < b$. If $f:[a, b] \to \mathbb{R}$ is a continuous function, then $f$ is bounded.

**PROOF** Suppose $f$ is not bounded. Then either $f$ is not bounded below or $f$ is not bounded above. Since $f$ is continuous at $a$, to the positive number 1 there corresponds a $\delta > 0$ such that $a + \delta < b$ and

$$|f(x) - f(a)| < 1 \; \forall x \in [a, a + \delta]$$

so that

$$|f(x)| < 1 + |f(a)| \; \forall x \in [a, a + \delta]$$

Thus $|f|$ and hence $f$ is bounded in $[a, a + \delta]$. Let

$$S = \{c \in (a, b) | f \text{ is bounded in } (a, c)\}$$

Then clearly $S$ is non-empty because $a + \delta \in S$ and also $S$ is bounded above. Let $\lambda$ be the l.u.b. $S$. We observe that $c \in (a, \lambda) \Rightarrow c \in S$. Suppose $\lambda < b$. Since $f$ is continuous at $\lambda$, corresponding to 1, there exists a $\delta_1 > 0$ such that

$$a < \lambda - \delta_1 < \lambda + \delta_1 < b$$

and
$$|f(x) - f(\lambda)| < 1 \quad \forall x \in [\lambda - \delta_1, \lambda + \delta_1]$$

Therefore, $f$ is bounded on $[\lambda - \delta_1, \lambda + \delta_1]$ and hence it is bounded on $[a, \lambda + \delta_1]$. Thus $\lambda - \delta_1 \in S$ such that $\lambda + \delta_1 \le \lambda$, a contradiction (because $\lambda$ is the l.u.b. $S$). Therefore $\lambda \nless b$ and hence $\lambda = b$.

Again since $f$ is continuous at $b$, to 1 there corresponds a $\delta_2 > 0$ such that $a < b - \delta_2 < b$ and

$$|f(x) - f(b)| < 1 \quad \forall x \in [b - \delta_2, b]$$

so that $f$ is bounded on $[b - \delta_2, b]$. Clearly $b - \delta_2 \in S$ because $b = \lambda$ is the l.u.b. $S$. Hence $b \in S$. Thus $f$ is bounded on $[a, b]$. ■

***Note:*** The following examples illustrate that a continuous function defined on an open interval $(a, b)$ or semi-open interval $(a, b]$ or $[a, b)$ need not be bounded.

## Examples

1. Define $f:(0, 1] \to \mathbb{R}$ by $f(x) = 1/x$.
2. Define $g:[0, 1) \to \mathbb{R}$ by $g(x) = 1/1 - x$.
3. Define $h:(0, 1) \to \mathbb{R}$ by $h(x) \to 1/x(1 - x)$.

One can see that $f$, $g$, $h$ are continuous functions but not bounded.

**Theorem 1.32 (Intermediate Value Theorem)** Suppose $f:[a,b]\to\mathbb{R}$ is a continuous function and $f(a)\neq f(b)$. Then $f$ assumes every value between $f(a)$ and $f(b)$.

**Proof** Without loss of generality we may assume that $f(a)<f(b)$. Suppose $f(a)<\lambda<f(b)$. We now show that, there exists $x_0\in(a,b)$ such that $f(x_0)=\lambda$. Let

$$\varepsilon=\frac{\lambda-f(a)}{2}>0$$

Since $f$ is continuous at $a$, there exists $\delta_1>0$ such that

$$\begin{aligned} x\in[a,a+\delta_1] &\Rightarrow |f(x)-f(a)|<\varepsilon \\ &\Rightarrow f(x)-f(a)<\varepsilon=\frac{\lambda-f(a)}{2} \\ &\Rightarrow f(x)<f(a)+\frac{\lambda-f(a)}{2}=\frac{\lambda+f(a)}{2}<\lambda \end{aligned}$$

Now, let

$$S=\{t\in[a,b]\,|\,f(x)<\lambda\ \forall\, x\in[a,t]\}$$

Clearly, from what we have established above, it follows that

$$a+\delta_1\in S \quad \text{and} \quad \text{hence } S\neq\phi \tag{1.28}$$

Further,

$$t\in S, a\leq s<t\Rightarrow s\in S \tag{1.29}$$

Clearly $S$ is bounded above by $b$. Let $\alpha=$ l.u.b. $S$ so that by Eq. (1.29)

$$[a\,,\alpha)\subset S \tag{1.30}$$

From Eq. (1.28),

$$a<a+\delta_1\leq\alpha \tag{1.31}$$

Since $f$ is continuous at $b$, corresponding to the positive number $[f(b)-\lambda]/2$, there exists $\delta_2>0$ such that $a<b-\delta_2<b$ and $x\in[b-\delta_2,b]$. This implies

$$\begin{aligned} |f(x)-f(b)| &< \frac{f(b)-\lambda}{2} \\ \Rightarrow f(b)-f(x) &< \frac{f(b)-\lambda}{2} \\ \Rightarrow \frac{f(b)+\lambda}{2} &< f(x) \end{aligned}$$

This shows that $t\in[b-\delta_2,b]\Rightarrow t\notin S$. Hence

$$\alpha\leq b-\delta_2<b \tag{1.32}$$

Thus $a<\alpha<b$ and further

$$a\leq t<\alpha\Rightarrow t\in S\Rightarrow f(t)<\lambda\Rightarrow f(\alpha)\leq\lambda$$

Now, $f(\alpha)<\lambda$ as in Eq. (1.28), there exists $\delta_3>0$ such that

$$f(x)<\frac{\lambda+f(\alpha)}{2}<\lambda\ \forall\, x\in[\alpha,\alpha+\delta_3]$$

so that by Eq. (1.30), $\alpha+\delta_3\in S$, which is a contradiction. Hence $f(\alpha)=\lambda$. Thus $f$ assumes the value $\lambda$. ■

**THEOREM 1.33** Suppose $f:[a,b]\to\mathbb{R}$ is a continuous function, $M = \underset{x\in[a,b]}{\text{l.u.b.}}\ f(x)$ and $m = \underset{x\in[a,b]}{\text{g.l.b.}}\ f(x)$. Then there exists $c, d$ in $[a,b]$ such that $f(c) = M$ and $f(d) = m$.

**PROOF** Suppose $f(x) \neq M\ \forall x \in [a,b]$. Then $M - f(x) > 0$ for all $x \in [a,b]$. Write $g(x) = M - f(x)$. Then $g(x)$ is continuous on $[a,b]$, and does not vanish on $[a,b]$. Hence $1/g$ is continuous on $[a,b]$ [see part (3) of Theorem 1.28] and hence $1/g$ is bounded on $[a,b]$. If $\lambda$ is a bound of $1/g$ on $[a,b]$, then $1/\lambda > 0$. Now,

$$\frac{1}{M-f(x)} = \frac{1}{g} < \lambda \Rightarrow \frac{1}{\lambda} < M - f(x)\quad \forall x\in[a,b]$$

$$\Rightarrow f(x) < M - \frac{1}{\lambda}\,\forall x\in[a,b]$$

$$\Rightarrow M \le M - \frac{1}{\lambda}\quad \left(\because M = \underset{x\in[a,b]}{\text{l.u.b.}}\ f(x)\right)$$

$$\Rightarrow \frac{1}{\lambda} \le 0,\ \text{which is a contradiction}$$

Hence $M - f(x) = 0$ for some $x \in [a,b]$. That is, there exists $c \in [a,b]$ such that $f(c) = M$. Similarly, we can show that, there exists $d \in [a,b]$, such that $f(d) = m$. ■

**QUICK LOOK 8**

1. From Theorems 1.31, 1.32 and 1.33 we have that every real-valued continuous function on a closed interval $[a,b]$, where $a$ and $b$ are finite real numbers, is bounded, attains its bounds and assumes every value between $\underset{x\in[a,b]}{\text{g.l.b.}}\ f(x)$ and $\underset{x\in[a,b]}{\text{l.u.b.}}\ f(x)$.

2. Theorem 1.33 may not be true if the interval is not a closed interval. For example, the function $f(x) = x$ $\forall x \in (0,1)$ is clearly continuous and $\underset{x\in(0,1)}{\text{g.l.b.}}\ f(x) = 0$ and $\underset{x\in(0,1)}{\text{l.u.b.}}\ f(x) = 1$ which cannot be attained by $f(x)$.

**COROLLARY 1.10** Suppose $f:[a,b]\to\mathbb{R}$ is continuous and $f(a)\,f(b) < 0$. Then, there exists $x \in (a,b)$ such that $f(x) = 0$.

**PROOF** Since $f(a)\,f(b) < 0$, $m = \underset{x\in[a,b]}{\text{g.l.b.}}\ f(x) < 0$ and $M = \underset{x\in[a,b]}{\text{l.u.b.}}\ f(x) > 0$, hence $0 \in [m, M]$. Now the result follows from Quick Look 8.

***Note:*** Graphically $f(a)\,f(b) < 0$ means that the graph of $y = f(x)$ (see Fig. 1.19) must cross the $x$-axis in between $a$ and $b$. ■

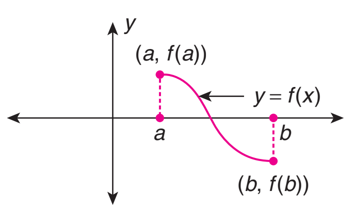



**FIGURE 1.22** Corollary 1.10.

The following theorem provides useful information.

**THEOREM 1.34** If $f: [0,1] \to [0,1]$ is continuous, then for some $x_0$ in $[0,1]$ $f(x_0) = x_0$. [Generally, for any function $f(x)$, a point $x_0$ such that $f(x_0) = x_0$ is called a **fixed point** of $f$.]

**PROOF** Define $g(x) = f(x) - x\ \forall x \in [0,1]$ which is also continuous on $[0,1]$ with $g(0) = f(0) \ge 0$ and $g(1) = f(1) - 1 \le 0$. Hence by the intermediate value theorem (Theorem 1.32), $g(x)$ must assume the value 0.

Thus $g(x_0) = x_0$ for some $x_0 \in [0, 1]$. Graphically, the graph of $y = f(x)$ (see Fig. 1.23) lies in the unit square for which the line $y = x$ is a diagonal so that the curve must intersect the diagonal $y = x$.

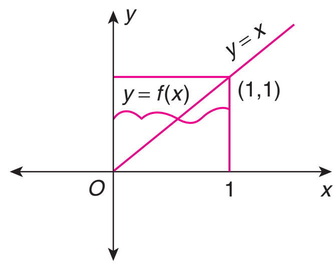



**FIGURE 1.23** Theorem 1.34. ■

**THEOREM 1.35** Let $f$ and $g$ be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Then $f(x_1) = g(x_1)$ for some $x_1 \in [a, b]$.

**PROOF** Define $h(x) = f(x) - g(x)$ for $x \in [a, b]$ so that $h$ is continuous on $[a, b]$ and

$$h(a) = f(a) - g(a) \geq 0$$

and
$$h(b) = f(b) - g(b) \leq 0$$

If either $h(a) = 0$ or $h(b) = 0$, then the theorem is proved. If $h(a) \neq 0$ $h$ and $h(b) \neq 0$, then by Corollary 1.10, there exists $x_1 \in (a, b)$ such that $h(x_1) = 0$. Hence $f(x_1) = g(x_1)$. ■

## 1.10 | Infinite Limits

So far we considered the limit of a function at a point, keeping in mind that the limit is a real number, so that the function is bound in a deleted neighbourhood of that point (Corollary 1.1). We now consider cases where (i) $f$ is not bounded or (ii) $f$ is bounded, but the neighbourhood is infinite.

**DEFINITION 1.27** Suppose $a < c < b$ and $f$ is a function defined on $[a, b]$ except possibly at $c$.

1. Suppose to a positive number $\varepsilon$ (however large), there corresponds $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and $x \in (c - \delta, c + \delta), x \neq c \Rightarrow f(x) > \varepsilon$. Then we say that $f(x)$ *tends to infinity as* $x \to c$ and write $\lim_{x \to c} f(x) = \infty$ or $f(x) \to \infty$ as $x \to c$.
2. Suppose to a positive number $\varepsilon$ (however large), there corresponds $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and $x \in (c - \delta, c + \delta), x \neq c \Rightarrow f(x) < -\varepsilon$. Then we say that $f(x)$ *tends to minus infinity as* $x \to c$ and we write $\lim_{x \to c} f(x) = -\infty$ or $f(x) \to -\infty$ as $x \to c$.
3. If to $\varepsilon > 0$, there corresponds $\delta > 0$ such that $(c - \delta, c) \subset [a, b]$ and $x \in (c - \delta, c) \Rightarrow f(x) > \varepsilon$, then we say that $f(x)$ tends to $\infty$ as $x$ tends $c$ from the left side and write $\lim_{x \to c-0} f(x) = \infty$ or $f(x) \to \infty$ as $x \to c - 0$. In a similar way, we define the right limit at $c$.
4. If to $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $(c - \delta, c) \subset [a, b]$ and $x \in (c - \delta, c) \Rightarrow f(x) < -\varepsilon$, then we say that $f(x)$ tends to $-\infty$ as $x \to c - 0$ and write $\lim_{x \to c-0} f(x) = -\infty$.
5. The notions of left infinite limit can be extended to the right infinite limit as follows:
   **(a)** If to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $(c, c + \delta) \subset [a, b]$ and $x \in (c, c + \delta) \Rightarrow f(x) > \varepsilon$, then we write $\lim_{x \to c+0} f(x) = \infty$ or $f(x) \to \infty$ as $x \to c + 0$.
   **(b)** If to each $\varepsilon > 0$, there corresponds a $\delta > 0$, such that $(c, c + \delta) \subset [a, b]$ and $x \in (c, c + \delta) \Rightarrow f(x) < -\varepsilon$, then we write $\lim_{x \to c+0} f(x) = -\infty$ or $f(x) \to -\infty$ as $x \to c + 0$.

***Note:*** $-\infty$ and $\infty$ are only symbols, but not real numbers.

**Example 1.38**

1. Define $f(x) = 1/|x|$ for $x \in [-1,1]$ and $x \neq 0$. Show that $\lim_{x\to 0} f(x) = \infty$.

***Solution:*** Let $\varepsilon > 0$ be a large number, we may suppose that $\varepsilon > 1$. Take $\delta = 1/2\ \varepsilon$. Then $\delta < 1$, $(0 - \delta, 0 + \delta) \subset [-1, 1]$ and

$$x \in (0-\delta, 0+\delta),\ x \neq 0 \Rightarrow |x| < \delta$$
$$\Rightarrow \frac{1}{|x|} > \frac{1}{\delta} = 2\varepsilon > \varepsilon$$
$$\Rightarrow f(x) = \frac{1}{|x|} > \varepsilon$$

Therefore $\lim_{x\to 0} f(x) = \infty$.

2. Define $f(x) = -1/|x|$ for $x \in [-1, 1], x \neq 0$. Then as in (1), $\lim_{x\to 0} f(x) = -\infty$.
3. Define $f(x) = 1/x$ for $x \in [-1, 1]$ $x \neq 0$. Then it can be seen that $\lim_{x\to 0-0} f(x) = -\infty$ and $\lim_{x\to 0+0} f(x) = \infty$.
4. Define $f(x) = 1/(1-x)$ for $x \in [0,2], x \neq 1$. By the substitution $y = 1 - x$ so that $y \in [-1, 1]$, $y \neq 0$ we have by (3), $\lim_{x\to 1-0} f(x) = \infty$ at $\lim_{x\to 1+0} f(x) = -\infty$.

**DEFINITION 1.28** Suppose $f$ is defined on the infinite interval $[a, \infty)$ and $l \in \mathbb{R}$. If to each $\varepsilon > 0$, there corresponds a $\delta > 0$ (large enough) such that $x > \delta > a \Rightarrow |f(x) - l| < \varepsilon$ then we say that $f(x)$ tends to $l$ as $x$ tends to $\infty$ and we write $\lim_{x\to\infty} f(x) = l$. If $f$ is defined on $(-\infty, a]$, we can similarly define $\lim_{x\to-\infty} f(x) = l$.

**Examples**

1. Define $f(x) = 1/x$ for $x \geq 1$. Then to $\varepsilon > 0$, there corresponds $\delta = 1/\varepsilon$ such that
$$x > \delta,\ x > 1 \Rightarrow f(x) = 1/x < 1/\delta = \varepsilon$$
$$\Rightarrow |f(x) - 0| < \varepsilon$$
Hence $\lim_{x\to\infty} f(x) = 0$
2. Define $f(x) = 1/x$ for $x \leq -1$. Then as above, we can see that $\lim_{x\to-\infty} f(x) = 0$.
3. If $f(x) = 1/(1-x)$ for $x \neq 1$, then it can be verified that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

**DEFINITION 1.29** Suppose $a < c < b$ and $f$ is defined on $[a, b]$ except possibly at $c$. If $f$ does not tend to either a finite limit or an infinite limit as $x \to c$, then we say that $f$ oscillates at $c$. The function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

oscillates at every real number, because in any neighbourhood of a real number, there are infinitely many rationals and infinitely many irrationals.

## 1.11 | Sequences and Series

In this section we define a sequence, limit of a sequence, obtain tests for convergence of a sequence and Cauchy's general principle of convergence. We see a sequence as a function from the set $\mathbb{N}$ (or $\mathbb{Z}^+$) of positive integers (sometimes from the set $W$ of non-negative integers) into the set $\mathbb{R}$ of real numbers.

As a sequel, we define series as a sum of elements of a sequence, attach a meaning to the sum and obtain some tests for convergence of series.

In this process, we also find a link between the convergence of a function to a limit at a point and convergence of a sequence of functional values to that limit at that point. Finally, we end this with a condition for a function to be continuous at a point of its domain with the aid of sequence.

**DEFINITION 1.30** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers. Suppose $f : \mathbb{N} \to \mathbb{R}$ is a function. Then the functional values $f(1), f(2), f(3), \ldots$, denoted by $\{f(n)\}$ is called a *sequence*. $f(n)$ is called the $n$th term of the sequence.

It is customary to denote the functional value $f(n)$ by $x_n$ ($x$ may be replaced by any other letter). Thus $\{x_n\}$ is a sequence in $\mathbb{R}$.

## Examples

1. If $f : \mathbb{N} \to \mathbb{R}$ is defined by $f(n) = n^2$ or $x_n = n^2$, then $\{1^2, 2^2, 3^2, \ldots\} = \{f(1), f(2), f(3), \ldots\} = \{x_1, x_2, x_3, \ldots\}$ is a sequence.
2. Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = n + 1$. Then $\{f(1), f(2), f(3), \ldots\} = \{2, 3, 4, 5, \ldots\}$ is a sequence.
3. Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = 1/n$. Then $\{1, 1/2, 1/3, 1/4, \ldots\}$ is a sequence.
4. Define $x_n = -n^2 + \sqrt{3} \cdot n, n = 1, 2, 3, \ldots$. Then $\{-n^2 + \sqrt{3}n\}$, $n = 1, 2, 3, \ldots$ is a sequence.
5. Define
$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$
Then $\{1, -1, 1, -1, \ldots\}$ is a sequence in $\mathbb{R}$.
6. Let $a \in \mathbb{R}$. Then the sequence $a, -a, 0, a, -a, 0, \ldots$ is given by
$$x_n = \begin{cases} a & \text{if } n-1 \text{ is divisible by } 3 \\ -a & \text{if } n+1 \text{ is divisible by } 3 \\ 0 & \text{otherwise} \end{cases}$$

**DEFINITION 1.31** **Limit of a Sequence** Let $\{x_n\}$ be a sequence is $\mathbb{R}$ and $l \in \mathbb{R}$. Suppose to each $\varepsilon > 0$, there corresponds a positive integer $N$ such that $n \geq N \Rightarrow |x_n - l| < \varepsilon$. Then we say that $\{x_n\}$ converges to $l$ as $n \to \infty$ or $x_n \to l$ as $n \to \infty$ and write $\lim_{x\to\infty} x_n = l$.

**THEOREM 1.36 (UNIQUENESS OF LIMIT)** Suppose $x_n \to l$ and $x_n \to l'$ as $n \to \infty$. Then $l = l'$.

**PROOF** Suppose $l = l'$. Let $\varepsilon = |l - l'|/2 > 0$. Then to this $\varepsilon$, there corresponds a positive integer $N_1$ such that
$$n \geq N_1 \Rightarrow |x_n - l| < \varepsilon \quad (\because \lim_{x\to\infty} x_n = l)$$
Similarly, there exists positive integer $N_2$ such that
$$|x_n - l'| < \varepsilon \quad (\because \lim_{x\to\infty} x_n = l')$$
Let $N = \text{Max}\{N_1, N_2\}$. Then $n \geq N \Rightarrow |x_n - l| < \varepsilon$ and $|x_n - l'| < \varepsilon$. Now,
$$|l - l'| = |l - x_n + x_n - l'| \leq |l - x_n| + |x_n - l'| < \varepsilon + \varepsilon = 2\varepsilon = |l - l'| \text{ for } n \geq N$$
which is a contradiction. Hence $l = l'$. ■

**THEOREM 1.37** Every convergent sequence is bounded. That is, if $x_n \to l$ as $n \to \infty$, the $\{x_n\}$ is bounded. In other words, if $x_n \to l$ as $n \to \infty$, then there exists $k > 0$ such that $|x_n| \leq k$ for $n = 1, 2, 3, \ldots$.

**PROOF** Since $x_n \to l$ as $n \to \infty$, to the positive number 1, there corresponds a positive integer $N$ such that $|x_n - l| < 1$ for $n \geq N$ so that
$$|x_n| = |x_n - l + l| \leq |x_n - l| + |l| < 1 + |l| \text{ for } n \geq N \tag{1.33}$$
Let
$$M = \text{Max}\{|x_1|, |x_2|, \ldots, |x_{N-1}|\} \tag{1.34}$$
Take $k = \text{Max}\{1 + |l|, M\}$ so that from Eqs. (1.33) and (1.34), we have $|x_n| \leq k$ for $n = 1, 2, 3, \ldots$. Thus $\{x_n\}$ is bounded. ■

**QUICK LOOK 9**

If a sequence is not bounded, then it is not convergent. For example, the sequence $\{x_n\}$ where $x_n = n + 1$, $n = 1, 2, 3, \ldots$, is not bounded and hence it is not convergent.

**THEOREM 1.38** If $\{x_n\}$ converges to a limit $l$ and $l \neq 0$, then there exists a positive number $K$ and a positive integer $N$ such that $|x_n| \geq K$ for $n \geq N$.

**PROOF** Take $\varepsilon = |l|/2 > 0$. Then there exists a positive integer $N$ such that $|x_n - l| < \varepsilon$ for $n \geq N$ so that

$$\big||l| - |x_n|\big| \leq |l - x_n| \leq \varepsilon \text{ for } n \geq N$$

Therefore

$$|l| - \varepsilon \leq |x_n| \leq |l| + \varepsilon \text{ for } n \geq N$$

That is,

$$0 < \frac{|l|}{2} \leq |x_n| \leq \frac{3}{2}|l| \text{ for } n \geq N \quad \left(\because \varepsilon = \frac{|l|}{2}\right)$$

Hence $|x_n| \geq |l|/2$ for $n \geq N$. Take $k = |l|/2$. ■

***Note:*** If $x_n = 1/n$ for $n = 1, 2, 3, \ldots$, then one can see that $x_n > 0\ \forall n = 1, 2, 3 \ldots$ and $x_n \to 0$ as $n \to \infty$ which shows that the converse of the above theorem is not true.

**THEOREM 1.39** If $\{x_n\}$ converges to $l$, then $\{|x_n|\}$ converges to $|l|$.

**PROOF** Since $x_n \to l$ as $n \to \infty$, to $\varepsilon > 0$, there corresponds a positive integer $N$ such that $|x_n - l| < \varepsilon$ for $n \geq N$. Hence

$$\big||x_n| - |l|\big| \leq |x_n - l| < \varepsilon \text{ for } n \geq N$$

Thus $|x_n| \to |l|$ as $n \to \infty$. ■

***Note:*** The converse of the above theorem is not true. Example 1.39 illustrates the same.

## Example 1.39

Show that the converse of Theorem 1.39 does not hold.

***Solution:*** Take $x_n = (-1)^n$ so that $\{x_n\} = \{-1, 1, -1, 1, \ldots\}$. Suppose $x_n \to l$ as $n \to \infty$. Then to $\varepsilon = 1/2$, there corresponds a positive integer $N$ such that $|x_n - l| < 1/2$ for $n \geq N$ so that

$$|1 - l| < 1/2 \quad \text{if } n \text{ is even and } n \geq N$$

and

$$|1 + l| < 1/2 \quad \text{if } n \text{ is odd and } n \geq N$$

Hence

$$2 = |1 - l - (-1 - l)| \leq |1 - l| + |-1 - l| < 1/2 + 1/2 = 1$$

which is a contradiction. However $|x_n| = 1$ for all $n = 1, 2, 3, \ldots$ converges to 1 as $n \to \infty$.

Thus a sequence may not be convergent, but the sequence of its absolute values may converge.

**DEFINITION 1.32** Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences.

1. If $z_n = x_n + y_n$, then $\{z_n\}$ is called *sum of the sequences* $\{x_n\}$ and $\{y_n\}$.
2. If $t_n = x_n \cdot y_n$, then $\{t_n\}$ is called the *product* of $\{x_n\}$ and $\{y_n\}$.
3. If $y_n \neq 0$ for *large* $n$, and $w_n = x_n/y_n$ then $\{w_n\}$ is called the *quotient* of $\{x_n\}$ and $\{y_n\}$.

The following results are analogues to the results on functions and hence we state them without giving proofs.

**Theorem 1.40** Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences such that $\{x_n\}$ converges to $l$ and $\{y_n\}$ converges to $m$ as $n \to \infty$. Then

1. $\{x_n + y_n\}$ converges to $l + m$.
2. $\{x_n \cdot y_n\}$ converges to $l \cdot m$.
3. If $m \neq 0$, then $\{1/y_n\}$ converges to $1/m$.
4. If $m \neq 0$, then $\{x_n/y_n\}$ converges to $l/m$.
5. If $\lambda \in \mathbb{R}$, then $\{\lambda x_n\}$ converges to $\lambda l$.

**DEFINITION 1.33** **Cauchy Sequence** A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be a *Cauchy sequence*, if to each $\varepsilon > 0$, there corresponds a positive integer $N$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.

**Theorem 1.41 (Cauchy's General Principle of Convergence)**

1. Every convergent sequence in $\mathbb{R}$ is a Cauchy sequence.
2. Every Cauchy sequence in $\mathbb{R}$ is convergent.

**Proof** We prove (1) and assume the validity of (2) because its proof needs a little bit mechanism involving l.u.b. and g.l.b. of sets of real numbers.

1. Suppose $\{x_n\}$ is a convergent in $\mathbb{R}$ and let $\lim\limits_{n\to\infty} x_n = l$. Then to $\varepsilon > 0$, there corresponds a positive integer $N$ such that $|x_n - l| < \varepsilon/2$ for $n \geq N$. Now,

$$\begin{aligned} m, n \geq N \Rightarrow |x_n - x_m| &= |x_n - l + l - x_m| \\ &\leq |x_n - l| + |l - x_m| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Thus $\{x_n\}$ is a sequence. ■

***Note:*** To prove that a sequence in $\mathbb{R}$ is not convergent, we can use Cauchy's general principle of convergence. See the following example.

## Example 1.40

Let $x_n = (-1)^n$ for $n = 1, 2, 3, \ldots$. Show that the sequence is not convergent.

***Solution:*** This sequence is bounded. We show that this is not convergent. If this sequence is convergent then by part (1) of Theorem 1.41, corresponding to 1, there exists a positive integer $N$ such that

$$|x_n - x_m| < 1 \text{ for } n, m \geq N$$

Taking $m = n + 1$, we have

$$\begin{aligned} &|x_n - x_{n+1}| < 1 \\ &\Rightarrow |(-1)^n - (-1)^{n+1}| < 1 \\ &\Rightarrow |(-1)^n \,(1 - (-1))| < 1 \\ &\Rightarrow 2 < 1, \text{ a contradiction} \end{aligned}$$

Thus $\{(-1)^n\}$ is not convergent.

**DEFINITION 1.34** **Increasing and Decreasing Sequences**

1. A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be *monotonic increasing* (or simply increasing) if $x_n \leq x_{n+1}$ for $n = 1, 2, 3, \ldots$.
2. A sequence $\{x_n\}$ is said to be *strictly increasing* if $x_n < x_{n+1}$ for $n = 1, 2, 3, \ldots$.

3. A sequence $\{x_n\}$ is said to be *monotonic decreasing* (or simply decreasing) if $x_n \geq x_{n+1}$ for $n = 1, 2, 3, \ldots$.
4. A sequence $\{x_n\}$ is said to be *strictly decreasing* if $x_n > x_{n+1}$ for $n = 1, 2, 3, \ldots$.

## Examples

1. Let $x_{2n} = x_{2n-1} = 1/n$ for $n = 1, 2, 3, \ldots$. Then $\{x_n\} = \left\{1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots\right\}$ is decreasing but not strictly decreasing.
2. Let $x_{2n} = x_{2n-1} = n$ for $n = 1, 2, 3, \ldots$. Then $\{x_n\} = \{1, 1, 2, 2, 3, 3, \ldots\}$ is increasing but not strictly increasing.
3. Let $x_n = 1 - (1/n)$ for $n = 1, 2, 3, \ldots$. Then $\{x_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$ is strictly increasing.
4. If $x_n = 1/n$, then $\{x_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is strictly decreasing.

The following theorem on convergence of a monotonic sequence (i.e., either increasing or decreasing) is useful.

**THEOREM 1.42**

1. Every increasing sequence, which is bounded above, is convergent and converges to its l.u.b.
2. Every decreasing sequence, which is bounded below, is convergent and converges to its g.l.b.

**PROOF**

1. Let $\{x_n\}$ be an increasing sequence so that $x_1 \leq x_2 \leq x_3 \leq \ldots$. Thus $\{x_n\}$ is bounded below. Now, suppose $\{x_n\}$ is bounded above and let $\alpha$ be the l.u.b. $\{x_n\}$. Let $\varepsilon > 0$. Then $\alpha - \varepsilon < \alpha$ and $\alpha - \varepsilon$ is not an upper bound of $\{x_n\}$. Hence there exists a positive integer $N$ such that $\alpha - \varepsilon < x_N$ so that $\alpha - \varepsilon < x_N < \alpha < \alpha + \varepsilon$. Since $\{x_n\}$ is increasing with l.u.b. $\alpha$, $x_N \leq x_n \leq \alpha$ for $n \geq N$. Hence
$$|x_n - \alpha| = \alpha - x_n \leq \alpha - x_N < \varepsilon \ \forall\, n \geq N$$
Therefore $\lim\limits_{n\to\infty} x_n = \alpha$.
2. Suppose $\{x_n\}$ is a decreasing sequence which is bounded below. Then one can see that $\{-x_n\}$ is an increasing sequence bounded above and l.u.b. $\{-x_n\} = -$g.l.b. $\{x_n\}$. Let $\beta =$ g.l.b. $\{x_n\}$. Therefore by (1) $-x_n \to -\beta$ as $n \to \infty$ and hence $x_n \to \beta$ as $n \to \infty$. Thus $\lim\limits_{n\to\infty} x_n = \beta$ which is g.l.b. $\{x_n\}$. ■

**Notation:** If $\{x_n\}$ is an increasing sequence, we write $\{x_n\}\uparrow$ and if it is decreasing, then we write $\{x_n\}\downarrow$.

**DEFINITION 1.35** **Subsequence** Let $\{n_k\}$ be a strictly increasing sequence of positive integers so that $n_1 < n_2 < n_3 < \cdots$. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then $\{x_{n_k}\}, k = 1, 2, 3, \ldots$ is called a *subsequence* of $\{x_n\}$.

## Example

Let $x_n = n^2, n = 1, 2, 3 \ldots$. Then $\{x_n\}$ is a sequence in $\mathbb{R}$. Let $n_1 = 2 \cdot 1, n_2 = 2 \cdot 2, n_3 = 2 \cdot 3, \ldots, n_k = 2 \cdot k$, so that $n_1 < n_2 < n_3 < \cdots < n_k < 2 \ldots$. Then
$$\{x_{n_k}\} = \{x_{2k}\} = \{(2k)^2, k = 1, 2, 3, \ldots\}$$
is a subsequence of $\{x_n\}$.

**THEOREM 1.43** Every subsequence of a convergent sequence converges to the same limit.

**PROOF** Let $\{x_n\}$ be a convergent sequence, $x_n \to l$ as $n \to \infty$ and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. To $\varepsilon > 0$, there corresponds a positive integer $N$ such that $|x_n - l| < \varepsilon$ for all $n \geq N$. Since $\{n_k\}$ is an increasing sequence of integers, there exists a positive integer $k$ such that $n_k > N$ so that $n_p \geq n_k > N \ \forall\, p \geq k$. Hence
$$p \geq k \Rightarrow |x_{n_p} - l| < \varepsilon$$
Thus $x_{n_k} \to l$ as $k \to \infty$. ■

**Theorem 1.44** If a subsequence of a monotonically increasing sequence converges to a limit, then the sequence itself converges to the same limit.

**Proof** Let $\{x_n\}$ be a monotonically increasing sequence and $\{x_{n_k}\}$ a subsequence of $\{x_n\}$. Suppose $x_{n_k} \to l$ as $k \to \infty$. Since $\{x_{n_k}\}$ is also increasing and $\lim_{k\to\infty} x_{n_k} = l$, we have $l = \text{l.u.b. } x_{n_k}$ (Theorem 1.42) so that $x_{n_k} \le l$ for $k = 1, 2, 3, \ldots$. Therefore to $\varepsilon > 0$, there corresponds $k_1$ such that

$$|x_{n_k} - l| < \varepsilon \quad \text{for } k \le k_1$$

Let $N = n_{k_1}$. Then $n \ge N \Rightarrow n \ge n_{k_1}$ and $n \le n_p$ for some $p \ge k_1$. Therefore

$$x_{n_k} \le x_n \le x_{n_p} \le l \Rightarrow |x_n - l| \le |x_{n_k} - l| < \varepsilon$$

for $n \ge N$. Hence $x_n \to l$ as $n \to \infty$. ■

***Note:*** In Theorem 1.44, the increasing nature of the sequence is essential.

The following is an example of a sequence which is neither increasing nor decreasing, but contains convergent subsequences.

## Example

Let $x_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

Then $\{x_n\} = \{0, 1, 0, 1, 0, \ldots\}$ which contains two subsequences

$$\{x_{2n+1}\} = \{0, 0, 0, \ldots\} \quad \text{and} \quad \{x_{2n}\} = \{1, 1, 1, \ldots\}$$

which converge to 0 and 1, respectively. Hence $\{x_n\}$ does not converge according to Theorem 1.43.

We now introduce the notion of divergence of a sequence and discuss few tests for the convergence of a sequence.

**DEFINITION 1.36**

1. A sequence $\{x_n\}$ in $\mathbb{R}$ is said to *diverge* to $+\infty$, if to each $\varepsilon > 0$, there corresponds a positive integer $N$ such that $x_n > \varepsilon$ for all $n \ge N$, We write $x_n \to +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$.
2. A sequence $\{x_n\}$ is $\mathbb{R}$ is said to *diverge* to $-\infty$, if to each $\varepsilon > 0$, there corresponds a positive integer $N$ such that $x_n < -\varepsilon$ for all $n \ge N$. We write $x_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$.

**QUICK LOOK 10**

$$x_n \to +\infty \text{ if and only if } -x_n \to -\infty \text{ as } n \to \infty.$$

***Note:***

1. $x_n \to +\infty$ as $n \to \infty \Rightarrow \{x_n\}$ is bounded below but not bounded above.
2. $x_n \to -\infty$ as $n \to \infty \Rightarrow \{x_n\}$ is bounded above but not bounded below.
3. If $\{x_n\}$ is an increasing sequence and not bounded above, then $x_n \to +\infty$ as $n \to \infty$.
4. If $\{x_n\}$ is a decreasing sequence and not bounded below, then $x_n \to -\infty$ as $n \to \infty$.
5. A sequence may neither be bounded below nor bounded above. For example

$$x_n = \begin{cases} n & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases}$$

## Examples

1. Let $x_n = n^2$ for $n = 1, 2, 3, \ldots$. That is $\{x_n\} = \{1^2, 2^2, 3^2, \ldots\}$. Then, given $\varepsilon > 0$, take $N = [\sqrt{\varepsilon}] + 1$, where $[\sqrt{\varepsilon}]$ denotes the integral part of $\sqrt{\varepsilon}$. Then

$$n \ge N \Rightarrow n^2 \ge N^2 > (\sqrt{\varepsilon})^2 = \varepsilon$$

Hence $x_n \to +\infty$ as $n \to \infty$.

**2.** Let $x_n = -n$ for $n = 1, 2, 3, \ldots$. Here, given $\varepsilon > 0$, take $n = [\varepsilon] + 1$ so that

$$n \geq N \Rightarrow -n \leq -N = -([\varepsilon]+1) < -\varepsilon \quad (\because [\varepsilon]+1 > \varepsilon)$$

Thus $x_n \to -\infty$ as $n \to \infty$. Also according to Quick Look 10, $-n^2 \to -\infty$ as $n \to \infty$.

The following theorem is called **Cauchy's first theorem on limits**.

**Theorem 1.45 (Cauchy's First Theorem on Limits)** Suppose $\{x_n\}$ is a sequence in $\mathbb{R}$ and $x_n \to l$ as $n \to x$. Then the sequence $\{y_n\}$ given by

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to $l$ as $n \to \infty$.

**Proof** It is enough if we prove the theorem for the case $l = 0$. Suppose $x_n \to 0$ as $n \to \infty$ and

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

We now show that $y_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since $x_n \to 0$ as $n \to \infty$, to $\varepsilon > 0$, there corresponds a positive integer $N$ such that

$$|x_n| < \frac{\varepsilon}{2} \quad \text{for } n \geq N$$

By Theorem 1.39 since $\{x_n\}$ converges so does $\{|x_n|\}$. Hence by Theorem 1.37 both $\{x_n\}$ and $\{|x_n|\}$ are bounded. Let $k$ be a bound of $\{|x_n|\}$. Take

$$M = \text{Max}\left\{N, \frac{2kN}{\varepsilon} + 1\right\}$$

Then for $n \geq M$ we have

$$\begin{aligned}
|y_n| &= \left|\frac{x_1 + x_2 + \cdots + x_n}{n}\right| \\
&= \left|\frac{x_1 + x_2 + \cdots + x_M}{n} + \frac{x_{M+1} + x_{M+2} + \cdots + x_n}{n}\right| \\
&< \frac{|x_1| + |x_2| + \cdots + |x_M|}{n} + \frac{|x_{M+1} + x_{M+2} + \cdots + x_n|}{n} \\
&< \frac{KM}{n} + \frac{\varepsilon}{2}\left(\frac{n-M}{n}\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{(by choice of } M\text{)}
\end{aligned}$$

Therefore $y_n \to 0$ as $n \to \infty$.

To prove the main result, suppose $x_n \to l$ as $n \to \infty$. Let

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Write

$$a_n = x_n - l$$

and

$$b_n = \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_n - l)}{n}$$

Now $a_n \to 0$ as $n \to \infty \Rightarrow b_n \to 0$ as $n \to \infty$. But $b_n = y_n - l$. Therefore

$$b_n \to 0 \text{ as } n \to \infty \Rightarrow y_n \to l \text{ as } n \to \infty$$

■

The following two theorems are tests for convergence of a sequence.

**Theorem 1.46 (Comparison Test for Sequences)** Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of non-negative real numbers and $x_n \le y_n\ \forall n$. Then

1. $y_n \to 0$ as $n \to \infty \Rightarrow x_n \to 0$ as $n \to \infty$.
2. $x_n \to \infty$ as $n \to \infty \Rightarrow y_n \to \infty$ as $n \to \infty$.

**Proof**

1. Suppose $y_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since $y_n \to 0$ as $n \to \infty$, there exists a positive integer $N$ such that $|y_n| < \varepsilon$ for $n \ge N$. That is, $0 \le y_n < \varepsilon$ from $n \ge N$ ($\because y_n \ge 0$). Hence (by hypothesis)
$$0 \le x_n \le y_n < \varepsilon\ \forall n \ge N$$
Therefore $x_n \to 0$ as $n \to \infty$.
2. Suppose $x_n \to \infty$ as $n \to \infty$. Let $\varepsilon > 0$. Therefore there exists a positive integer $N$ such that $x_n > \varepsilon\ \forall n \ge N$. Now
$$y_n \ge x_n > \varepsilon\ \forall n > N \Rightarrow y_n \to \infty \quad \text{as } n \to \infty$$
■

**Corollary 1.11** If $\{x_n\}$ and $\{y_n\}$ are two sequences in $\mathbb{R}$ such that $|x_n| \le |y_n|$ for all $n$, then $y_n \to 0$ as $n \to \infty \Rightarrow x_n \to 0$ as $n \to \infty$.

**Proof** Using Theorem 1.39, we have $y_n \to 0$ as $n \to \infty \Rightarrow |y_n| \to 0$ as $n \to \infty$. Therefore $|x_n| \le |y_n| \to 0$ as $n \to \infty \Rightarrow |x_n| \to 0$ as $n \to \infty$. Hence
$$\lim_{n\to\infty} x_n = 0$$
■

To prove the Ratio Test for sequences we need some simple results which are stated below as examples.

## Example 1.41

Show that if $x > 1$, then the sequence $\{x^n\}$, $n = 1, 2, 3, \ldots$, diverges to $+\infty$ as $n \to \infty$.

***Solution:*** Write $x = 1 + h$ where $h > 0$. Using either part (2) of Theorem 1.20 or Binomial theorem (Vol. 1) for positive integral index, we have
$$x^n = (1+h)^n > 1 + nh \to \infty \text{ as } n \to \infty$$
Hence from part (2) of Theorem 1.46, $x_n \to +\infty$ as $n \to \infty$. The converse can be provid similarly.

## Example 1.42

If $a_n > 0$ for $n = 1, 2, 3, \ldots$, then show that $a_n \to 0$ as $n \to \infty$ if and only if $1/a_n \to \infty$ as $n \to \infty$.

***Solution:*** Suppose $a_n \to 0$ as $n \to \infty$ and let $\varepsilon > 0$. Therefore there exists a positive integer $N$ such that
$$a_n < \frac{1}{\varepsilon} \text{ for } n \ge N \Rightarrow \frac{1}{a_n} > \varepsilon \quad \text{for } n \ge N$$
Hence $a_n \to \infty$ as $n \to \infty$.
The converse can be proved similarly.

## Example 1.43

If $0 < x < 1$, then show that $x^n \to 0$ as $n \to \infty$.

***Solution:*** We have
$$0 < x < 1 \Rightarrow \frac{1}{x} > 1$$
$$\Rightarrow \left(\frac{1}{x}\right)^n \to \infty \quad \text{as } n \to \infty \text{ (by Example 1.41)}$$
$$\Rightarrow x^n \to 0 \quad \text{as } n \to \infty \text{ (by Example 1.42)}$$

**QUICK LOOK 11**

If $-1 < x < 1$ (i.e., $|x| < 1$), then $x^n \to 0$ as $n \to \infty$.

**Theorem 1.47 (Ratio Test for Sequences)** Suppose $\{x_n\}$ is a sequence of non-zero real numbers such that

$$\frac{x_{n+1}}{x_n} \to l \text{ as } n \to \infty$$

Then,

1. $x_n \to 0$ as $n \to \infty$, if $|l| < 1$.
2. $|x_n| \to \infty$ as $n \to \infty$, if $|l| > 1$.

**Proof**

1. Suppose $|l| < 1$. Let

$$k = \frac{1+|l|}{2}$$

so that $0 < k < 1$. Write

$$\varepsilon = \frac{1-|l|}{2}$$

so that $k = |l| + \varepsilon$. Then there exists $N$ such that

$$\left|\frac{x_{n+1}}{x_n} - l\right| < \varepsilon \quad \text{for } n > N$$

$$\Rightarrow \left|\frac{x_{n+1}}{x_n}\right| < |l| + \varepsilon = k \quad \text{for } n > N$$

$$\Rightarrow |x_{n+1}| < k|x_n| \quad \text{for } n > N$$

$$\Rightarrow |x_{n+1}| < k|x_n| < k^2|x_n| < \cdots < k^{n-N+1}|x_n| \quad \text{for } n > 1 \quad (\because\ 0 < k < 1)$$

$$= k^{n+1}\frac{|x_n|}{k^N}$$

Since $0 < k < 1$, $k^n \to 0$ as $n \to \infty$ (by Example 1.43). Therefore by part (1) of Theorem 1.46, $x_n \to 0$ as $n \to \infty$.

2. Suppose $|l| > 1$. Write $y_n = 1/x_n$ so that

$$\frac{y_{n+1}}{y_n} = \frac{x_n}{x_{n+1}} \to \frac{1}{l} \quad \text{as } n \to \infty \quad \text{and} \quad \frac{1}{|l|} < 1$$

Hence from (1), $y_n \to 0$ as $n \to \infty$ so that $|y_n| \to 0$ as $n \to \infty$. Consequently $|x_n| = 1/|y_n| \to \infty$ as $n \to \infty$ (by Example 1.42). ■

We assume the validity of the following without the proof.

**Theorem 1.48 (Cauchy's Second Theorem on Limits)** Suppose $\{x_n\}$ is a sequence of positive terms such that

$$\frac{x_{n+1}}{x_n} \to l \quad \text{as } n \to \infty$$

where $l$ is a finite number. Then the sequence $(x_n)^{1/n} \to l$ as $n \to \infty$.

The following theorem is a major consequence of the **density property of $\boldsymbol{Q}$** (Theorem 0.14).

**THEOREM 1.49** Let $\alpha \in \mathbb{R}$. Then, there exists strictly decreasing sequence $\{x_n\}$ of *rational numbers* such that $x_n \to \alpha$ as $n \to \infty$.

**PROOF** We already know (pre-requisites, Theorem 0.14) that between any two real numbers, there is a rational number. Let $x_1$ be a rational number such that $\alpha < x_1 < \alpha + 1$. Let

$$y_1 = \text{Min}\left\{x_1, \alpha + \frac{1}{2}\right\}$$

There exists a rational number $x_2$ such that $\alpha < x_2 < y_1$. In general, suppose we have defined rational numbers $x_1, x_2, \ldots, x_n$ such that $x_1 > x_2 > x_3 > \cdots > x_n > \alpha$ and

$$y_c = \text{Min}\left\{x_i, \alpha + \frac{1}{i+1}\right\}, i = 1, 2, \ldots, n$$

Then, there exists a rational number $x_{n+1}$ such that $\alpha < x_{n+1} < y_n$, so that

$$\alpha < x_{n+1} < \alpha + \frac{1}{n+1} \quad \text{and} \quad \alpha < x_{n+1} < x_n$$

Thus, we have constructed a sequence $\{x_n\}$ of rational numbers such that

$$\alpha < x_n < \alpha + \frac{1}{n} \quad \text{and} \quad x_1 > x_2 > \cdots > x_{n-1} > x_n \tag{1.35}$$

Now,

$$\begin{aligned} |\alpha - x_n| &= x_n - \alpha \quad (\because \alpha < x_n) \\ &< \left(\alpha + \frac{1}{n}\right) - \alpha \quad \text{(by construction)} \\ &= \frac{1}{n} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

Consequently $x_n \to \alpha$ as $n \to \infty$. Hence by Eq. (1.35), $\{x_n\}$ is a strictly decreasing sequence of rational numbers such that $x_n \to \alpha$ as $n \to \infty$. ■

***Note:*** In a similar way, given $\alpha \in \mathbb{R}$, we can show the existence of *strictly increasing sequence* $\{y_n\}$ *of rational numbers* such that $y_n \to \alpha$ as $n \to \infty$. In fact by Theorem 1.49, there exists *strictly increasing sequence* $\{u_n\}$ *of rational numbers* such that $-u_n \to \alpha$ as $n \to \infty$.

We now turn our discussion to establish a link connecting the limit of a sequence and limit of a function for which we assume that **"Every sequence of real numbers contains a monotonic subsequence"** (i.e., either increasing or decreasing).

**THEOREM 1.50** Suppose $a < c < b$, $f : [a, b] \to \mathbb{R}$ is a function and $f(x) \to l$ as $x \to c$. Further suppose that $\{x_n\}$ is a sequence in $[a, b]$ such that $x_n \neq c \; \forall \; n$ and $x_n \to c$ as $n \to \infty$. Then $f(x_n) \to l$ as $n \to \infty$.

**PROOF** Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and

$$x \,\varepsilon\, (c - \delta, c + \delta), x \neq c \Rightarrow |f(x) - l| < \varepsilon$$

To this $\delta > 0$, there corresponds a positive integer $N$ such that $|x_n - c| < \delta$ for $n \geq N$. Now,

$$\begin{aligned} n \geq N &\Rightarrow |x_n - c| < \delta \\ &\Rightarrow x_n \,\varepsilon\, (c - \delta, c + \delta), x_n \neq c \\ &\Rightarrow |f(x_n) - l| < \varepsilon \end{aligned}$$

Hence $f(x_n) \to l$ as $n \to \infty$. ■

**Corollary 1.12** Suppose $a < c < b$, $f:[a, b] \to \mathbb{R}$ is a function and $f$ is *continuous* at $c$. If $\{x_n\}$ is a sequence in $[a, b]$ and $x_n \to c$ as $n \to \infty$, then $f(x_n) \to f(c)$ as $n \to \infty$.

**Proof** In Theorem 1.50 take $l = f(c)$. Then the result follows even if $x_n = c$ for some $n$ s. ■

**Theorem 1.51** Suppose $a < c < b$, $f:[a, b] \to \mathbb{R}$ is a function and $l \in \mathbb{R}$. Further suppose that whenever a sequence $\{x_n\} \subset [a, b]$, $x_n \neq c$, is such that $x_n \to c$ as $n \to \infty$ implies that $f(x_n) \to l$ as $x \to c$, then $f(x) \to l$ as $x \to c$.

**Proof** Suppose that $f(x)$ does not tend to $l$ as $x \to c$. Then there exists a positive real number $\varepsilon$ such that for any $\delta > 0$ with $(c - \delta, c + \delta) \subset [a, b]$, there exists $x_\delta\, \varepsilon\, (c - \delta, c + \delta)$, $x_\delta \neq c$ such that

$$|f(x_\delta) - l| > \varepsilon \tag{1.36}$$

Now take $\delta = 1/n$ and write $x_\delta = x_n$. Then

$$x_n \in \left(c - \frac{1}{n}, c + \frac{1}{n}\right), x_n \neq c$$

Let $\eta > 0$. Take

$$N = \left[\frac{1}{\eta}\right] + 1 > \frac{1}{\eta}$$

where $[1/\eta]$ denote the integral part of $1/\eta$ so that

$$n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \eta$$

Hence

$$n \geq N \Rightarrow |x_n - c| < \frac{1}{n} < \eta$$

so that $x_n \to c$ as $n \to \infty$. Since, by hypothesis $f(x_n) \to l$ as $n \to \infty$, there exists a positive integer $N$ such that $|f(x_n) - l| < \varepsilon$ for $n \geq N$. According to Eq. (1.36), this contradicts the choice of $x_n$. Hence $f(x) \to l$ as $x \to c$. ■

**Corollary 1.13** Suppose $a < c < b$, $f:[a, b] \to \mathbb{R}$ is a function such that $f(x_n) \to f(c)$ whenever $\{x_n\}$ is a sequence in $[a, b]$ with $x_n \to c$ as $n \to \infty$. Then $f$ is *continuous* at $c$.

**Proof** In Theorem 1.51 take $l = f(c)$. ■

Combining Corollaries 1.12 and 1.13, we have the following theorem which gives an equivalent definition for the continuity of a function at a point of its domain.

**Theorem 1.52** Suppose $a < c < b$, $f:[a, b] \to \mathbb{R}$ is a function. Then $f$ is continuous at $c$ if and only if $f(x_n) \to f(c)$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in $[a, b]$ with $x_n \to c$ as $n \to \infty$.

**Proof** Suppose $f$ is continuous at $c$ and $x_n \to c$ as $n \to \infty$. Then by Corollary 1.12, $f(x_n) \to f(c)$ as $n \to \infty$. Now, suppose that $f(x_n) \to f(c)$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in $[a, b]$ with $x_n \to c$ as $n \to \infty$. We may suppose that $x_n \neq c\ \forall n$. Then by Corollary 1.13, $f$ is continuous at $c$. ■

## 1.12 | Infinite Series

In this section, we introduce the concept of an infinite series, its sum, convergent and divergent series with the aid of the sequences. Also we give few tests to determine convergence or divergence of a series.

**DEFINITION 1.37** **Infinite Series** Suppose $\{x_n\}$ is a sequence of real numbers. Then the series $x_1 + x_2 + x_3 + \cdots$ is called an infinite series. $\sum_{n=1}^{\infty} x_n$ stands for the infinite series $x_1 + x_2 + x_3 + \cdots$. Here $n$ is only a variable and we may equally write $\sum_{m=1}^{\infty} x_m$.

***Note:*** $\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$ is a finite sum. As it is, the symbol $\sum_{n=1}^{\infty} x_n$ has no meaning, but we attach a meaning to this symbol in certain situation as follows.

**DEFINITION 1.38** Let $\sum_{n=1}^{\infty} x_n$ be an infinite series. Write

$$s_n = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

which is the sum of the first $n$ terms $x_1 + x_2 + \cdots + x_n$. This sequence $\{s_n\}$ is called the *sequence of partial sums* of the series $\sum_{n=1}^{\infty} x_n$.

1. If the sequence $\{s_n\}$ converges, say to $s$, then we say that the infinite series converges to $s$ and has the sum $s$. In this case we write $\Sigma x_n = s$.
2. If $s_n$ diverges to $+\infty$ (or $-\infty$) then we say that $\Sigma x_n$ diverges to $+\infty$ (or $-\infty$).

**Notation:** Suppose $\{x_n\}, \{y_n\}$ are two sequences of real numbers. Then $\Sigma(x_n + y_n)$, $\Sigma\lambda x_n$, $\lambda \in \mathbb{R}$ are the series associated with the sequences $\{x_n + y_n\}$ and $\{\lambda x_n\}$.

***Note:*** It can be shown, as we did in the case of sequences, that if $\Sigma x_n = s$ and $\Sigma y_n = t$, then $\Sigma(x_n + y_n) = s + t$ and $\Sigma\lambda x_n = \lambda s$.

## Example 1.44

Consider the geometric series $a + ar + ar^2 + \cdots$ where $-1 < r < 1$. Under what conditions the series converges and diverges.

***Solution:*** For the given series

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1-r^n)}{1-r}$$

Now

$$\lim_{x\to\infty} s_n = \lim_{x\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

$$= \frac{a}{1-r} - \lim_{x\to\infty}\left(\frac{ar^n}{1-r}\right)$$

$$= \frac{a}{1-r} - 0 \quad \text{(Example 1.43 and Quick Look 11)}$$

$$= \frac{a}{1-r}$$

In particular the series $1 + (1/2) + (1/2^2) + \cdots$ (here $x_n = 1/2^n$ for $n = 0, 1, 2, \ldots$ is convergent and its sum is

$$\frac{1}{1-(1/2)} = 2$$

Similarly if $r > 1$, then the

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

diverges because $r_n$ diverges and hence $\Sigma ar^n$ diverges if $r > 1$.

## Example 1.45

Let

$$x_n = \begin{cases} (-1)^{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Does this series converge or diverge?

***Solution:*** The series $\Sigma x_n = 1 + 0 + 1 + 0 + \cdots$ diverges because if $s_n = x_1 + x_2 + \cdots + x_n$, then $s_{2n-1} = s_{2n} = n$ for $n = 1, 2, 3, \ldots$.

## Example 1.46

Let $x_n = (-1)^{n+1}$ for $n = 1, 2, 3, \ldots$. Find the sum of this series.

***Solution:*** For the given series, we have $\Sigma x_n = 1 - 1 + 1 - 1 + \cdots$ so that

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Then $s_n$ does not converge and hence the series $\Sigma x_n$ has no sum.

## Example 1.47

Let

$$x_n = \frac{1}{n(n+1)}$$

for $n = 1, 2, 3, \ldots$. Find its sum.

***Solution:*** We have

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Now $\dfrac{1}{n+1} \to 0$ as $n \to \infty \Rightarrow s_n \to 1$ as $n \to \infty$. Hence

$$\sum \frac{1}{n(n+1)} = 1$$

## Example 1.48

Show that the series $1\lfloor 1 + 2\lfloor 2 + 3\lfloor 3 + \cdots$ diverges.

***Solution:*** The series $1\lfloor 1 + 2\lfloor 2 + 3\lfloor 3 + \cdots$ diverges because

$$x_n = n\lfloor n = (n+1-1)\lfloor n = \lfloor n+1 - \lfloor n$$

and hence

$$\begin{aligned} s_n &= x_1 + x_2 + \cdots + x_n \\ &= (\lfloor 2 - \lfloor 1) + (\lfloor 3 - \lfloor 2) + (\lfloor 4 - \lfloor 3) + \cdots \\ &= \lfloor n+1 - 1 \to \infty \quad \text{as } n \to \infty \end{aligned}$$

Hence, the series $\Sigma(n\lfloor n)$ diverges.

**Theorem 1.53 (Cauchy's General Principle of Convergence for Series)** Let $\Sigma x_n$ be the series. Then a necessary and sufficient condition for $\Sigma x_n$ to converge is that to $\varepsilon > 0$, there corresponds a positive integer $N(\varepsilon)$ such that for any positive integers $m, n$ with $m > n > N(\varepsilon)$, $|x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon$. (In other words, the sequence $\{s_n\}$ of partial sums is a Cauchy sequence.)

**Proof** Let $s_n = x_1 + x_2 + \cdots + x_n$. Then $\Sigma x_n$ converges if and only if $s_n$ converges and $\{s_n\}$ converges; that is, by Theorem 1.48, if and only if to $\varepsilon > 0$, there corresponds a positive integer $N(\varepsilon)$ such that $|s_m - s_n| < \varepsilon$ whenever $m > n > N(\varepsilon)$. This implies

$$|x_{n+1} + x_{n+2} + \cdots + x_m| = |s_m - s_n| \text{ for } m > n > N(\varepsilon)$$

Thus the result follows. ■

The following theorem is a necessary but not sufficient condition for the convergence of a series.

**Theorem 1.54** If $\Sigma x_n$ converges, then $x_n \to 0$ as $n \to \infty$.

**Proof** Suppose $\Sigma x_n$ converges. Let $\varepsilon > 0$. Then there exists a positive integer $N(\varepsilon)$ such that $|s_m - s_n| < \varepsilon$ for $m > n > N(\varepsilon)$. Now, take $m = n+1, n > N(\varepsilon)$. Then

$$|x_{n+1}| = |s_{n+1} - s_n| < \varepsilon \text{ for } n > N(\varepsilon)$$

Hence $x_n \to 0$ as $n \to \infty$. ■

The following example shows that the converse of Theorem 1.54 is not true.

## Example 1.49

Show that the series $\Sigma(1/n)$ is not convergent while the $n$th term $(1/n) \to 0$ as $n \to \infty$.

***Solution:*** Let

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Then

$$\begin{aligned} |s_{2n} - s_n| &= s_{2n} - s_n \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\ &> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} \\ &= n\left(\frac{1}{2n}\right) > \frac{1}{2} \end{aligned}$$

Consequently by Theorem 1.53, $\Sigma(1/n)$ is not convergent.

**DEFINITION 1.39** If $x_n \geq 0\ \forall\, n$, then $\Sigma x_n$ is called *series of non-negative terms* and if $x_n > 0\ \forall\, n$ then $\Sigma x_n$ is called *series of positive terms.*

***Note:*** If $\Sigma x_n$ is a series of non-negative terms, then the sequence $\{s_n\}$ of partial sums is an increasing sequence of non-negative terms so that either $\{s_n\}$ converges in which case $\Sigma x_n$ converges or $\{s_n\}$ diverges to $+\infty$ in which case $\Sigma x_n$ diverges to $+\infty$.

The following theorem is called comparison test for the convergence or divergence of series.

**THEOREM 1.55 (COMPARISON TEST)** Let $\Sigma x_n$ and $\Sigma y_n$ be two series of non-negative terms such that $x_n \leq y_n\ \forall\, n \in \mathbb{N}$. Then

1. $\Sigma y_n$ converges $\Rightarrow \Sigma x_n$ converges.
2. $\Sigma x_n$ diverges to $+\infty \Rightarrow \Sigma y_n$ diverges to $+\infty$.

**PROOF** Let

$$s_n = x_1 + x_2 + \cdots + x_n \text{ and } t_n = y_1 + y_2 + \cdots + y_n$$

Then $x_n \leq y_n\ \forall\, n \Rightarrow s_n \leq t_n$ and $\{s_n\}, \{t_n\}$ are increasing sequences. Therefore by Theorem 1.47

$$\{t_n\} \text{ converges} \Rightarrow \{s_n\} \text{ converges}$$

$$\{s_n\} \text{ diverges to } +\infty \Rightarrow \{t_n\} \text{ diverges to } +\infty$$

■

**THEOREM 1.56** Let $\Sigma x_n$ be a series of non-negative terms and $\Sigma y_n$ a series of positive terms. If

$$\frac{x_n}{y_n} \to l \neq 0 \quad \text{as } n \to \infty$$

then either both $\Sigma x_n$ and $\Sigma y_n$ are both convergent or both divergent.

**PROOF** We have $l \neq 0 \Rightarrow l > 0$. Then there exists a positive integer $N$ such that

$$\left|\frac{x_n}{y_n} - l\right| < \frac{l}{2} \text{ for } n \geq N$$

That is

$$l - \frac{l}{2} < \frac{x_n}{y_n} < l + \frac{l}{2} \text{ for } n \geq N$$

$$\Rightarrow \left(\frac{l}{2}\right) y_n \leq x_n \leq \left(\frac{3}{2}\right) y_n\ \forall\, n \geq N$$

Hence

$$\Sigma x_n \text{ converges} \Rightarrow \frac{l}{2}\Sigma y_n \text{ converges} \Rightarrow \Sigma y_n \text{ converges}$$

Also,

$$\Sigma x_n \text{ diverges} \Rightarrow \frac{3}{2}\Sigma y_n \text{ also diverges} \Rightarrow \Sigma y_n \text{ diverges}$$

Thus $\Sigma x_n$ and $\Sigma y_n$ either both converge or both diverge simultaneously. ■

**THEOREM 1.57 (CAUCHY'S ROOT TEST)** Suppose $\Sigma x_n$ is a series of non-negative terms.

1. Let $0 < \lambda < 1$. If there exists a positive integer $N$ such that $x_n^{1/n} \le \lambda$ for $n \ge N$, then $\Sigma x_n$ converges.
2. If there exists a positive integer $N$ such that $x_n^{1/n} \ge 1$ for $n \ge N$, then $\Sigma x_n$ diverges.

**PROOF**

1. Suppose $x_n^{1/n} \le \lambda \in (0, 1)$ for $n \ge N$. Then $x_n \le \lambda^n$ for $n \ge N$ and $\Sigma \lambda^n$ is convergent (Example 1.44 taking $a = 1$). Therefore by Theorem 1.55, $\Sigma x_n$ is convergent.
2. Suppose $x_n^{1/n} \ge 1$ for $n \ge N$ so that $x_n \ge 1$ for $n \ge N$. Hence, $x_n$ does not tend to zero. Consequently, $\Sigma x_n$ diverges (by Theorem 1.54). ■

**THEOREM 1.58 (D'ALEMBERT'S TEST)** Suppose $\Sigma x_n$ is a series of positive terms.

1. Let $0 < \lambda < 1$. Suppose $N$ is a positive integer such that $x_{n+1}/x_n < \lambda$ for $n > N$. Then $\Sigma x_n$ converges.
2. Suppose $x_{n+1}/x_n \ge 1$ for $n > N$. Then $\Sigma x_n$ diverges.

**PROOF**

1. By hypothesis
$$x_{n+1} < \lambda x_n < \lambda^2 x_{n-1} < \cdots < \lambda^{n-N} x_{N+1} \text{ for } n > N$$
so that $x_{n+1} > \lambda^{n-N} x_{N+1}$. Therefore $\Sigma x_n$ converges since $\Sigma \lambda^{n-N} x_{N+1} = x_{N+1} \Sigma \lambda^{n-N}$ converges.
2. By hypothesis $x_{n+1} > x_n > x_{n-1} > \cdots > x_{N+1} > 0$ for $n > N$. So $\Sigma x_{N+1} < \Sigma x_{n+1}$ and $\Sigma x_{n+1}$ diverges. Hence $\Sigma x_{n+1}$ diverges and consequently $\Sigma x_n$ diverges. ■

## Example 1.50

Show that the series $\Sigma 1/n^p$ is convergent if $p > 1$ and divergent if $p \le 1$.

*Solution:*

1. The given series is
$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots \quad (1.37)$$
where $p > 1$. We observe that
$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$$
$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p} = \frac{1}{2^{2(p-1)}}$$
$$\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{2^{3(p-1)}}$$
Thus the $n$th group of Eq. (1.37) $< 1/2^{n(p-1)}$ and because $p > 1$, $1/2^{n(p-1)} < 1$. Hence $\Sigma 1/2^{n(p-1)}$ is convergent (since it is a geometric series with common ratio $1/2^{p-1} < 1$). Therefore, by comparison test (Theorem 1.55), the given series is also convergent.

**2.** When $p = 1$. In this case the given series becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ which is divergent (Example 1.67).

**3.** When $p < 1$. In this case $1/n^p > 1/n$ and $\Sigma(1/n)$ is divergent. Hence again by the comparison test the given series is divergent.

***Note:*** In the case $p > 1$, we have assumed that the convergence of a series does not change by grouping the terms of the series.

## Example 1.51

Show that the series $\Sigma \frac{1}{n^2+a^2}$ is convergent.

***Solution:*** Let

$$x_n = \frac{1}{n^2+a^2} \quad \text{and} \quad y_n = \frac{1}{n^2}$$

So $x_n < y_n$ and $\Sigma y_n$ is convergent (by Example 1.50). Hence by comparison test $\Sigma x_n$ is also convergent or

$$\frac{x_n}{y_n} = \frac{n^2}{n^2+a^2} = \frac{1}{1+(a^2/n^2)} \to 1 \quad \text{as } n \to \infty$$

Using Theorem 1.56 and the convergence of the series $\Sigma y_n$, we have that $\Sigma x_n$ is also convergent.

## Example 1.52

Show that the series $\Sigma \frac{1}{\sqrt{n}+\sqrt{n+1}}$ is divergent.

***Solution:*** Let

$$x_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} \quad \text{and} \quad y_n = \frac{1}{2\sqrt{n}}$$

Therefore

$$\frac{x_n}{y_n} = \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{2}{1+\sqrt{(n+1)/n}} \to 1 \quad \text{as } n \to \infty$$

By Example 1.50,

$$\Sigma \frac{1}{\sqrt{n}} \left( p = \frac{1}{2} < 1 \right) \text{is divergent}$$

Hence by Theorem 1.56, $\Sigma x_n$ is also divergent.

## Example 1.53

Show that the series $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots+\frac{n+1}{n^p}+\cdots$ is convergent if $p > 2$ and divergent if $p \le 2$.

***Solution:*** We have

$$x_n = \frac{n+1}{n^p}$$

Let

$$y_n = \frac{n}{n^p} = \frac{1}{n^{p-1}}$$

Clearly

$$\frac{x_n}{y_n} = 1+\frac{1}{n} \to 1 \quad \text{as } n \to \infty$$

1. If $p - 1 > 1$ (i.e., $p > 2$), then $\Sigma y_n$ is convergent and hence $\Sigma x_n$ is also convergent.
2. If $p - 1 \le 1$ (i.e., $p \le 2$), then $\Sigma y_n$ is divergent and hence $\Sigma x_n$ is also divergent.

# WORKED-OUT PROBLEMS

***Note:*** **IMPORTANT FORMULAE** The following limits are frequently used and are to be assumed.

1. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim\limits_{y\to 0}(1+y)^{1/y}$ where $e = 2.71828$
2. $\lim\limits_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$ where $a > 0$ and $a \ne 1$

   In particular $\frac{\log_e(1+x)}{x} \to \log_e e = 1$ as $x \to 0$

3. $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$, where $a>0$. In particular,

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=\log_e e=1$$

4. For all elementary standard functions (i.e., functions that are continuous at a point of their domains, see Definition 1.23)

$$\lim_{x\to a} f(x)=f(a)$$

5. If $a>0$, then $\frac{a}{1+a}<\log(1+a)<a$ will be assumed.

Note that using this inequality and the squeezing theorem (Theorem 1.14) we can show that

$$\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$$

6. In (1) if we replace $x$ with $n\in N$, we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

7. If $\lim_{x\to a} f(x)=\alpha>0$ and $\lim_{x\to a} g(x)=\beta$ (finite number), then $\lim_{x\to a}(f(x))^{g(x)}=\alpha^\beta$.

## *Single Correct Choice Type Questions*

### Functions and Limits

1. The domain of the function

$$f(x)=\sqrt{x^2-3x+2}+\frac{1}{\sqrt{3+2x-x^2}}$$

is

(A) $[-1,1]\cup[2,3]$ (B) $(-1,1]\cup[2,3)$
(C) $(-\infty,1]$ (D) $[2,\infty)$

**Solution:** We have

$$\begin{aligned}\sqrt{x^2-3x+2}\text{ is real} &\Leftrightarrow x^2-3x+2\ge 0\\ &\Leftrightarrow (x-1)(x-2)\ge 0\\ &\Leftrightarrow x\le 1 \quad\text{or}\quad x\ge 2 \qquad (1.38)\end{aligned}$$

Again

$$\begin{aligned}\frac{1}{\sqrt{3+2x-x^2}}\text{ is defined} &\Leftrightarrow 3+2x-x^2>0\\ &\Leftrightarrow -(x+1)(x-3)>0\\ &\Leftrightarrow (x+1)(x-3)<0\\ &\Leftrightarrow -1<x<3 \qquad (1.39)\end{aligned}$$

From Eqs. (1.38) and (1.39), it follows that $f(x)$ is defined for $x\in(-1,1]\cup[2,3)$.

**Answer: (B)**

2. The graph in Fig. 1.24 represents a function whose domain is

(A) $(-\infty,-1)\cup(1,2)$ (B) $(-\infty,-1]\cup[1,2)$
(C) $(-\infty,-1)\cup[1,2]$ (D) $(-\infty,-1]\cup[1,2]$

**Solution:** The given graph is represented by the function

$$f(x)=\begin{cases}\dfrac{1}{x+1}+1 & \text{for } x<-1\\ x & \text{for } 1<x<2\end{cases}$$

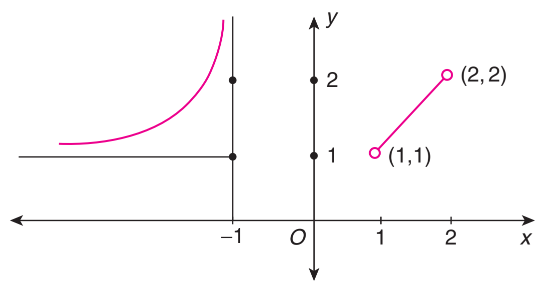



**FIGURE 1.24** Single correct choice type question 2.

**Answer: (A)**

3. If $f(x)=x^2+\lambda x+1$ for all real $x$ and $f$ is an even function, then the value of $\lambda$ is

(A) 1 (B) −1
(C) any real number (D) 0

**Solution:** Since $f$ is even, we have $f(1)=f(-1)$. Therefore $2+\lambda=2-\lambda$, which implies that $\lambda=0$.

**Answer: (D)**

4. Let $C_1$ be the graph of the curve represented by the equation $x^2-13x+4y=1$. If $C_2$ is the new curve obtained when $C_1$ is reflected in the origin, then the equation of $C_2$ is

(A) $x^2 - 13x - 4y = 1$ (B) $x^2 + 13x + 4y = 1$
(C) $x^2 - 13x - 4y = -1$ (D) $x^2 + 13x - 4y = 1$

**Solution:** Since $C_2$ is the reflection of $C_1$ in the origin, we have

$$\begin{aligned}(x, y) \in C_1 &\Leftrightarrow (-x, -y) \in C_2\\ &\Leftrightarrow (-x)^2 - 13(-x) + 4(-y) = 1\\ &\Leftrightarrow x^2 + 13x - 4y = 1\end{aligned}$$

**Answer: (D)**

**5.** Let $f(x) = x - 3$ and

$$g(x) = \begin{cases} \dfrac{x^2 - 9}{x+3} & \text{if } x \neq -3 \\ k & \text{if } x = -3 \end{cases}$$

If $f(x) = g(x)$ for all real $x$, then the value of $k$ is

(A) 0 (B) 3
(C) $-6$ (D) no real value exists

**Solution:** Clearly

$$g(x) = \frac{x^2 - 9}{x+3} = x - 3 = f(x)$$

for $x \neq -3$ and $f(-3) = -6$. Hence $k = -6$.

**Answer: (C)**

**6.** The domain of the function $f(x) = \sqrt{|x-1| - 1}$ is

(A) $(-\infty, 1] \cup [2, \infty)$ (B) $(-\infty, 0] \cup [2, \infty)$
(C) $(-\infty, 0] \cup [1, \infty)$ (D) $[0, 1]$

**Solution:** We have

$$\begin{aligned}|x-1| \geq 1 &\Leftrightarrow x - 1 \leq -1 \quad \text{or} \quad x - 1 \geq 1\\ &\Leftrightarrow x \leq 0 \quad \text{or} \quad x \geq 2\end{aligned}$$

**Answer: (B)**

**7.** If $[\cdot]$ denotes the greatest integer function, then the domain of the function

$$f(x) = \frac{1}{\sqrt{|[|x| - 1]| - 5}}$$

is

(A) $(-\infty, 7]$ (B) $(-\infty, -7] \cup [7, \infty)$
(C) $(-7, 7)$ (D) $(-\infty, \infty)$

**Solution:** We have that

$$\begin{aligned}f(x) \text{ is defined } &\Leftrightarrow |[|x| - 1]| - 5 > 0\\ &\Leftrightarrow [|x| - 1] < -5 \quad \text{or} \quad [|x| - 1] > 5\\ &\Leftrightarrow |x| - 1 < -5 \quad \text{or} \quad |x| - 1 \geq 6\\ &\Leftrightarrow |x| < -4 \quad \text{or} \quad |x| \geq 7\\ &\Leftrightarrow x \leq -7 \quad \text{or} \quad x \geq 7\\ &\Leftrightarrow x \in (-\infty, -7] \cup [7, \infty)\end{aligned}$$

**Answer: (B)**

**8.** The domain of the function

$$\text{Cos}^{-1}\left(\frac{|x| - 3}{2}\right) + \frac{1}{\log_{10}(4 - x)}$$

is

(A) $[-5, -1] \cup [1, 3) \cup (3, 4)$ (B) $[-5, 5]$
(C) $[1, 3] \cup (3, 4)$ (D) $[1, 3] \cup [3, 4]$

**Solution:** $\text{Cos}^{-1}[(|x| - 3)/2]$ is defined for

$$\begin{aligned}&-1 \leq \frac{|x| - 3}{2} \leq 1\\ &\Rightarrow -2 \leq |x| - 3 \leq 2\\ &\Rightarrow 1 \leq |x| \leq 5\end{aligned}$$

Therefore

$$x \in [-5, -1] \cup [1, 5] \tag{1.40}$$

Again $1/\log_{10}(4 - x)$ is defined for $x < 4$ and $x \neq 3$. That is

$$x \in (-\infty, 3) \cup (3, 4) \tag{1.41}$$

Therefore, from Eqs. (1.40) and (1.41), the domain of the given function is

$$[-5, -1] \cup [1, 3) \cup (3, 4)$$

**Answer: (A)**

**9.** The domain of the function

$$e^{\text{Sin}^{-1}x} + \frac{1}{[x]} + \frac{1}{\sqrt{x+1}}$$

is

(A) $(-1, 0) \cup (0, 1]$ (B) $[-1, 1] - \{0\}$
(C) $(-1, 0) \cup \{1\}$ (D) $(0, 1]$, where $[x]$ is the integral part of $x$

**Solution:** We have

$$\text{Sin}^{-1}x \text{ is defined for } -1 \leq x \leq 1 \tag{1.42}$$

Now $1/[x]$ is defined when $[x] \neq 0$, that is

$$x \notin [0, 1) \tag{1.43}$$

Also $1/\sqrt{x+1}$ is defined only when $x + 1 > 0$, that is

$$x > -1 \tag{1.44}$$

From Eqs. (1.42)–(1.44), the domain of the given function is $(-1, 0) \cup \{1\}$.

**Answer: (C)**

**10.** The domain of the function

$$e^x + \mathrm{Sin}^{-1}\left(\frac{x}{2}-1\right)+\log(\sqrt{x-[x]})$$

where $[x]$ is the integral part of $x$ is

(A) $[0, 4]$ (B) $[0, 3] \cup \{1\}$

(C) $(0, 4)$ (D) $(0, 4) - \{1, 2, 3\}$

**Solution:**

1. $e^x$ is defined for all $x$ (real or complex). In this context $x$ is real.
2. $\sin^{-1}[(x/2) - 1]$ is defined for $-1 \le (x/2) - 1 \le 1$, that is, $0 \le x \le 4$.
3. $x \ge [x] \Rightarrow \sqrt{x-[x]}$ is defined for all real $x$. Therefore, $\log(\sqrt{x-[x]})$ is defined when $x$ is not an integer, because $x = [x]$ when $x$ is an integer.

Therefore from the above, the domain of the given function is $(0, 4) - \{1, 2, 3\}$.

**Answer: (D)**

**11.** The domain of the function

$$f(x)=\sqrt{\log_{10}\frac{(3x-x^2)}{2}}$$

is

(A) $(0, 3)$ (B) $(0, 1) \cup [2, \infty)$

(C) $(1, 2)$ (D) $[1, 2]$

**Solution:** $\log_{10}[(3x - x^2)/2]$ is defined only when

$$\frac{3x-x^2}{2}>0$$

$$\Rightarrow x^2 - 3x < 0$$

$$\Rightarrow 0 < x < 3 \tag{1.45}$$

Now $\sqrt{\log_{10}[(3x-x^2)/2]}$ is defined when

$$\frac{3x-x^2}{2}\ge 1$$

$$\Rightarrow x^2 - 3x + 2 \le 0$$

$$\Rightarrow (x-1)(x-2) \le 0$$

$$\Rightarrow 1 \le x \le 2 \tag{1.46}$$

From Eqs. (1.45) and (1.46), the domain of $f(x)$ is $[1, 2]$.

**Answer: (D)**

**12.** Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x$, $y$ belonging to $\mathbb{R}$. If $m$ and $n$ are integers, then $f(m/n)$ is equal to

(A) $\dfrac{f(m)}{f(n)}$ (B) $\dfrac{m}{n}$

(C) $\left(\dfrac{m}{n}\right)f(1)$ (D) $f(m) - f(n)$

**Solution:** Clearly $f(0) = f(0 + 0) = f(0) + f(0)$ implies $f(0) = 0$. Also

$$0 = f(0) = f(x - x) = f(x + (-x)) = f(x) + f(-x)$$

Therefore

$$f(-x) = -f(x) \tag{1.47}$$

**Case I:** Suppose $x$ is a positive integer. Now

$$f(2) = f(1 + 1) = f(1) + f(1) = 2f(1)$$

$$f(3) = f(2 + 1) = f(2) + f(1) = 2f(1) + f(1) = 3f(1)$$

Therefore, by induction,

$$f(x) = xf(1) \tag{1.48}$$

**Case II:** Suppose $x$ is a negative integer, say $x = -y$ where $y$ is a positive integer. Now from Eqs. (1.47) and (1.48) we have

$$f(x) = f(-y) = -f(y) = -yf(1) = xf(1)$$

**Case III:** Suppose $m$ and $n$ are integers and $x = m/n$ and $n$ is positive. Now

$$\begin{aligned} mf(1) &= f(m) \text{ [By cases (1) and (2)]} \\ &= f(nx) \\ &= f(x + x + \cdots + n \text{ times}) \\ &= f(x) + f(x) + \cdots \text{upto } n \text{ times} \\ &= n\,f(x) \end{aligned}$$

Therefore

$$f(x)=\left(\frac{m}{n}\right)f(1)$$

Hence, $f(x) = xf(1)$ for all rational numbers $x$.

**Answer: (C)**

***Note:*** In the above problem, if $f$ is also continuous, then $f(x) = xf(1)$ for all real $x$ which we discuss later.

**13.** If $[x]$ denotes the integer part of $x$, then the domain of the function $f(x)=\dfrac{e^{-x}}{1+[x]}$ is

(A) $\mathbb{R} - [-1, 0)$ (B) $\mathbb{R}$

(C) $\mathbb{R} - [0, 1]$ (D) $(1, \infty)$

**Solution:** $f(x)$ is defined when $1 + [x] \ne 0$.

$$\begin{aligned} 1 + [x] = 0 &\Leftrightarrow [x] = -1 \\ &\Leftrightarrow -1 \le x < 0 \\ &\Leftrightarrow x \in [-1, 0) \end{aligned}$$

**Answer: (A)**

**14.** The domain of the function

$$f(x)=\frac{x|x-3|}{(x^2-x-6)|x|},\ x\in\mathbb{R}$$

is

(A) $\mathbb{R}$ (B) $\mathbb{R}-\{3\}$

(C) $\mathbb{R}-\{2,3\}$ (D) $\mathbb{R}-\{-2,0,3\}$

**Solution:** The given function can be written as

$$f(x)=\frac{x|x-3|}{(x-3)(x+2)|x|}$$

Therefore, $f$ is defined for $x\neq -2, 0, 3$.

**Answer: (D)**

**15.** Domain of the function

$$f(x)=2\,\mathrm{Sin}^{-1}\sqrt{1-x}+\mathrm{Sin}^{-1}[2\sqrt{x(1-x)}]$$

is

(A) $0\le x\le 1$ (B) $0\le x\le 1/2$

(C) $\frac{1}{\sqrt{2}}\le x\le\frac{1}{2}$ (D) $\frac{1}{2}\le x\le 1$

**Solution:** $1-x\ge 0$ and $x(1-x)\ge 0$ when $0\le x\le 1$. In this case, $0\le 2\sqrt{(1-x)x}\le 1$.

**Answer: (A)**

**16.** Let $f:\mathbb{R}\to\mathbb{R}-\{3\}$ be a function such that for some $p>0$,

$$f(x+p)=\frac{f(x)-5}{f(x)-3}$$

for all $x\in\mathbb{R}$. Then, period of $f$ is

(A) $2p$ (B) $3p$

(C) $4p$ (D) $5p$

**Solution:** 3 does not belong to the range of $f$ implies 2 also cannot belong to range of $f$ because, if $f(x)=2$ for some $x\in\mathbb{R}$. Then

$$f(x+p)=\frac{2-5}{2-3}=3$$

which is not in the range of $f$. Hence 2 and 3 are not in the range of $f$. If $f(x+2p)=f(x)$, this implies

$$\begin{aligned}f(x)&=f(x+p+p)\\&=\frac{f(x+p)-5}{f(x+p)-3}\\&=\frac{\dfrac{f(x)-5}{f(x)-3}-5}{\dfrac{f(x)-5}{f(x)-3}-3}\\&=\frac{-4f(x)+10}{-2f(x)+4}=\frac{2f(x)-5}{f(x)-2}\end{aligned}$$

so that $[f(x)-2]^2=-1$ which is absurd. Therefore, $2p$ is not a period. Again

$$\begin{aligned}f(x+3p)&=\frac{2f(x+p)-5}{f(x+p)-2}\\&=\frac{3f(x)-5}{f(x)-1}\neq f(x)\end{aligned}$$

Now

$$\begin{aligned}f(x+4p)&=f(x+3p+p)\\&=\frac{f(x+3p)-5}{f(x+3p)-3}\\&=\frac{\dfrac{3f(x)-5}{f(x)-1}-5}{\dfrac{3f(x)-5}{f(x)-1}-3}\\&=\frac{-2f(x)}{-2}=f(x)\end{aligned}$$

Therefore $4p$ is a period.

**Answer: (C)**

**17.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that

$$f(x+y)+f(x-y)=2f(x)\,f(y)$$

for all $x, y\in\mathbb{R}$ and $f(\alpha)=-1$ for some $\alpha\in\mathbb{R}$. Then period of $f$ is

(A) $2\alpha$ (B) $3\alpha$

(C) $5\alpha$ (D) $7\alpha$

**Solution:** Given that

$$f(x+y)+f(x-y)=2f(x)f(y)\ \ \forall x,y\in\mathbb{R}$$

Substituting $x=0$ and $y=0$, we have

$$f(0)=(f(0))^2\Rightarrow f(0)=0\quad\text{or}\quad 1$$

If $f(0)=0$, then

$$f(x+0)+f(x-0)=2f(x)f(0)$$

and hence $f(x)=0$ for all $x\in\mathbb{R}$ which contradicts the fact that $f(\alpha)=-1$. Therefore

$$f(0)=1$$

Now, replacing $x$ with $x+2\alpha$ and $y$ with $x-2\alpha$ in the given relation, we have

$$f(2x)+f(4\alpha)=2f(x+2\alpha)f(x-2\alpha)\tag{1.49}$$

Also in the given relation, if we put $y=x$, then we have

$$f(2x)+f(0)=2f(x)\,f(x)$$

Therefore

$$f(2x) = 2[f(x)]^2 - 1 \qquad (1.50)$$

In Eq. (1.50), if we replace $x$ with $2\alpha$, then

$$f(4\alpha) = 2[f(2\alpha)]^2 - 1 \qquad (1.51)$$

But

$$\begin{aligned} x = y = \alpha &\Rightarrow f(2\alpha) + f(0) = 2[f(\alpha)]^2 \\ &\Rightarrow f(2\alpha) = 2[f(\alpha)]^2 - 1 \\ &= 2(1) - 1 \quad [\because f(\alpha) = -1] \\ &= 1 \qquad (1.52) \end{aligned}$$

From Eqs. (1.51) and (1.52) we have

$$f(4\alpha) = 1 \qquad (1.53)$$

From Eqs. (1.49), (1.50) and (1.53), we get that

$$f(x+2\alpha)\, f(x-2\alpha) = (f(x))^2 \qquad (1.54)$$

Similarly if we put $y = 2\alpha$ in the given relation, we have

$$f(x+2\alpha) + f(x-2\alpha) = 2f(x) \qquad (1.55)$$

From Eqs. (1.54) and (1.55), we have

$$f(x-2\alpha) = f(x+2\alpha) = f(x)$$

Therefore $2\alpha$ is a period of $f(x)$.

**Answer: (A)**

**QUICK LOOK**

In Question 17, $f(x) = \cos x$ satisfies the conditions of $f(x)$ with $\alpha = \pi$.

**18.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the relation

$$f(x+1) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$$

for all $x \in \mathbb{R}$. Then period of $f(x)$ is

(A) 2 (B) 3

(C) $\frac{3}{2}$ (D) 5

**Solution:** Observe that $f(x) \geq 1/2$ for all $x \in \mathbb{R}$. Now

$$\begin{aligned} f(x+2) &= f(x+1+1) \\ &= \frac{1}{2} + \sqrt{f(x+1) - (f(x+1))^2} \\ &= \frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{f(x) - (f(x))^2} - \frac{1}{4} - \sqrt{f(x) - (f(x))^2} - [f(x) - (f(x))^2]} \\ &= \frac{1}{2} + \sqrt{\frac{1}{4} - f(x) + [f(x)]^2} \\ &= \frac{1}{2} + \left|\frac{1}{2} - f(x)\right| \\ &= \frac{1}{2} - \frac{1}{2} + f(x) \qquad \left(\because f(x) \geq \frac{1}{2}\ \forall x\right) \\ &= f(x) \end{aligned}$$

Therefore, 2 is period of $f(x)$.

**Answer: (A)**

**Try it out** In Question 18, replace 1 with $a > 0$ and prove that $2a$ is period of $f$. (See Worked-Out Problem 24, page 63, Vol. 1.)

**19.** The period of the function $f(x) = \mathrm{Tan}^{-1}(\tan x)$ is

(A) $\pi$ (B) $\frac{\pi}{2}$

(C) $\frac{3\pi}{2}$ (D) $\frac{3\pi}{4}$

**Solution:** $f(x+\pi) = \mathrm{Tan}^{-1}(\tan(\pi + x)) = \mathrm{Tan}^{-1}(\tan x) = f(x)$

This implies $\pi$ is the least period of $f(x)$.

**Answer: (A)**

**20.** The period of

$$f(x) = 2\cos\left(\frac{x-\pi}{3}\right)$$

is

(A) $2\pi$ (B) $4\pi$

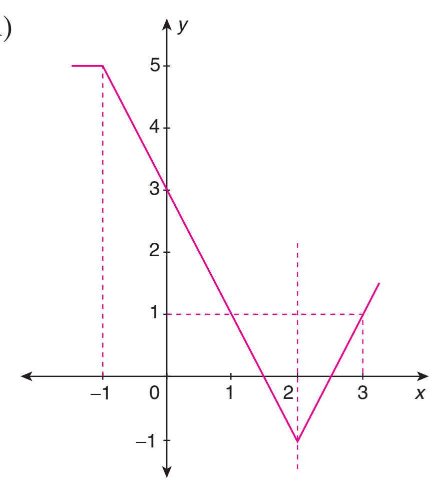
(C) $5\pi$ (D) $6\pi$

**Solution:** Period of $a\cos(ax+b)$ is $2\pi$ when $a$ is an integer and the period of $\cos(kx) = 2\pi/k$ when $k > 0$ integer. Therefore, period of $f(x)$ is $6\pi$.

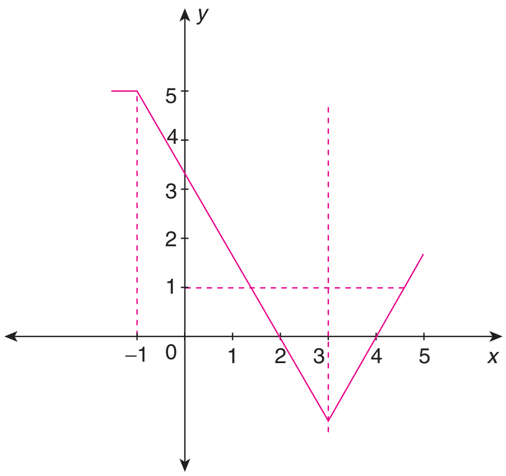
**Answer: (D)**

**21.** Which of the following graphs represents $y = 2|x-2| - |x+1| + x$?

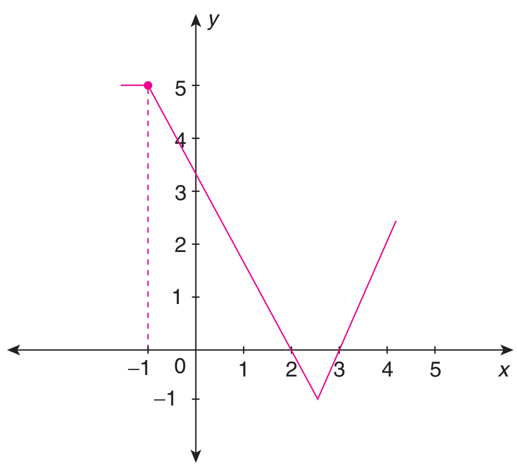
(A)



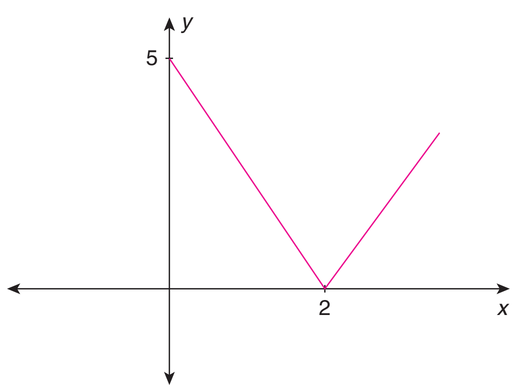
(B)




(C)




(D)




**Solution:** It can be seen that

$$f(x) = \begin{cases} 5 & \text{if } x \le -1 \\ 3 - 2x & \text{if } -1 \le x \le 2 \\ 2x - 5 & \text{if } x \ge 2 \end{cases}$$

**Answer: (A)**

**22.** The graph in Fig. 1.25 is represented by which of the following function?

(A) $f(x) = \sqrt{|x| - 2}$ (B) $f(x) = \sqrt{|x-1| - 1}$

(C) $f(x) = \sqrt{|x| - 1}$ (D) $f(x) = \sqrt{|x+1| - 1}$

**Solution:** The domain of $f(x) = \sqrt{|x-1| - 1}$ is $(-\infty, 0] \cup [2, \infty)$ because $|x-1| \ge 1$. Now

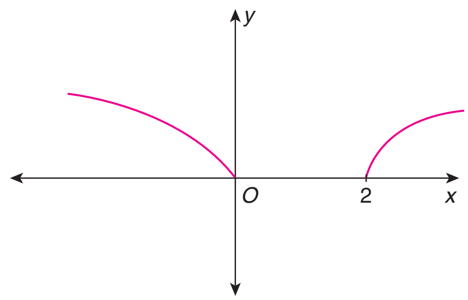



**FIGURE 1.25** Single correct choice type question 22.

$$x \le 0 \Rightarrow f(x) = \sqrt{|x-1| - 1} = \sqrt{-x}$$

So the graph must be upper half the parabola $y^2 = -x$ with vertex at origin. Again

$$x \ge 2 \Rightarrow y = \sqrt{x-2} \Rightarrow y^2 = x - 2$$

which represents parabola in the upper half of the $x$-axis with vertex at $(2, 0)$.

**Answer: (B)**

**23.** $\lim_{x \to 3+0} f(x)$ of the graph of the function given in Fig. 1.26 is

(A) 1 (B) 3

(C) 2 (D) 4

**Solution:** As per the graph, $\lim_{x \to 3+0} f(x) = 2$

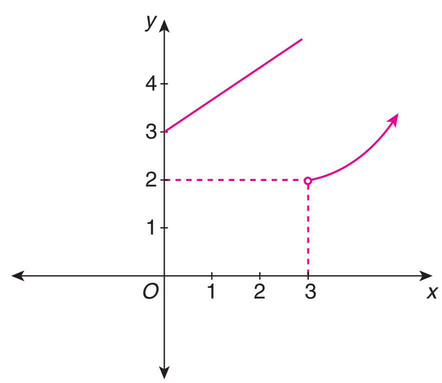



**FIGURE 1.26** Single correct choice type question 23.

**Answer: (C)**

**24.** If $f(x) = 3x^2 - x$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

(A) $3x - 1$ (B) $4x - 1$

(C) $8x - 1$ (D) $6x - 1$

**Solution:** We have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 - (x+h)] - (3x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - h}{h}$$

$$= \lim_{h \to 0} [6x + 3h - 1] = 6x - 1$$

**Answer: (D)**

**25.** Let

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2x - 1 & \text{if } 1 \le x \le 2x \end{cases}$$

Then $\lim_{x \to 1} f(x)$ is equal to

(A) 1 (B) 2
(C) ±1 (D) does not exist

**Solution:** We have

$$\lim_{x \to 1-0} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h) = 1$$

$$\lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [2(1+h) - 1]$$

$$= \lim_{h \to 0} (2 + 2h - 1) = 1$$

Therefore

$$\lim_{x \to 1-0} f(x) = \lim_{x \to 1+0} f(x) = 1$$

$$\Rightarrow \lim_{x \to 1} f(x) = 1$$

**Answer: (A)**

**26.** Let

$$f(x) = \begin{cases} x - 1 & \text{if } x \le 1 \\ 2x - 1 & \text{if } 1 < x < 2 \\ x + 1 & \text{if } x \ge 2 \end{cases}$$

Then $\lim_{x \to 1} f(x)$

(A) is equal to 1 (B) is equal to 0
(C) is equal to 3 (D) does not exist

**Solution:** We have

$$\lim_{x \to 1-0} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [(1-h) - 1] = 0$$

$$\lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [2(1+h) - 1] = 1$$

$$\lim_{x \to 1-0} f(x) \ne \lim_{x \to 1+0} f(x) \Rightarrow \lim_{x \to 1} f(x) \text{ does not exist}$$

**Answer: (D)**

**27.** Let

$$f(x) = \begin{cases} 7x - 3 & \text{if } x \ge 2 \\ 3x + 5 & \text{if } x < 2 \end{cases}$$

Then $\lim_{x \to 2} f(x)$ equals

(A) 11 (B) 14
(C) 12 (D) does not exist

**Solution:** We have

$$\lim_{x \to 2-0} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} (3(2+h) + 5) = 6 + 5 = 11$$

$$\lim_{x \to 2+0} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (7(2+h) - 3) = 14 - 3 = 11$$

Therefore

$$\lim_{x \to 2-0} f(x) = \lim_{x \to 2+0} f(x) = 11 \Rightarrow \lim_{x \to 2} f(x) = 11$$

**Answer: (A)**

**28.** $\lim_{x \to 2} \left( \frac{1}{x-2} - \frac{4}{x^2 - 4} \right)$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$
(C) $\infty$ (D) does not exist

**Solution:** This is the case where $\lim_{x \to a} (f+g)(x)$ exists even though $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist as we said in Theorem 1.6. Therefore

$$\lim_{x \to 2} \left( \frac{1}{x-2} - \frac{4}{x^2 - 4} \right) = \lim_{x \to 2} \left( \frac{x + 2 - 4}{x^2 - 4} \right) = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$$

**Answer: (B)**

**29.** $\lim_{x \to 1} \left( \frac{1}{1-x} - \frac{3}{1-x^3} \right)$ is equal to

(A) 0 (B) 1
(C) does not exist (D) –1

**Solution:** Observe that individually $\lim_{x \to 1} 1/(1-x)$ and $\lim_{x \to 1} 1/(1-x^3)$ do not exist. But

$$\lim_{x \to 1} \left( \frac{1}{1-x} - \frac{3}{1-x^3} \right) = \lim_{x \to 1} \left( \frac{1 + x + x^2 - 3}{1 - x^3} \right)$$

$$= \lim_{x \to 1} \frac{(x+2)(x-1)}{1 - x^3}$$

$$= \lim_{x \to 1} \left( \frac{-(x+2)}{1 + x + x^2} \right)$$

$$= \frac{-(1+2)}{1+1+1} = -1$$

**Answer: (D)**

**30.** Let $f(x) = x[x]$, where $[x]$ denotes integral part of $x$. If $a$ is not an integer, then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} =$$

(A) $a$ (B) $2[a]$

(C) $[a]$ (D) does not exist

**Solution:** We have

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{(a+h)[a+h]-a[a]}{h}$$

$$=\lim_{h\to 0}\frac{(a+h)[a]-a[a]}{h}$$ ($\because a$ is not an integer, $[a+h]=[a]$ for small values of $h$)

$$=\lim_{h\to 0}[a]=[a]$$

**Answer: (C)**

**31.** Let

$$f(x)=\begin{cases}\dfrac{\sin(x^2)}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

Then $\lim_{x\to 0} f(x)$ is

(A) 1 (B) 0
(C) $\infty$ (D) does not exist

**Solution:** We have

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x\left(\frac{\sin x^2}{x^2}\right)$$
$$=0\times 1 \quad \text{(By Theorem 1.27)}$$
$$=0$$

**Answer: (B)**

**32.** $\lim_{x\to 0} x^2\left(\sin\frac{1}{x}\right)$ equals

(A) 0 (B) 1
(C) does not exist (D) $\infty$

**Solution:** By Example 1.16 $\lim_{x\to 0}(x^2)=0$ and $\sin(1/x)$ is bounded. Hence by Corollary 1.4,

$$\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)=0$$

**Answer: (A)**

**33.** $\lim_{x\to 0}\frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1}$ is equal to

(A) 0 (B) $\frac{3}{2}$
(C) 1 (D) $\infty$

**Solution:** Let

$$f(x)=\frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1}$$
$$=\left(\frac{(1+x)^{1/2}-1}{1+x-1}\right)\div\frac{(1+x)^{1/3}-1}{1+x-1}$$
$$=\frac{y^{1/2}-1}{y-1}\div\frac{y^{1/3}-1}{y-1}$$

where $y=1+x$. Now $y\to 1$ as $x\to 0$. Therefore

$$\lim_{x\to 0} f(x)=\lim_{y\to 1}\left[\left(\frac{y^{1/2}-1}{y-1}\right)\div\left(\frac{y^{1/3}-1}{y-1}\right)\right]$$
$$=\lim_{y\to 1}\left(\frac{y^{1/2}-1}{y-1}\right)\div\lim_{y\to 1}\left(\frac{y^{1/3}-1}{y-1}\right)$$
$$=\frac{1}{2}\div\frac{1}{3} \quad \text{(By Theorem 1.26)}$$
$$=\frac{3}{2}$$

**Answer: (B)**

**34.** $\lim_{x\to 0}\frac{\sin(a+x)-\sin(a-x)}{x}$ equals

(A) $2\sin a$ (B) $\cos a$
(C) $-2\cos a$ (D) $2\cos a$

**Solution:** Let

$$f(x)=\frac{\sin(a+x)-\sin(a-x)}{x}$$
$$=\frac{2\cos a\sin x}{x}$$
$$=(2\cos a)\left(\frac{\sin x}{x}\right)$$

Therefore

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(2\cos a)\left(\frac{\sin x}{x}\right)$$
$$=(2\cos a)\lim_{x\to a}\left(\frac{\sin x}{x}\right) \quad \text{(By Corollary 1.2)}$$
$$=(2\cos a)\times 1 \quad \text{(By Theorem 1.27)}$$
$$=2\cos a$$

**Answer: (D)**

**35.** $\lim_{x\to a}\left(\frac{\tan x-\tan a}{x-a}\right)$ is

(A) $2\sin a$ (B) $2\cos a$
(C) $\operatorname{cosec}^2 a$ (D) $\sec^2 a$

**Solution:** We have

$$f(x) = \frac{\tan x - \tan a}{x - a}$$

$$= \frac{\sin x \cos a - \cos x \sin a}{(x - a)\cos x \cos a}$$

$$= \left(\frac{\sin(x - a)}{x - a}\right)\frac{1}{\cos x \cos a}$$

Therefore

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(\frac{\sin(x-a)}{x-a}\right) \times \lim_{x\to a}\left(\frac{1}{\cos x \cos a}\right)$$

$$= 1 \times \frac{1}{\cos a \cos a}$$

$$= \sec^2 a$$

**Answer: (D)**

**36.** $\lim_{x\to 0} \dfrac{\sin 2x}{x \cos x}$ is equal to

(A) 2 (B) 1
(C) 0 (D) does not exist

**Solution:** We have

$$\lim_{x\to 0} \frac{\sin 2x}{x\cos x} = \lim_{x\to 0} 2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{\cos x}\right)$$

$$= 2 \times 1 \times \frac{1}{1}$$

$$= 2$$

**Answer: (A)**

**37.** Let

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x + 2 & \text{if } 1 \le x \le 2 \\ 8 - 2x & \text{if } 2 < x \le 3 \\ 2x - 4 & \text{if } x > 3 \end{cases}$$

Which one of the following is *not* correct?

(A) $\lim_{x\to 1} f(x) = 1$ (B) $\lim_{x\to 2} f(x) = 4$
(C) $\lim_{x\to 3} f(x) = 2$ (D) $\lim_{x\to 4} f(x) = 4$

**Solution:** We have

$$\lim_{x\to 1-0} f(x) = \lim_{h\to 0}(1 - h^2) = 1$$

and

$$\lim_{x\to 1+0} f(x) = \lim_{h\to 0}[(1+h)+2] = 3$$

Therefore $\lim_{x\to 1} f(x) = 1$ is not true.

**Answer: (A)**

**38.** Suppose $a$ and $b$ are positive and $[y]$ denotes the integral part of $y$ and

$$f(x) = \frac{x}{a}\left[\frac{b}{x}\right]$$

Then

(A) $\lim_{x\to 0+0} f(x) = 0$ (B) $\lim_{x\to 0+0} f(x) = 0$ and $\lim_{x\to 0-0} f(x) = \dfrac{b}{a}$

(C) $\lim_{x\to 0-0} f(x) = 0$ and $\lim_{x\to 0+0} f(x) = \dfrac{b}{a}$ (D) $\lim_{x\to 0} f(x) = \dfrac{b}{a}$

**Solution:** We have

$$\frac{b}{x} - 1 < \left[\frac{b}{x}\right] \le \frac{b}{x}$$

$$\Rightarrow \frac{x}{a}\left(\frac{b}{x} - 1\right) < \frac{x}{a}\left[\frac{b}{x}\right] \le \frac{b}{a}$$

Therefore (by Theorem 1.15)

$$\lim_{x\to 0+0} \frac{x}{a}\left(\frac{b}{x} - 1\right) \le \lim_{x\to 0+0} \frac{x}{a}\left[\frac{b}{x}\right] \le \frac{b}{a}$$

$$\frac{b}{a} \le \lim_{x\to 0+0} \frac{x}{a}\left[\frac{b}{x}\right] \le \frac{b}{a}$$

Therefore by squeezing theorem

$$\lim_{x\to 0+0} \frac{x}{a}\left[\frac{b}{x}\right] = \frac{b}{a}$$

When $x < 0$, we have

$$\frac{x}{a}\left(\frac{b}{x}\right) \le \frac{x}{a}\left[\frac{b}{x}\right] < \frac{x}{a}\left(\frac{b}{x} - 1\right)$$

Again

$$\frac{b}{a} \le \lim_{x\to 0-0} f(x) \le \frac{b}{a}$$

Therefore

$$\lim_{x\to 0-0} f(x) = \frac{b}{a}$$

Hence $\lim_{x\to 0-0} f(x) = \lim_{x\to 0+0} f(x) = b/a$. Thus

$$\lim_{x\to 0} f(x) = \frac{b}{a}$$

**Answer: (D)**

**39.** Let $f(x) = \cos[x]$ where $[x]$ is the integral part of $x$. Then

(A) $\lim_{x\to 0} f(x) = 1$  (B) $\lim_{x\to 0+0} f(x) = 1$

(C) $\lim_{x\to 0-0} f(x) = 1$  (D) $\lim_{x\to 0} f(x) = 0$

**Solution:** If $\delta$ is a positive number less than 1, then

$$x \in [0, \delta] \Rightarrow [x] = 0$$

and hence $f(x) = 1\ \forall x \in [0, 1)$. Therefore

$$\lim_{x\to 0+0} f(x) = 1$$

Again

$$x \in (-\delta, 0) \Rightarrow [x] = -1$$

and hence $f(x) = \cos(-1) = \cos 1$ for all $x \in (-\delta, 0)$. Thus $\lim_{x\to 0-0} f(x) = \cos 1$.

**Answer: (B)**

**40.** Let $f(x) = \sin x/x$ for $x \neq 0$ and $[\cdot]$ denote the greatest integer function. Then $\lim_{x\to 0}[f(x)]$ is

(A) 0  (B) 1

(C) does not exist  (D) $-1$

**Solution:** Let $0 < \delta < 1$. Then according to the proof of Theorem 1.27

$$0 < \frac{\sin x}{x} < 1 \quad \forall x \in (-\delta, \delta)$$

Therefore

$$\left[\frac{\sin x}{x}\right] = 0 \quad \forall x \in (-\delta, \delta)$$

Hence $\lim_{x\to 0}[f(x)] = 0$.

**Answer: (A)**

**41.** $\lim_{x\to\infty} \sqrt{\dfrac{x + \sin x}{x - \cos x}}$ is equal to

(A) 0  (B) $\infty$

(C) 1  (D) does not exist

**Solution:** We have

$$\frac{x + \sin x}{x - \cos x} = \frac{1 + \dfrac{\sin x}{x}}{1 - \dfrac{\cos x}{x}}$$

Put $y = 1/x$ so that $y \to 0$ as $x \to \infty$. Now

$$y \sin\frac{1}{y} \to 0 \quad \text{and} \quad y\cos\frac{1}{y} \to 0$$

as $y \to 0$ (see Example 1.20). Therefore

$$\lim_{x\to\infty} f(x) = \sqrt{\frac{1}{1}} = 1$$

**Answer: (C)**

**42.** Let

$$f(x) = \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$$

for $x \neq \pi/6$. Then $\lim_{x\to \pi/6} f(x)$ is

(A) $\dfrac{1}{24}$  (B) $\dfrac{1}{36}$

(C) $\dfrac{1}{12}$  (D) $\dfrac{1}{48}$

**Solution:** Put $y = 6x - \pi$ so that $y \to 0$ as $x \to \pi/6$. Also $x = (\pi + y)/6$. Therefore

$$f(x) = \frac{2 - \sqrt{3}\cos\left(\dfrac{\pi}{6} + \dfrac{y}{6}\right) - \sin\left(\dfrac{\pi}{6} + \dfrac{y}{6}\right)}{y^2}$$

$$= \frac{2 - \sqrt{3}\left[\dfrac{\sqrt{3}}{2}\cos\dfrac{y}{6} - \dfrac{1}{2}\sin\dfrac{y}{6}\right] - \left[\dfrac{1}{2}\cos\dfrac{y}{6} + \dfrac{\sqrt{3}}{2}\sin\dfrac{y}{6}\right]}{y^2}$$

$$= \frac{2 - 2\cos(y/6)}{y^2}$$

$$= \frac{4\sin^2(y/12)}{y^2}$$

$$= 4\left[\frac{\sin(y/12)}{y/12}\right]^2 \times \frac{1}{144}$$

$$= \frac{1}{36}\left(\frac{\sin\theta}{\theta}\right)^2$$

where $\theta = y/12 \to 0$ as $y \to 0$. Therefore

$$\lim_{x\to 0} f(x) = \frac{1}{36}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2$$

$$= \frac{1}{36} \times 1^2 = \frac{1}{36}$$

**Answer: (B)**

**43.** $\lim_{x\to 0} \dfrac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$ is equal to

(A) $\dfrac{1}{2\sqrt{2}}$  (B) $\dfrac{1}{4\sqrt{2}}$

(C) $\dfrac{1}{8\sqrt{2}}$ (D) $\dfrac{1}{12\sqrt{2}}$

**Solution:** We have

$$\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}=\frac{2-(1+\cos x)}{\sin^2 x}\times\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$
$$=\left(\frac{2\sin^2\frac{x}{2}}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}\right)\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$
$$=\frac{1}{2\cos^2\frac{x}{2}}\times\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

Therefore

$$\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}=\frac{1}{2}\times\frac{1}{\sqrt{2}+\sqrt{1+1}}=\frac{1}{4\sqrt{2}}$$

**Answer: (B)**

**44.** $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}$ is equal to

(A) 1 (B) $\dfrac{\pi}{2}$

(C) $-\pi$ (D) $\pi$

**Solution:** We have

$$\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}=\lim_{x\to 0}\frac{\sin(\pi(1-\sin^2 x))}{x^2}$$
$$=\lim_{x\to 0}\frac{\sin(\pi-\pi\sin^2 x)}{x^2}$$
$$=\lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{x^2}$$
$$=\lim_{x\to 0}\pi\cdot\frac{\sin(\pi\sin^2 x)}{(\pi\sin^2 x)}\cdot\frac{\sin^2 x}{x^2}$$
$$=\pi\lim_{x\to 0}\left(\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\right)\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$
$$=\pi\times 1\times 1=\pi$$

**Answer: (D)**

**45.** Let

$$f(x)=\begin{cases}\dfrac{\sin[x]}{[x]} & \text{if } [x]\neq 0\\ 0 & \text{if } [x]=0\end{cases}$$

where $[x]$ is the integral part of $x$. Then $\lim\limits_{x\to 0}f(x)$ equals

(A) 1 (B) 0

(C) $-1$ (D) does not exist

**Solution:** Observe that $[x]=0\Leftrightarrow 0<x<1$. Therefore if $0<\delta<1$, then

$$f(x)=\begin{cases}0 & \text{if } 0<x<\delta\\ \dfrac{\sin(-1)}{(-1)}=\sin 1 & \text{if } -\delta<x<0\end{cases}$$

So

$$\lim_{x\to 0+0}f(x)=0\quad\text{and}\quad\lim_{x\to 0-0}f(x)=\sin 1$$

Hence $\lim\limits_{x\to 0}f(x)$ does not exist.

**Answer: (D)**

**46.** $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(A) exists and equals $\sqrt{2}$

(B) exists and is equal to $-\sqrt{2}$

(C) does not exist because $x-1\to 0$ as $x\to 1$

(D) does not exist because the left limit at $x=1$ is not equal to the right limit at $x=1$

**Solution:** We have

$$\frac{\sqrt{1-\cos 2(x-1)}}{x-1}=\sqrt{2}\frac{|\sin(x-1)|}{x-1}$$

Therefore

$$\lim_{x\to 1-0}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}=\lim_{h\to 0}\frac{\sqrt{2}|\sin(1-h-1)|}{(1-h)-1}$$
$$=\lim_{h\to 0}\sqrt{2}\left(\frac{-\sin h}{h}\right)=-\sqrt{2}$$
$$\lim_{x\to 1+0}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}=\sqrt{2}\lim_{h\to 0}\frac{|\sin(1+h-1)|}{1+h-1}$$
$$=\sqrt{2}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$
$$=\sqrt{2}$$

Left limit $\neq$ Right limit

**Answer: (D)**

**47.** $\lim\limits_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ equals

(A) $\dfrac{1}{2}$ (B) 2

(C) $\dfrac{-1}{2}$ (D) $-2$

**Solution:** Let

$$f(x)=\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$$

$$=\frac{2x\tan x\left(\dfrac{1}{1-\tan^2 x}-1\right)}{4\sin^4 x}$$

$$=\frac{x\tan^3 x}{2\sin^4 x}\times\frac{1}{1-\tan^2 x}$$

$$=\frac{x\sin^3 x}{2\cos^3 x\sin^4 x}\times\frac{\cos^2 x}{\cos 2x}$$

$$=\frac{1}{2}\left(\frac{x}{\sin x}\right)\frac{1}{\cos x\cos 2x}$$

So

$$\lim_{x\to 0} f(x)=\frac{1}{2}(1)\times\frac{1}{1}=\frac{1}{2}$$

**Answer: (A)**

**48.** If $[t]$ denotes the integral part of $t$, then $\lim_{x\to 1}[x\sin \pi x]$

(A) equals 1 (B) equals –1
(C) equals 0 (D) does not exist

**Solution:** Use the concept of the integral part of a real number.

**Answer: (D)**

**49.** Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function satisfying the relation $f(x\cdot f(y))=f(xy)+x$ for all $x, y\in\mathbb{R}^+$. Then

$$\lim_{x\to 0}\left(\frac{f(x)^{1/3}-1}{f(x)^{1/2}-1}\right)=$$

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{2}{3}$ (D) $\frac{3}{2}$

**Solution:** Given relation is

$$f(x\cdot f(y))=f(xy)+x \tag{1.56}$$

Interchanging $x$ and $y$ in Eq. (1.56), we have

$$f(y\cdot f(x))=f(yx)+y \tag{1.57}$$

Again replacing $x$ with $f(x)$ in Eq. (1.56) we get

$$f(f(x)\cdot f(y))=f(y\cdot f(x))+f(x) \tag{1.58}$$

Therefore, Eqs. (1.56)–(1.58) imply

$$f(f(x)\cdot f(y))=f(xy)+y+f(x) \tag{1.59}$$

Again interchanging $x$ and $y$ in Eq. (1.59), we have

$$f(f(y)\cdot f(x))=f(yx)+x+f(y) \tag{1.60}$$

Equations (1.59) and (1.60) imply

$$f(xy)+y+f(x)=f(yx)+x+f(y) \tag{1.61}$$

$$\Rightarrow f(x)-x=f(y)-y\quad \forall x, y\in\mathbb{R}^+$$

Suppose

$$f(x)-x=f(y)-y=\lambda$$

Substituting $f(x)=\lambda+x$ in Eq. (1.56), we have

$$x\cdot f(y)+\lambda=(xy+\lambda)+x$$

$$\Rightarrow x\cdot f(y)=xy+x$$

Therefore

$$x(y+\lambda)=xy+x\quad [\because f(y)=\lambda+y]$$

$$\Rightarrow \lambda x=x$$

$$\Rightarrow \lambda=1\quad (\because x>0)$$

So

$$f(x)=x+\lambda=x+1$$

Hence

$$\lim_{x\to 0}\frac{(f(x))^{1/3}-1}{(f(x))^{1/2}-1}=\lim_{x\to 0}\frac{(1+x)^{1/3}-1}{(1+x)^{1/2}-1}$$

$$=\lim_{x\to 0}\left(\frac{(1+x)^{1/3}-1}{1+x-1}\right)\cdot\left(\frac{1+x-1}{(1+x)^{1/2}-1}\right)$$

$$=\frac{1/3}{1/2}=\frac{2}{3}$$

**Answer: (C)**

**50.** Let

$$f(x)=\begin{cases}\sin x & \text{if } x\neq n\pi, n\neq 0,\pm 1,\pm 2,\dots\\ 2 & \text{otherwise}\end{cases}$$

and

$$g(x)=\begin{cases}x^2+1, & x\neq 0, 2\\ 4, & x=0\\ 5, & x=2\end{cases}$$

Then $\lim_{x\to 0} g(f(x))$ is

(A) 1 (B) 6
(C) 5 (D) 7

**Solution:** If $0<\delta<1$, then $f(x)=\sin x$ for $x\in(-\delta,\delta)$, $x\neq 0$. Therefore

$$\lim_{x\to 0} g(f(x))=\lim_{x\to 0} g(\sin x)=\lim_{x\to 0}(\sin^2 x+1)=1$$

**Answer: (A)**

**51.** $\lim_{x\to1}(1-x)\tan\frac{\pi x}{2} =$

(A) $\pi$ (B) $\frac{\pi}{2}$

(C) $\frac{2}{\pi}$ (D) $\frac{1}{\pi}$

**Solution:** Put $1-x=\theta$ so that $\theta\to0$ as $x\to1$. Therefore

$$\lim_{x\to1}(1-x)\tan\frac{\pi x}{2} = \lim_{\theta\to0}\theta\tan\frac{\pi}{2}(1-\theta)$$
$$= \lim_{\theta\to0}\theta\cot\left(\frac{\pi\theta}{2}\right)$$
$$= \lim_{\theta\to0}\frac{\theta\cos\left(\frac{\pi}{2}\theta\right)}{\sin\left(\frac{\pi}{2}\theta\right)}$$
$$= \lim_{\theta\to0}\frac{2}{\pi}\cdot\left(\frac{\frac{\pi}{2}\theta}{\sin\left(\frac{\pi}{2}\theta\right)}\right)\cos\left(\frac{\pi}{2}\theta\right)$$
$$= \frac{2}{\pi}\times1\times1 = \frac{2}{\pi}$$

**Answer: (C)**

**52.** Let

$$f(x) = \begin{cases} |x-3| & \text{if } x\ge1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & \text{if } x<1 \end{cases}$$

Then $\lim_{x\to1}f(x)$ is equal to

(A) 1 (B) 0

(C) –2 (D) 2

**Solution:** We have

$$\lim_{x\to1-0}f(x) = \lim_{h\to0}\left[\frac{(1-h)^2}{4}-\frac{3}{2}(1-h)+\frac{13}{4}\right] = \frac{1}{4}-\frac{3}{2}+\frac{13}{4} = 2$$
$$\lim_{x\to1+0}f(x) = \lim_{h\to0}|1+h-3| = 2$$

Therefore

$$\lim_{x\to1}f(x) = 2$$

**Answer: (D)**

**53.** Let $f(x) = \text{Min}\{x, x^2\}$. Then

$$\lim_{h\to0}\frac{f(1+h)-f(1)}{h} =$$

(A) 0 (B) 1

(C) 2 (D) does not exist

**Solution:** We can see that (Fig. 1.27)

$$f(x) = \begin{cases} x & \text{if } x\le0 \\ x^2 & \text{if } 0<x\le1 \\ x & \text{if } x>1 \end{cases}$$

Now

$$\lim_{h\to0-0}\frac{f(1+h)-f(1)}{-h} = \lim_{h\to0}\frac{(1-h)^2-1}{-h}$$
$$= \lim_{h\to0}\frac{-2h+h^2}{-h} = 2$$

Now

$$\lim_{h\to0+0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{(1+0+h)-1}{0+h} = \lim_{h\to0}\left(\frac{h}{h}\right) = 1$$

Therefore the required limit does not exist.

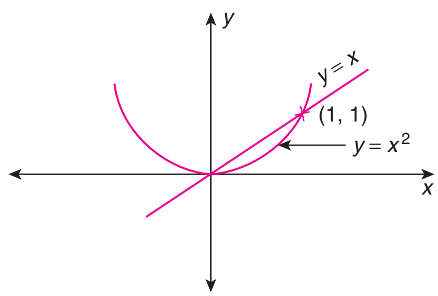



**FIGURE 1.27** Single correct choice type question 53.

**Answer: (D)**

**54.** $\lim_{x\to0}\frac{\tan x-\sin x}{x^3} =$

(A) 1 (B) $\frac{1}{2}$

(C) 0 (D) $\infty$

**Solution:** We have

$$\lim_{x\to0}\frac{\tan x-\sin x}{x^3} = \lim_{x\to0}\frac{\sin x(1-\cos x)}{x^3\cos x}$$
$$= \lim_{x\to0}\frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\left(2\sin^2\frac{x}{2}\right)}{x^3\cos x}$$
$$= \lim_{x\to0}4\left(\frac{\sin\frac{x}{2}}{\left(\frac{x}{2}\right)}\right)^3\cdot\left(\frac{\cos\frac{x}{2}}{8\cos x}\right)$$
$$= \frac{1}{2}\times1^3\times1 = \frac{1}{2}$$

**Answer: (B)**

**55.** $\lim_{x\to -1} \frac{x+1}{(17+x)^{1/4}-2} =$

(A) 8 (B) 16
(C) 32 (D) 64

**Solution:** Let

$$f(x) = \frac{x+1}{(17+x)^{1/4}-2}$$

Put $(17+x)^{1/4} = y$ so that $y \to 2$ as $x \to -1$. Therefore

$$f(x) = \frac{y^4-16}{y-2} = \frac{y^4-2^4}{y-2}$$

So

$$\lim_{x\to -1} f(x) = \lim_{y\to 2}\left(\frac{y^4-2^4}{y-2}\right) = 4\times 2^3 = 32$$

**Answer: (C)**

**56.** $\lim_{x\to -1} \frac{1+\sqrt[n]{x}}{1+\sqrt[m]{x}}$ ($m$, $n$ are odd positive integers) is

(A) $\frac{m}{n}$ (B) $\frac{n}{m}$
(C) 1 if $n > m$ (D) $\infty$ if $n < m$

**Solution:** Put $x = z^{mn}$ so that $x \to -1 \Rightarrow z \to -1$ and $x^{1/n} = z^m, x^{1/m} = z^n$. Therefore

$$\lim_{x\to -1} \frac{1+\sqrt[n]{x}}{1+\sqrt[m]{x}} = \lim_{z-1} \frac{z^m+1}{z^n+1}$$

$$= \lim_{z\to -1}\left(\frac{z^m-(-1)^m}{z+1}\cdot\frac{z+1}{z^n-(-1)^n}\right)$$

($\because m, n$ are odd)

$$= m(-1)^{m-1}\cdot\frac{1}{n(-1)^{n-1}}$$

$$= \frac{m}{n} \quad (\because m, n \text{ are odd})$$

**Answer: (A)**

**57.** $\lim_{x\to \frac{\pi}{6}}\left(\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}\right) =$

(A) 1 (B) $-1$
(C) $-3$ (D) 3

**Solution:** Given limit is

$$\lim_{x\to \frac{\pi}{6}} \frac{(2\sin x-1)(\sin x+1)}{(2\sin x-1)(\sin x-1)} = \lim_{x\to \frac{\pi}{6}}\left(\frac{\sin x+1}{\sin x-1}\right)$$

$$= \frac{(1/2)+1}{(1/2)-1} = -3$$

**Answer: (C)**

**58.** $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx} =$

(A) $\frac{m^2}{n^2}$ (B) $\frac{n^2}{m^2}$
(C) $\infty$ (D) does not exist

**Solution:** We have

$$\lim_{x\to 0}\left(\frac{1-\cos mx}{1-\cos nx}\right) = \lim_{x\to 0} \frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)}$$

$$= \lim_{x\to 0}\left(\frac{\sin\left(\frac{mx}{2}\right)}{mx/2}\cdot\frac{mx}{2}\right)^2\left(\frac{nx/2}{\sin\left(\frac{nx}{2}\right)}\cdot\frac{1}{nx/2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right)^2\cdot\lim_{x\to 0}\left(\frac{\frac{nx}{2}}{\sin\frac{nx}{2}}\right)^2\cdot\frac{m^2}{n^2}$$

$$= 1^2\times 1^2\times\frac{m^2}{n^2} = \frac{m^2}{n^2}$$

**Answer: (A)**

**59.** $\lim_{x\to 0}\left(\frac{\mathrm{Sin}^{-1} x}{x}\right) =$

(A) 0 (B) 1
(C) $\infty$ (D) does not exist

**Solution:** Put $\theta = \mathrm{Sin}^{-1} x$ so that $\theta \to 0$ as $x \to 0$ and $x = \sin\theta$. Therefore

$$\lim_{x\to 0}\left(\frac{\mathrm{Sin}^{-1} x}{x}\right) = \lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1$$

**Answer: (B)**

**60.** $\lim_{x\to 1}\left(\frac{1+\cos\pi x}{\tan^2\pi x}\right) =$

(A) 0 (B) $-\frac{1}{2}$

(C) 1 (D) $\frac{1}{2}$

**Solution:** Let

$$f(x) = \frac{1+\cos\pi x}{\tan^2\pi x} = \frac{(1+\cos\pi x)\cos^2\pi x}{\sin^2\pi x}$$

$$= \frac{(1+\cos \pi x)\cos^2 \pi x}{1-\cos^2 \pi x} = \frac{\cos^2 \pi x}{1-\cos \pi x}$$

Therefore

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\cos^2 \pi x}{1-\cos \pi x}$$

$$= \frac{\cos^2(\pi)}{1-\cos \pi}$$

$$= \frac{(-1)^2}{1-(-1)} = \frac{1}{2}$$

**Answer: (D)**

**61.** $\lim_{x\to 1}\left(\frac{x^{-1/3}-1}{x^{-2/3}-1}\right) =$

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{6}$ (D) $-\frac{1}{6}$

**Solution:** We have

$$\frac{x^{-1/3}-1}{x^{-2/3}-1} = \frac{x^{-1/3}-1}{x-1}\cdot\frac{x-1}{x^{-2/3}-1}$$

Therefore

$$\lim_{x\to 1}\left(\frac{x^{-1/3}-1}{x^{-2/3}-1}\right) = \lim_{x\to 1}\left(\frac{x^{-1/3}-1}{x-1}\right)\cdot\lim_{x\to 1}\left(\frac{x-1}{x^{-2/3}-1}\right)$$

$$= \frac{-\frac{1}{3}(1)^{(-1/3)-1}}{-\frac{2}{3}(1)^{(-2/3)-1}} = \frac{1}{2}$$

**Answer: (A)**

**62.** $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x) =$

(A) 1 (B) 0

(C) $\infty$ (D) 1/2

**Solution:** We have

$$\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to \frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right)$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{1-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{4}-\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right)} = 0$$

**ALITER**

$$\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to \frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right)$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\sqrt{1-\sin^2 x}}$$

$$= \lim_{x\to \frac{\pi}{2}} \sqrt{\frac{1-\sin x}{1+\sin x}}$$

$$= \sqrt{\frac{1-1}{1+1}} = 0$$

**Answer: (B)**

**63.** If

$$f(x) = \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$$

then $\lim_{x\to 1} f(x)$ is equal to

(A) 0 (B) 10

(C) $\frac{1}{10}$ (D) $-\frac{1}{10}$

**Solution:** $f(x)$ is defined in a deleted neighbourhood 1. Also

$$f(x) = \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)} = \frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}$$

Therefore

$$\lim_{x\to 1} f(x) = \frac{2-3}{(2+3)(1+1)} = -\frac{1}{10}$$

**Answer: (D)**

**64.** Let

$$g(x) = \frac{\tan^3 x - \tan x}{\cos[x+(\pi/4)]}$$

Then $\lim_{x\to \frac{\pi}{4}} g(x)$ is

(A) $2\sqrt{2}$ (B) $-4\sqrt{2}$
(C) $-4$ (D) 8

**Solution:** $g(x)$ is defined in *a* neighbourhood of $\pi/4$ except at $\pi/4$. Now,

$$g(x)=\frac{\tan x(\tan x+1)(\tan x-1)}{(1/\sqrt{2})(\cos x-\sin x)}$$

$$=\frac{\sqrt{2}\tan x(\tan x+1)(\sin x-\cos x)}{\cos x(\cos x-\sin x)}$$

$$=\frac{-\sqrt{2}\tan x(\tan x+1)}{\cos x}$$

Therefore

$$\lim_{x\to\frac{\pi}{4}} g(x)=\frac{-\sqrt{2}(1)(1+1)}{1/\sqrt{2}}=-2(2)=-4$$

**Answer: (C)**

**65.** $\lim\limits_{x\to+\infty} x(\sqrt{x^2+1}-x)=$

(A) $\frac{1}{2}$ (B) $+\infty$
(C) 0 (D) 1

**Solution:** We have

$$\lim_{x\to+\infty} x(\sqrt{x^2+1}-x)=\lim_{x\to+\infty}\frac{x(x^2+1-x^2)}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}+1}$$

$$=\frac{1}{1+1}=\frac{1}{2}$$

**Answer: (A)**

**66.** $\lim\limits_{x\to-\infty} (\sqrt{2x^2-3}-5x)$ is

(A) $-\infty$ (B) $+\infty$
(C) 0 (D) 10

**Solution:** We have

$$\lim_{x\to-\infty} (\sqrt{2x^2-3}-5x)=\lim_{x\to-\infty}\left(\frac{2x^2-3-25x^2}{\sqrt{2x^2-3}+5x}\right)$$

$$=-\lim_{x\to-\infty}\left(\frac{23x^2+3}{\sqrt{2x^2-3}+5x}\right)$$

$$=-\lim_{x\to-\infty}\left(\frac{23x+\frac{3}{x}}{\sqrt{2-\frac{3}{x^2}}+5x}\right)$$

$$=\frac{-(-\infty)}{\sqrt{2}+5}=+\infty$$

**Answer: (B)**

**67.** $\lim\limits_{x\to+\infty}\dfrac{\sqrt{2x^2+3}}{4x+3}=$

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2\sqrt{2}}$
(C) $+\infty$ (D) does not exist

**Solution:** As $x$ is positive, we have

$$\lim_{x\to+\infty}\left(\frac{\sqrt{2x^2+3}}{4x+3}\right)=\lim_{x\to+\infty}\frac{\sqrt{2+\frac{3}{x^2}}}{4+\frac{3}{x}}=\frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}}$$

**Answer: (B)**

**68.** $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{(1-\sin x)^{2/3}}=$

(A) 0 (B) 1
(C) $\infty$ (D) $-1$

**Solution:** Let

$$f(x)=\frac{\cos x}{(1-\sin x)^{2/3}}$$

Put $\theta=(\pi/2)-x$ so that $\theta\to 0$ as $x\to\pi/2$. Now

$$f(x)=\frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\left(1-\sin\left(\frac{\pi}{2}-\theta\right)\right)^{2/3}}$$

$$=\frac{\sin\theta}{(1-\cos\theta)^{2/3}}$$

$$=\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2^{2/3}\cdot\sin^{4/3}\left(\frac{\theta}{2}\right)}$$

$$=\frac{2^{1/3}\cos\theta}{\sin^{1/3}\left(\frac{\theta}{2}\right)}\to\infty \quad \text{as } \theta\to 0$$

**Answer: (C)**

**69.** $\lim\limits_{x\to\infty}\dfrac{4x^5+9x+7}{3x^5+x^4+x^2+1}=$

(A) $\infty$ (B) 0
(C) 4/3 (D) 7

**Solution:** We have

$$\lim_{x\to\infty}\left(\frac{4x^5+9x+7}{3x^5+x^4+x^2+1}\right)=\lim_{x\to\infty}\left[\frac{4+\frac{9}{x^4}+\frac{7}{x^5}}{3+\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^5}}\right]=\frac{4}{3}$$

$$\left(\because \lim_{x\to\infty}\left(\frac{1}{x^n}\right)=0 \text{ when } n \text{ is a positive integer}\right)$$

**Answer: (C)**

**QUICK LOOK**

Let

$$P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n,\ a_n\neq 0$$

and $Q(x)=b_0+b_1x+b_2x^2+\cdots+b_mx^m,\ b_m\neq 0$

Then

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\begin{cases}\dfrac{a_n}{b_m} & \text{if } n=m\\ 0 & \text{if } n<m\end{cases}$$

When $n>m$, then

$$\lim_{x\to+\infty}\frac{P(x)}{Q(x)}=\begin{cases}\infty & \text{if } \dfrac{a_n}{b_m}>0\\ -\infty & \text{if } \dfrac{a_n}{b_m}<0\end{cases}$$

**70.** $\lim_{x\to+\infty}\dfrac{x-8}{3-x+10x^2}=$

(A) 0
(B) $+\infty$
(C) $-\infty$
(D) $-1$

**Solution:** We have

$$\lim_{x\to+\infty}\left(\frac{x-8}{3-x+10x^2}\right)=\lim_{x\to+\infty}\left(\frac{\frac{1}{x}-\frac{8}{x^2}}{\frac{3}{x^2}-\frac{1}{x}+10}\right)=\frac{0}{0-0+10}=0$$

**Answer: (A)**

**71.** $\lim_{x\to+\infty}\dfrac{\sqrt{9x^2+2}-x}{4x+11}=$

(A) 1/4
(B) 3
(C) 3/4
(D) 1/2

**Solution:**

$$\lim_{x\to+\infty}\frac{\sqrt{9x^2+2}-x}{4x+11}=\lim_{x\to+\infty}\frac{\sqrt{9+\frac{2}{x^2}}-1}{4+\frac{11}{x}}=\frac{\sqrt{9+0}-1}{4+0}=\frac{1}{2}$$

**Answer: (D)**

**72.** Let $f(x)=\sqrt{x}(\sqrt{x+4}-\sqrt{x})$. Then $\lim_{x\to+\infty}f(x)$

(A) exists and is equal to 2
(B) exists and is equal to 0
(C) exists and equals to $\frac{1}{2}$
(D) does not exist

**Solution:** $f(x)$ is defined for all $x\geq 0$. We have

$$f(x)=\frac{\sqrt{x}(x+4-x)}{\sqrt{x+4}+\sqrt{x}}=\frac{4\sqrt{x}}{\sqrt{x+4}+\sqrt{x}}=\frac{4}{\sqrt{1+\frac{4}{x}}+1}$$

Therefore

$$\lim_{x\to+\infty}f(x)=\frac{4}{\sqrt{1+0}+1}=\frac{4}{2}=2$$

**Answer: (A)**

**73.** Let $f(x)=x^3\{\sqrt{x^2+\sqrt{x^4+1}}-x\sqrt{2}\}$. Then $\lim_{x\to\infty}f(x)$ is equal to

(A) $\dfrac{1}{2\sqrt{2}}$
(B) $\dfrac{1}{4\sqrt{2}}$
(C) $\dfrac{3}{4\sqrt{2}}$
(D) does not exist

**Solution:** We have

$$\begin{aligned}f(x)&=\frac{x^3\{x^2+\sqrt{x^4+1}-2x^2\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}\\&=\frac{x^3\{\sqrt{x^4+1}-x^2\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}\\&=\frac{x^3(x^4+1-x^4)}{[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}][\sqrt{x^4+1}+x^2]}\\&=\frac{x^3}{[\sqrt{x^2\sqrt{x^4+1}}+x\sqrt{2}][\sqrt{x^4+1}+x^2]}\\&=\frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}\end{aligned}$$

$$= \frac{1}{(\sqrt{1+\sqrt{1}+\sqrt{2}})(\sqrt{1}+1)}$$

$$= \frac{1}{2\sqrt{2}(2)} = \frac{1}{4\sqrt{2}}$$

**Answer: (B)**

**74.** $\lim\limits_{x\to+\infty} \dfrac{20+2\sqrt{x}+3x^{1/3}}{2+\sqrt{4x-3}+(8x-4)^{1/3}} =$

(A) 1 (B) 0

(C) $\dfrac{3}{2}$ (D) 10

**Solution:** Let

$$f(x) = \frac{20+2\sqrt{x}+3x^{1/3}}{2+\sqrt{4x-3}+(8x-4)^{1/3}}$$

Dividing numerator and denominator by $\sqrt{x}$ we get

$$f(x) = \frac{\dfrac{20}{\sqrt{x}}+2+\dfrac{3}{\sqrt[6]{x}}}{\dfrac{2}{\sqrt{x}}+\sqrt{4-\dfrac{3}{\sqrt{x}}}+\left[\dfrac{8}{x}-\dfrac{4}{x\sqrt{x}}\right]^{1/3}}$$

Therefore

$$\lim_{x\to+\infty} f(x) = \frac{0+2+0}{0+\sqrt{4-0}+(0-0)^{1/3}} = \frac{2}{2} = 1$$

**Answer: (A)**

**75.** $\lim\limits_{x\to\infty}[x\sqrt{x^2+4}-\sqrt{x^4+16}] =$

(A) 4 (B) 8

(C) 2 (D) 16

**Solution:** We have

$$f(x) = x\sqrt{x^2+4}-\sqrt{x^4+16}$$

$$f(x) = \frac{x^2(x^2+4)-(x^4+16)}{x\sqrt{x^2+4}+\sqrt{x^4+16}}$$

$$= \frac{4x^2-16}{x\sqrt{x^2+4}+\sqrt{x^4+16}}$$

$$= \frac{4-\dfrac{16}{x^2}}{\sqrt{1+\dfrac{4}{x^2}}+\sqrt{1+\dfrac{16}{x^4}}}$$

$$= \frac{4-0}{\sqrt{1+0}+\sqrt{1+0}}$$

$$= \frac{4}{2} = 2$$

**Answer: (C)**

**Try it out** In place of 4 (= $2^2$) and 16 (= $2^4$) if there are $a^2$ and $a^4$, respectively, then the answer is $a^2/2$.

**76.** $\lim\limits_{x\to+\infty} \dfrac{\sqrt{3x^2-1}-\sqrt{x^2-1}}{2x+3} =$

(A) $\dfrac{\sqrt{3}}{2}$ (B) $\dfrac{\sqrt{3}-1}{2}$

(C) $\dfrac{\sqrt{3}+1}{2}$ (D) does not exist

**Solution:** We have

$$\lim_{x\to+\infty} \frac{\sqrt{3x^2-1}-\sqrt{x^2-1}}{2x+3} = \lim_{x\to+\infty} \frac{\sqrt{3+\dfrac{1}{x^2}}-\sqrt{1-\dfrac{1}{x^2}}}{2+\dfrac{3}{x}}$$

$$= \frac{\sqrt{3+0}-\sqrt{1-0}}{2+0} = \frac{\sqrt{3}-1}{2}$$

**Answer: (B)**

**Try it out** If $x < 0$, then $\sqrt{x^2} = -x$, and so the limit of the above function as $x \to -\infty$ is $(1-\sqrt{3})/2$. Hence,

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{x^2-1}}{2x+3}$$

is meaningless.

**77.** $\lim\limits_{x\to 0}\left[\dfrac{1}{x}\mathrm{Sin}^{-1}\left(\dfrac{2x}{1+x^2}\right)\right] =$

(A) $\dfrac{1}{2}$ (B) 1

(C) 2 (D) $-2$

**Solution:** In a neighbourhood of zero,

$$\mathrm{Sin}^{-1}\left(\frac{2x}{1+x^2}\right) = 2\mathrm{Tan}^{-1}x$$

Therefore

$$\lim_{x\to 0}\left(\frac{1}{x}\mathrm{Sin}^{-1}x\right) = \lim_{x\to 0}\left(\frac{2\mathrm{Tan}^{-1}x}{x}\right)$$

$$= 2\lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) \quad \text{where } \theta = \mathrm{Tan}^{-1}x$$

$$= 2\times 1 = 2$$

**Answer: (C)**

**78.** $\lim_{x\to+\infty}\left[2^{x-1}\tan\left(\frac{3}{2^x}\right)\right]=$

(A) 1 (B) $\frac{3}{2}$

(C) $\frac{2}{3}$ (D) $\infty$

**Solution:** We have

$$\lim_{x\to+\infty}\left(2^{x-1}\tan\left(\frac{3}{2^x}\right)\right)=\lim_{x\to+\infty}\left(\frac{1}{2}\frac{\tan\left(\frac{3}{2^x}\right)}{3/2^x}\cdot 3\right)$$

$$=\frac{3}{2}\lim_{\theta\to0}\left(\frac{\tan\theta}{\theta}\right)\quad\left(\text{where } \theta=\frac{3}{2^x}\right)$$

$$=\frac{3}{2}\times1=\frac{3}{2}$$

**Answer: (B)**

**79.** $\lim_{x\to0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$ is equal to $(a>0)$

(A) $\log_e a$ (B) 1

(C) 0 (D) $\infty$

**Solution:** We have

$$\lim_{x\to0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}=\lim_{x\to0}a^{\sin x}\left(\frac{a^{\tan x-\sin x}-1}{\tan x-\sin x}\right)$$

$$=\lim_{x\to0}(a^{\sin x})\times\lim_{t\to0}\left(\frac{a^t-1}{t}\right)\quad(\text{where } t=\tan x-\sin x)$$

$$=a^0\times\log_e a=\log_e a\quad[\text{By part (3) of Important Formulae}]$$

**Answer: (A)**

**80.** $\lim_{x\to0}\left(\frac{\log(6+x)-\log(6-x)}{x}\right)=$

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) 1 (D) 2

**Solution:** We have

$$\lim_{x\to0}\left(\frac{\log(6+x)-\log(6-x)}{x}\right)=\lim_{x\to0}\frac{\frac{1}{6}\log\left(1+\frac{x}{6}\right)}{x/6}+\lim_{x\to0}\frac{\frac{1}{6}\log\left(1-\frac{x}{6}\right)}{-x/6}$$

$$=\frac{1}{6}\times1+\frac{1}{6}\times1=\frac{1}{3}$$

Here we have made use of part (2) of Important Formulae.

**Answer: (B)**

**81.** $\lim_{x\to0}\frac{12^x-3^x-4^x+1}{\sqrt{2\cos x+7}-3}=$

(A) $\log_e 12$ (B) $\log_e 3\cdot\log_e 4$

(C) $(\log_e 3\cdot\log_e 4)^{-1}$ (D) $-6\log_e 3\cdot\log_e 4$

**Solution:** The given limit can be written as

$$\lim_{x\to0}\frac{(3^x-1)(4^x-1)(\sqrt{2\cos x+7}+3)}{2\cos x-2}$$

$$=\lim_{x\to0}\frac{\left(\frac{3^x-1}{x}\right)\left(\frac{4^x-1}{x}\right)(\sqrt{2\cos x+7}+3)x^2}{-4\sin^2\frac{x}{2}}$$

$$=\lim_{x\to0}\frac{\left(\frac{3^x-1}{x}\right)\left(\frac{4^x-1}{x}\right)(\sqrt{2\cos x+7}+3)}{-[\sin(x/2)/(x/2)]^2}$$

$$=-\log_e 3\times\log_e 4\times6$$

We have used part (3) of Important Formulae.

**Answer: (D)**

**82.** $\lim_{x\to0}\left(\frac{1-2^x-5^x+10^x}{x\sin x}\right)=$

(A) $\log_e 10$ (B) $\log_e 5-\log_e 2$

(C) $\log_e 5\cdot\log_e 2$ (D) $(\log_e 5\cdot\log_e 2)^{-1}$

**Solution:** We have

$$\text{Given limit}=\lim_{x\to0}\frac{(5^x-1)(2^x-1)}{x\sin x}$$

$$=\lim_{x\to0}\left(\frac{5^x-1}{x}\right)\left(\frac{2^x-1}{x}\right)\left(\frac{x}{\sin x}\right)$$

$$=\log_e 5\cdot\log_e 2\cdot1$$

$$=\log_e 5\cdot\log_e 2$$

**Answer: (C)**

**83.** If $G(x)=-\sqrt{25-x^2}$, then

$$\lim_{x\to1}\left(\frac{G(x)-G(1)}{x-1}\right)=$$

(A) $\frac{1}{2\sqrt{6}}$ (B) $-\frac{1}{\sqrt{24}}$

(C) $\frac{1}{2}$ (D) $\frac{1}{2\sqrt{3}}$

**Solution:** We have

$$\frac{G(x)-G(1)}{x-1}=-\frac{\sqrt{25-x^2}-(-\sqrt{24})}{x-1}$$

$$= -\frac{[\sqrt{25-x^2}-\sqrt{24}]}{x-1}$$

$$= -\frac{[25-x^2-24]}{x-1} - \frac{1}{(\sqrt{25-x^2}+\sqrt{24})}$$

$$= \frac{x+1}{\sqrt{25-x^2}+\sqrt{24}}$$

Therefore

$$\lim_{x\to 1}\left(\frac{G(x)-G(1)}{x-1}\right) = \frac{1+1}{\sqrt{24}+\sqrt{24}} = \frac{1}{\sqrt{24}} = \frac{1}{2\sqrt{6}}$$

**Answer: (A)**

**84.** $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4} =$

(A) $e^6$ (B) $e^5$
(C) $e^4$ (D) $\infty$

**Solution:** We have

$$\left(\frac{x+6}{x+1}\right)^{x+4} = \frac{\left(1+\frac{6}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x}\cdot\left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^4$$

$$= \frac{\left[\left(1+\frac{6}{x}\right)^{x/6}\right]^6}{\left(1+\frac{1}{x}\right)^x}\cdot\left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^4$$

Therefore

$$\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4} = \frac{e^6}{e}\cdot\left(\frac{1+0}{1+0}\right) = e^5$$

We have used part (1) of Important Formulae.

**Answer: (B)**

**85.** $\lim_{x\to 0}\left(\frac{1+6x^2}{1+3x^2}\right)^{1/x^2} =$

(A) $e^2$ (B) $e^9$
(C) $e^3$ (D) $\infty$

**Solution:** We have

$$\left(\frac{1+6x^2}{1+3x^2}\right)^{1/x^2} = \frac{[(1+6x^2)^{1/6x^2}]^6}{[(1+3x^2)^{1/3x^2}]^3}$$

Therefore using part (1) of Important Formulae we get

$$\lim_{x\to 0}\left(\frac{1+6x^2}{1+3x^2}\right)^{1/x^2} = \frac{e^6}{e^3} = e^3$$

**Answer: (C)**

**86.** $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x} =$

(A) $e$ (B) $e^2$
(C) 1 (D) $\infty$

**Solution:** Let

$$f(x) = \left[\tan\left(\frac{\pi}{4}+x\right)\right]^{1/x}$$

$$= \left(\frac{1+\tan x}{1-\tan x}\right)^{1/x}$$

$$= \frac{\left[(1+\tan x)^{1/\tan x}\right]^{\tan x/x}}{\left[(1-\tan x)^{-1/\tan x}\right]^{-\tan x/x}}$$

Using parts (1) and (7) of Important Formulae we get

$$\lim_{x\to 0} f(x) = \frac{e^1}{e^{-1}} = e^2$$

**Answer: (B)**

**87.** $\lim_{x\to 0}(\cos x)^{\cot x} =$

(A) 1 (B) $\infty$
(C) $e$ (D) 0

**Solution:** We have

$$(\cos x)^{\cot x} = [(1+\cos x - 1)^{1/(\cos x-1)}]^{(\cos x-1)/\tan x}$$

Take

$$f(x) = [1+(\cos x-1)]^{1/(\cos x-1)} \text{ and } g(x) = \frac{\cos x - 1}{\tan x}$$

We know that [by part (1) of Important Formulae]

$$\lim_{x\to 0} f(x) = e$$

Now,

$$g(x) = \frac{-2\sin^2\frac{x}{2}\cos x}{\sin x} = -\frac{\sin\frac{x}{2}\cos x}{\cos\frac{x}{2}} \to 0 \quad \text{as } x\to 0$$

Therefore, by part (7) of Important Formulae

$$\lim_{x\to 0}(f(x))^{g(x)} = e^0 = 1$$

**Answer: (A)**

**88.** $\lim_{x\to\frac{\pi}{2}} \dfrac{(1-\sin x)(8x^3-\pi^3)\cos x}{(\pi-2x)^4} =$

(A) $-\dfrac{\pi^2}{16}$ (B) $\dfrac{3\pi^2}{16}$

(C) $\dfrac{\pi^2}{16}$ (D) $-\dfrac{3\pi^2}{16}$

**Solution:** Let

$$f(x)=\frac{(1-\sin x)(8x^3-\pi^3)\cos x}{(\pi-2x)^4}$$
$$=\frac{(1-\sin x)\cos x(2x-\pi)(4x^2+2\pi x+\pi^2)}{(2x-\pi)^4}$$
$$=\frac{(1-\sin x)\cos x(4x^2+2\pi x+\pi^2)}{(2x-\pi)^3}$$

Therefore

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{(2x-\pi)^3}\cdot(3\pi^2) \qquad (1.62)$$

Put $2x-\pi=y$ so that $y\to 0$ as $x\to\pi/2$. Therefore now

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3}=\frac{\left[1-\sin\left(\dfrac{\pi+y}{2}\right)\right]\cos\left(\dfrac{\pi+y}{2}\right)}{y^3}$$
$$=\frac{\left(1-\cos\dfrac{y}{2}\right)\left(-\sin\dfrac{y}{2}\right)}{y^3}$$
$$=-\left(\frac{2\sin^2\dfrac{y}{4}}{y^2}\right)\left(\frac{\sin\dfrac{y}{2}}{y}\right)$$
$$=-2\left(\frac{\sin^2\dfrac{y}{4}}{y/4}\right)^2\cdot\frac{1}{16}\cdot\left(\frac{\sin\dfrac{y}{2}}{y/2}\right)\cdot\frac{1}{2}$$
$$=-\frac{1}{16}\left(\frac{\sin\dfrac{y}{4}}{y/4}\right)^2\left(\frac{\sin\dfrac{y}{2}}{y/2}\right) \qquad (1.63)$$

Therefore from Eqs. (1.62) and (1.63)

$$\lim_{x\to 0} f(y)=\frac{-3\pi^2}{16}\times1\times1$$

**Answer: (D)**

**89.** $\lim_{x\to 1}\left(2-\dfrac{1}{x}\right)^{\tan(\pi/2x)} =$

(A) $e^{2\pi}$ (B) $e^{2/\pi}$

(C) $e^{-2/\pi}$ (D) $e^{-\pi/2}$

**Solution:** Suppose

$$f(x)=\left(2-\frac{1}{x}\right)^{\tan(\pi x/2)}=\left(1+\left(1-\frac{1}{x}\right)\right)^{\tan(\pi x/2)}$$

Put $y=1-(1/x)$ so that $y\to 0$ as $x\to 1$. Therefore

$$f(x)=(1+y)^{\tan(\pi/2)(1-y)}$$
$$=(1+y)^{\cot(\pi y/2)}$$
$$=[(1+y)^{1/y}]^{y\cot(\pi y/2)}$$
$$=[(1+y)^{1/y}]^{[(\pi y/2)/\tan(\pi y/2)]\times(2/\pi)}$$

Therefore using part (7) of Important Formulae we have

$$\text{Given limit}=e^{1\times 2/\pi}=e^{2/\pi}$$

**Answer: (B)**

**90.** If

$$\lim_{x\to 0}\frac{ae^x-b}{x}=2$$

then $(a, b)$ is equal to

(A) $(2,-2)$ (B) $(-2,2)$

(C) $(2,2)$ (D) $(-2,-2)$

**Solution:** We have

$$\lim_{x\to 0}\frac{ae^x-b}{x}=2$$

Since the denominator $x\to 0$ as $x\to 0$, by Corollary 1.2 we have

$$\lim_{x\to 0}(ae^x-b)=0$$

which implies that $a-b=0$. Therefore by part (3) of Important Formulae

$$2=\lim_{x\to 0}\left(\frac{ae^x-a}{x}\right)=\lim_{x\to 0} a\left(\frac{e^x-1}{x}\right)=a$$

Hence

$$a=2=b$$

**Answer: (C)**

**91.** If $\lim_{x\to 0}(1+ax+bx^2)^{1/x}=e^3$, then

(A) $a=3$ and $b$ is any real number

(B) $a=3/2$ and $b$ is any real number

(C) $b=3$ and $a$ is any real number

(D) $a=1$ and $b$ is any real number

**Solution:** We have

$$[1+(ax+bx^2)]^{1/x} = [1+(ax+bx^2)]^{(ax+bx^2)/x(ax+bx^2)}$$
$$= [1+(ax+bx^2)]^{[1/(ax+bx^2)][(ax+bx^2)/x]}$$

Using parts (1) and (7) of Important Formulae, the given limit $= e^a = e^3 \Rightarrow a = 3.$

**Answer: (A)**

**92.** If

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$$

then $(a, b)$ value is

(A) $(1,1)$ (B) $(1,-1)$
(C) $(-1,1)$ (D) $(1,0)$

**Solution:** Let

$$f(x)=\frac{x^2+1}{x+1}-ax-b$$
$$=\frac{(x^2+1)-(x+1)(ax+b)}{x+1}$$
$$=\frac{(1-a)x^2-(a+b)x+1-b}{x+1}$$

Put $y = 1/x$ so that $y \to 0$ as $x \to \infty$. Therefore

$$f(x)=f\left(\frac{1}{y}\right)=\frac{(1-a)-(a+b)y+(1-b)y^2}{y(1+y)}$$

Now,

$$\lim_{x\to\infty} f(x)=0$$
$$\Rightarrow \lim_{y\to 0}\frac{(1-a)-(a+b)y+(1-b)y^2}{y(y+1)}=0$$

Since the denominator tends to zero as $y \to 0$, by Corollary 1.2 the numerator must tend to zero as $y \to 0$. Therefore

$$1-a=0 \quad \text{or} \quad a=1$$
$$0=\lim_{y\to 0}\frac{[-(a+b)+(1-b)y]}{y+1}=-(a+b)$$
$$a=-b$$

Hence $(a, b) = (1,-1)$.

**Answer: (B)**

***Note:*** Under the given hypothesis, the function given in Problem 91 becomes

$$\frac{x^2+1}{x+1}-x+1=\frac{2}{x+1}\to 0 \quad \text{as } x\to\infty$$

**93.** $\lim_{x\to\infty}\dfrac{x^3\sin(1/x)+2x^2}{1+3x^2}=$

(A) 0 (B) 1
(C) $\infty$ (D) $\dfrac{2}{3}$

**Solution:** We have

$$\lim_{x\to\infty}\frac{x^3\sin\frac{1}{x}+2x^2}{1+3x^2}=\lim_{x\to\infty}\frac{x\sin\frac{1}{x}+2}{\frac{1}{x^2}+3}$$
$$=\frac{1+2}{0+3}\quad\left[\because \lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=1\right]$$
$$=\frac{3}{3}=1$$

**Answer: (B)**

**94.** If $n$ is a fixed positive integer then

$$\lim_{x\to\frac{\pi}{2}}(1^{\sec^2 x}+2^{\sec^2 x}+3^{\sec^2 x}+\cdots+n^{\sec^2 x})^{\cos^2 x}=$$

(A) $n$ (B) $+\infty$
(C) $\infty$ (D) $n^2$

**Solution:** It is known that $K^{\sec^2 x}\le n^{\sec^2 x}$ for $K=1,2,3,\ldots,n$. Therefore

$$(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x})^{\cos^2 x}\le(n\cdot n^{\sec^2 x})^{\cos^2 x}=n\cdot n^{\cos^2 x}$$

Again

$$(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x})^{\cos^2 x}\ge(n^{\sec^2 x})^{\cos^2 x}=n$$

Therefore

$$n=\lim_{x\to\frac{\pi}{2}}(n)\le\lim_{x\to\frac{\pi}{2}}(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x})^{\cos^2 x}$$
$$\le\lim_{x\to\frac{\pi}{2}}(n\cdot n^{\cos^2 x})=n$$

By squeezing theorem, we have

$$\lim_{x\to\frac{\pi}{2}}(1^{\sec^2 x}+2^{\sec^2 x}+\cdots+n^{\sec^2 x})^{\cos^2 x}=n$$

**Answer: (A)**

**Try it out**

$$\lim_{x\to 0}(1^{\operatorname{cosec}^2 x}+2^{\operatorname{cosec}^2 x}+\cdots+n^{\operatorname{cosec}^2 x})^{\sin^2 x}=n$$

## Limits of Sequences

**95.** Let

$$x_n = \frac{1+2+3+\cdots+n}{n^2}$$

for $n = 1, 2, 3, \ldots$. Then the value of $\lim_{n\to\infty} x_n$ is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) $+\infty$

**Solution:** We have

$$x_n = \frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right) \to \frac{1}{2} \quad \text{as } n \to \infty$$

**Answer: (A)**

**96.** If

$$y_n = \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

then $\lim_{n\to\infty} y_n$ is equal to

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$

(C) $\frac{2}{3}$ (D) $+\infty$

**Solution:** We have

$$y_n = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \to \frac{2}{6} = \frac{1}{3}$$

as $n \to \infty$.

**Answer: (B)**

**97.** Let

$$z_n = \frac{1^3+2^3+3^3+\cdots+n^3}{n^4}$$

Then $\lim_{n\to\infty} z_n$ is

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{1}{4}$ (D) 0

**Solution:** We have

$$z_n = \frac{n^2(n+1)^2}{4n^4} = \frac{1}{4}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{4} \quad \text{as } n \to \infty$$

**Answer: (C)**

**98.** Let $x$ be a real number and $[\cdot]$ denote the integral part function. Then

$$\lim_{n\to\infty} \frac{[1x]+[2x]+[3x]+\cdots+[nx]}{n^2} =$$

(A) $\frac{x}{2}$ (B) $x$

(C) 0 if $0 < x < 1$ (D) $+\infty$ if $x > 1$

**Solution:** By the definition of the integral part

$$x-1 < [x] \le x$$
$$2x-1 < [2x] \le 2x$$
$$3x-1 < [3x] \le 3x$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$nx-1 < [nx] \le nx$$

Adding all the inequalities we have

$$x(1+2+3+\cdots+n) - n < [1x]+[2x]+\cdots+[nx] \le x(1+2+\cdots+n)$$

$$\frac{x}{2}n(n+1) - n < [1x]+[2x]+\cdots+[nx] \le \frac{xn(n+1)}{2}$$

Dividing throughout by $n^2$, we get

$$\frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \frac{[1x]+[2x]+\cdots+[nx]}{n^2} \le \frac{x}{2}\left(1+\frac{1}{n}\right)$$

Taking limit as $n \to \infty$ and using squeezing theorem we have

$$\lim_{n\to\infty} \frac{[1x]+[2x]+\cdots+[nx]}{n^2} = \frac{x}{2}$$

**Answer: (A)**

**Try it out** On similar lines (i.e., by using the concept of $[x]$) we can show that

1. $\lim_{n\to\infty} \frac{[1^2x]+[2^2x]+\cdots+[n^2x]}{n^3} = \frac{x}{3}$
2. $\lim_{n\to\infty} \frac{[1^3x]+[2^3x]+\cdots+[n^3x]}{n^4} = \frac{x}{4}$

**99.** Let $x$ be a real number and

$$a_n = \frac{1}{n^2}\{[x+1]+[2x+2]+[3x+3]+\cdots+[nx+n]\}$$

then $\lim_{x\to\infty} a_n$ is equal to (where $[\cdot]$ denotes integral part)

(A) $\frac{x}{2}+1$ (B) $\frac{x+1}{2}$

(C) $\dfrac{x+2}{2}$ (D) $\dfrac{x-2}{2}$

**Solution:** We know that $[x+y]=[x]+[y]$ if one of $x$ or $y$ is an integer. Therefore

$$[x+1]=[x]+1$$
$$[2x+2]=[2x]+2$$
$$[3x+2]=[3x]+3$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$[nx+n]=[nx]+n$$

Adding all the equations and dividing by $n^2$, we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2}\{[x]+[2x]+\cdots+[nx]+\frac{1}{2}n(n+1)\}$$
$$=\lim_{n\to\infty}\frac{[x]+[2x]+\cdots+[nx]}{n^2}+\lim_{n\to\infty}\frac{n(n+1)}{2n^2}$$
$$=\frac{x}{2}+\frac{1}{2}\quad\text{(by Problem 98)}$$

**Answer: (B)**

**100.** Let

$$x_1=1\quad\text{and}\quad x_{n+1}=\frac{4+3x_n}{3+2x_n}\text{ for } n\geq 1$$

If $\lim\limits_{n\to\infty} x_n$ exists finitely, then the limit is equal to

(A) $\sqrt{2}$ (B) 1
(C) 2 (D) $\sqrt{2}+1$

**Solution:** We have

$$x_1=1,\ x_2=\frac{4+3}{3+2}=\frac{7}{5}$$

$$x_3=\frac{4+3x_2}{3+2x_2}=\frac{4+3\left(\frac{7}{5}\right)}{3+2\left(\frac{7}{5}\right)}=\frac{41}{29}>x_2$$

We can easily verify that $x_n<x_{n+1}$ and hence $\{x_n\}$ is strictly increasing sequence of positive terms. Let $\lim\limits_{n\to\infty} x_n = l$. Therefore

$$l=\lim_{n\to\infty} x_{n+1}$$
$$=\lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)$$
$$=\frac{4+3\lim\limits_{n\to\infty} x_n}{3+2\lim\limits_{n\to\infty} x_n}$$
$$=\frac{4+3l}{3+2l}$$

Hence

$$3l+2l^2=4+3l$$

or $$l^2=2\Rightarrow l=\sqrt{2}\quad(\because x_n>0\ \forall\ n)$$

**Answer: (A)**

***Note:*** It can be seen that $\sqrt{2}$ is the l.u.b. $\{x_n\}$ (see Theorem 1.42).

**101.** Let $x_0=0$, $x_1=1$ and $x_{n+1}=x_n+\sqrt{1+x_n^2}$ for $n\geq 1$. Then

$$\lim_{n\to\infty}\left(\frac{x_n}{2^{n-1}}\right)=$$

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{2}$
(C) $\dfrac{2}{\pi}$ (D) $\dfrac{4}{\pi}$

**Solution:** $\{x_n\}$ is an increasing sequence. Define

$$\theta_n=\mathrm{Cot}^{-1}(x_n)\quad\text{or}\quad x_n=\cot\theta_n$$

Now

$$x_0=0=\cot\frac{\pi}{2},\ x_1=1=\cot\frac{\pi}{2^2}$$

and in general

$$x_{n+1}=\cot\theta_n+\operatorname{cosec}\theta_n$$
$$=\frac{1+\cos\theta_n}{\sin\theta_n}$$
$$=\cot\left(\frac{\theta_n}{2}\right)$$
$$=\cot\left(\frac{\pi}{4\times 2^n}\right)$$
$$\left(\because x_p=1=\cot\left(\frac{\pi}{4}\right),\ x_2=\cot\left(\frac{\pi}{8}\right)=\cot\left(\frac{\pi}{4\times 2}\right),\text{ etc.}\right)$$

Therefore

$$\frac{x_n}{2^{n-1}}=\frac{\cot\left(\dfrac{\pi}{4\times 2^{n-1}}\right)}{2^{n-1}}$$
$$=\frac{1}{2^{n-1}\tan\left(\dfrac{\pi}{4\times 2^{n-1}}\right)}$$

$$= \frac{1}{\dfrac{\tan\left(\dfrac{\pi}{4\times 2^{n-1}}\right)}{\left(\dfrac{\pi}{4\times 2^{n-1}}\right)}\cdot\dfrac{\pi}{4}}$$

$$= \frac{4}{\pi}\frac{1}{\left(\dfrac{\tan x}{x}\right)}$$

where $x = \dfrac{\pi}{4\times 2^{n-1}} \to 0$ as $n\to\infty$. Therefore

$$\lim_{n\to\infty} x_n = \frac{4}{\pi}\cdot 1 = \frac{4}{\pi}$$

**Answer: (D)**

**102.** Let $a_0 = 1, a_1 = 2$ and

$$n(n+1)a_{n+1} = n(n-1)\,a_n - (n-2)\,a_{n-1}$$

for $n \ge 1$. Then $\lim\limits_{n\to\infty} a_n$ is

(A) 1 (B) 0

(C) $+\infty$ (D) $\dfrac{1}{e}$

**Solution:** From the given relation we have

$$1\cdot 2\cdot a_2 = 0-(1-2)a_0 = 1$$

$$\Rightarrow a_2 = \frac{1}{2} = \frac{1}{\lfloor 2}$$

and $$6\cdot a_3 = 2a_2 - 0\cdot a_1 = 2\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow a_3 = \frac{1}{6} = \frac{1}{\lfloor 3}$$

Assume that $a_k = 1/\lfloor k$ for all $k = 2, 3, \ldots, m$. Now

$$m(m+1)\cdot a_{m+1} = m(m-1)a_m - (m-2)a_{m-1}$$

$$= \frac{m(m-1)}{\lfloor m} - \frac{m-2}{\lfloor m-1}$$

$$= \frac{1}{\lfloor m-2} - \frac{m-2}{\lfloor m-1}$$

$$= \frac{(m-1)-(m-2)}{\lfloor m-1} = \frac{1}{\lfloor m-1}$$

Therefore

$$a_{m+1} = \frac{1}{m(m+1)\lfloor m-1} = \frac{1}{\lfloor m+1}$$

Hence by complete induction, $a_k = 1/\lfloor k$ for $k \ge 2$.

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{\lfloor n}\right) = 0$$

**Answer: (B)**

**103.** Let $x > 1$ and for each positive integer $n$, define

$$s_n(x) = \frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \cdots + \frac{x^{2^{n-1}}}{(x+1)(x^2+1)\cdots(x^{2^{n-1}}+1)}$$

Then $\lim\limits_{n\to\infty} s_n(x)$ is equal to

(A) 1 (B) 0

(C) $x$ (D) $\dfrac{1}{x-1}$

**Solution:** We have

$$\frac{s_n(x)}{x-1} = \frac{x}{x^2-1} + \frac{x^2}{x^4-1} + \frac{x^4}{x^{16}-1} + \cdots + \frac{x^{2^{n-1}}}{x^{2^n}-1}$$

$$= \left(\frac{1}{x-1} - \frac{1}{x^2-1}\right) + \left(\frac{1}{x^2-1} - \frac{1}{x^4-1}\right) + \cdots + \left(\frac{1}{x^{2^{n-1}}-1} - \frac{1}{x^{2^n}-1}\right)$$

$$= \frac{1}{x-1} - \frac{1}{x^{2^n}-1}$$

Therefore

$$s_n(x) = 1 - \frac{x-1}{x^{2^n}-1}$$

Since $x > 1$, we have $\lim\limits_{n\to\infty}(x^{2^n}-1) = +\infty$ so that

$$\frac{1}{x^{2^n}-1} \to 0 \quad \text{as } n\to\infty$$

Therefore

$$\lim_{n\to\infty} s_n(x) = 1 - 0 = 1$$

**Answer: (A)**

**104.** The sequence $\{a_n\}$ is defined as $a_1 = 1$ and $a_n = n(a_{n-1}+1)$ for $n \ge 2$. If

$$P_n = \left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\cdots\left(1+\frac{1}{a_n}\right)$$

then $\lim\limits_{n\to\infty} P_n$ equals

(A) 1 (B) $e-1$

(C) $e+1$ (D) $e$

**Solution:** We have

$$P_n = \left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right)\cdots\left(\frac{a_n+1}{a_n}\right)$$

$$= \left(\frac{a_2}{2a_1}\right)\left(\frac{a_3}{3a_2}\right)\cdots\left(\frac{a_n+1}{(n+1)a_n}\right) \quad \left(\because \frac{a_k}{k} = a_{k-1}+1 \text{ for } k \geq 2\right)$$

$$= \frac{a_{n+1}}{\lfloor n+1} \tag{1.64}$$

Now

$$\frac{a_n+1}{\lfloor n+1} = \frac{(n+1)(a_n+1)}{\lfloor n+1}$$

$$\Rightarrow \frac{a_n+1}{\lfloor n+1} - \frac{a_n}{\lfloor n} = \frac{1}{\lfloor n}$$

Therefore

$$\frac{a_2}{\lfloor 2} - \frac{a_1}{\lfloor 1} = \frac{1}{\lfloor 1}$$

$$\frac{a_3}{\lfloor 3} - \frac{a_2}{\lfloor 2} = \frac{1}{\lfloor 2}$$

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$$\frac{a_{n+1}}{\lfloor n+1} - \frac{a_n}{\lfloor n} = \frac{1}{\lfloor n}$$

Adding all the above equations we get

$$\frac{a_{n+1}}{\lfloor n+1} - \frac{a_1}{\lfloor 1} = \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots + \frac{1}{\lfloor n}$$

From Eq. (1.64) we have

$$P_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots + \frac{1}{\lfloor n} \quad (\because a_1 = 1)$$

But the number $e$ is the sum of the infinite series $\sum_{n=0}^{\infty} \frac{1}{\lfloor n}$ (this is to be assumed). Note that $\Sigma a_n$ is defined as the limit of the sequence of the partial sums $\{s_n\}$ where $s_n = a_1 + a_2 + \cdots + a_n$ (see Definition 1.38). Thus

$$\lim_{n\to\infty} P_n = e$$

**Answer: (D)**

**105.** Suppose $x$ is real. Define

$$f(x) = \frac{x}{2} + 1$$

$$f^{(2)}(x) = f(f(x))$$
$$f^{(3)}(x) = f(f^{(2)}(x))$$

and in general

$$f^{n+1}(x) = f(f^{(n)}(x))$$

for $n \geq 1$ where

$$f^{(1)}(x) = f(x) = \frac{x}{2} + 1$$

Then $\lim_{n\to\infty} f^{(n)}(x)$ is equal to

(A) 1 (B) 2

(C) $x$ (D) $\frac{1}{1-x}$

**Solution:** We have

$$f^{(2)}(x) = f(f^{(1)}(x)) = f(f(x)) = \frac{f(x)}{2} + 1 = \frac{x}{2^2} + \frac{1}{2} + 1$$

$$f^{(3)}(x) = f(f^{(2)}(x)) = \frac{1}{2}\left(\frac{x}{2^2} + \frac{1}{2} + 1\right) + 1 = \frac{x}{2^3} + \frac{1}{2^2} + \frac{1}{2} + 1$$

By induction we can see that

$$f^{(n)}(x) = \frac{x}{2^n} + \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2} + 1$$

$$= \frac{x}{2^n} + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

$$= \frac{x}{2^n} + 2 - \frac{1}{2^{n-1}}$$

Now

$$\lim_{n\to\infty} \frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty} f^{(n)}(x) = 0 + 2 - 0 = 2$$

**Answer: (B)**

**106.** Define $a_n = n^2 + 20$ for $n = 1, 2, 3, \ldots$. Let $d_n$ be the greatest common divisor of $a_n$ and $a_{n+1}$ and $d$ be the least common multiple of $d_n$ for $n = 1, 2, 3, \ldots$. Then the infinite sum of the series

$$d + \frac{d}{4} + \frac{d}{4^2} + \frac{d}{4^3} + \cdots$$

is

(A) 162 (B) 9

(C) 108 (D) 54

**Solution:** We have

$$a_{n+1}-a_n=2n+1$$

This implies $d_n$ divides $2n+1$ for all $n=1,2,3,\ldots$. Now

$$3a_n+a_{n+1}=4n^2+2n+81=2n\,(2n+1)+81$$

This implies $d_n$ divides 81 and 81 will be the L.C.M. of $d_n$ for $n=1,2,3,\ldots$ so that $d=81$. Now

$$d+\frac{d}{4}+\frac{d}{4^2}+\cdots\infty=\frac{d}{1-\frac{1}{4}}$$

$$=(81)\frac{4}{3}=108$$

**Answer: (C)**

**107.** Let $a_1=1, a_2=1+a_1, a_3=1+a_1a_2, \ldots, a_{n+1}=1+a_1a_2\ldots a_n$. Then $\sum_{n=1}^{\infty}\frac{1}{a_n}=$

(A) 1 (B) $e$

(C) 2 (D) $\frac{1}{2}$

**Solution:** Let

$$s_n=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$$

$$=2+\left(\frac{-1}{a_1}+\frac{1}{a_2}\right)+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\quad(\because a_1=1)$$

$$=2+\left(\frac{a_1-a_2}{a_1a_2}+\frac{1}{a_3}\right)+\frac{1}{a_4}+\cdots+\frac{1}{a_n}\quad(\because a_1-a_2=-1)$$

$$=2+\left(\frac{-1}{a_1a_2}+\frac{1}{a_3}\right)+\frac{1}{a_4}+\cdots+\frac{1}{a_n}$$

$$=2+\left(\frac{a_1a_2-a_3}{a_1a_2a_3}+\frac{1}{a_4}\right)+\cdots+\frac{1}{a_n}\quad(\because a_1a_2-a_3=-1)$$

$$=2+\left(\frac{-1}{a_1a_2a_3}+\frac{1}{a_4}\right)+\cdots+\frac{1}{a_n}$$

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Finally

$$s_n=2-\frac{1}{a_1a_2\cdots a_n}\quad\text{and}\quad a_1a_2\cdots a_n>1$$

Therefore

$$\lim_{n\to\infty}s_n=2-0=2$$

**Answer: (C)**

**108.** For each positive integer $n$, let

$$s_n=\frac{3}{1\cdot2\cdot4}+\frac{4}{2\cdot3\cdot5}+\frac{5}{3\cdot4\cdot6}+\cdots+\frac{n+2}{n(n+1)(n+3)}$$

Then $\lim_{n\to\infty}s_n$ equals

(A) $\frac{29}{6}$ (B) $\frac{29}{36}$

(C) 0 (D) $\frac{29}{18}$

**Solution:** Let

$$u_k=\frac{k+2}{k(k+1)(k+3)}$$

$$=\frac{(k+2)^2}{k(k+1)(k+2)(k+3)}$$

$$=\frac{k^2+4k+4}{k(k+1)(k+2)(k+3)}$$

$$=k(k+1)+3k+4$$

$$=\frac{1}{(k+2)(k+3)}+\frac{3}{(k+1)(k+2)(k+3)}+\frac{4}{k(k+1)(k+2)(k+3)}$$

$$=\left(\frac{1}{k+2}-\frac{1}{k+3}\right)-\frac{3}{2}\left[\frac{1}{(k+2)(k+3)}-\frac{1}{(k+1)(k+2)}\right]-\frac{4}{3}\left[\frac{1}{(k+1)(k+2)(k+3)}-\frac{1}{k(k+1)(k+2)}\right]$$

(see Sec. 5.5, Vol. 1, p. 225)

Now, put $k=1,2,3,\ldots,n$ and add. Thus

$$s_u=u_1+u_2+\cdots+u_n$$

$$=\left(\frac{1}{3}-\frac{1}{n+3}\right)-\frac{3}{2}\left[\frac{1}{(n+2)(n+3)}-\frac{1}{2\cdot3}\right]-\frac{4}{3}\left[\frac{1}{(n+1)(n+2)(n+3)}-\frac{1}{1\cdot2\cdot3}\right]$$

Therefore

$$\lim_{n\to\infty}s_n=\frac{1}{3}+\frac{3}{12}+\frac{4}{18}=\frac{29}{36}$$

**Answer: (B)**

**109.** $\lim_{n\to\infty}\left(1+\frac{1}{n^2+\cos n}\right)^{n^2+n}$ is equal to

(A) 1 (B) $e$

(C) $e^2$ (D) does not exist

**Solution:** Put $m=n^2+\cos n$ so that $m\to\infty=$ as $n\to\infty$.

Now

$$\frac{n^2+n}{m} = \frac{n^2+n}{n^2+\cos n}$$

$$= \frac{1+\dfrac{1}{n}}{1+\dfrac{\cos n}{n^2}} \to 1 \quad \text{as } n \to \infty$$

because

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\cos n}{n^2} = 0$$

Therefore

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+\cos n}\right)^{n^2+n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^{(n^2+n)/m} = e$$

**Answer: (B)**

**110.** $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n!}$ equals

(A) $e$ (B) $e^2$

(C) $2e$ (D) $\infty$

**Solution:** Let

$$u_n = \frac{n^2}{\lfloor n} = \frac{n}{\lfloor n-1} = \frac{n-1+1}{\lfloor n-1} = \frac{1}{\lfloor n-2} + \frac{1}{\lfloor n-1}$$

for $n \geq 2$. Therefore

$$s_n = \sum_{n=1}^{n} u_k = 1+\left(1+\frac{1}{\lfloor 1}\right)+\left(\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}\right)+\cdots+\left(\frac{1}{\lfloor n-2}+\frac{1}{\lfloor n-1}\right)$$

$$= 2\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\cdots+\frac{1}{\lfloor n-1}\right)-\frac{1}{\lfloor n-1}$$

Hence

$$\lim_{n\to\infty} s_n = 2e$$

**Answer: (C)**

**111.** Let $x_n = 2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}$ ($n$ square roots). Then $\lim_{n\to\infty} x_n$ is equal to

(A) 8 (B) 16

(C) $e^2$ (D) 4

**Solution:** We have $x_n = 2+\sqrt{x_{n-1}}$. Suppose $\lim_{n\to\infty} = l$. Then

$$l = 2+\sqrt{l}$$

$$\Rightarrow l^2 - 5l + 4 = 0$$

$$\Rightarrow (l-1)(l-4) = 0$$

$$\Rightarrow l = 1 \text{ or } 4$$

But $x_n > 2$ and $\{x_n\}$ is an increasing sequence and hence $l \neq 1$. Thus $l = 4$.

**Answer: (D)**

**112.** $\lim_{n\to\infty}[2\log(3n) - \log(n^2+1)]$ is equal to

(A) 0 (B) log 3

(C) 2 log 3 (D) 4 log 6

**Solution:** The given limit is

$$\lim_{n\to\infty}[\log(9n^2)-\log(n^2+1)] = \lim_{n\to\infty}\log\frac{9n^2}{n^2+1}$$

$$= \lim_{n\to\infty}\log\left(\frac{9}{1+\dfrac{1}{n^2}}\right)$$

$$= \log\left(\frac{9}{1+0}\right)$$

$$= \log 9 = 2\log 3$$

**Answer: (C)**

**113.** $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{\frac{8n^3}{\pi}\sin\left(\frac{\pi}{2n}\right)}$ is equal to

(A) 4 (B) $e^4$

(C) 1 (D) $\infty$

**Solution:** We have

$$\text{Given limit} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\frac{4\left(\sin\frac{\pi}{2n}\right)}{\pi/2n}} = e^{4\times 1} = e^4$$

**Answer: (B)**

**114.** If $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ is

(A) $\dfrac{\pi^2}{8}$ (B) $\dfrac{\pi^2}{6}$

(C) $\dfrac{\pi^2}{3}$ (D) $\dfrac{\pi^2}{24}$

**Solution:** We have

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$$

$$= \frac{\pi^2}{6} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$$

**115.** If

$$\alpha = \lim_{n\to\infty}\left(\frac{1^2+2^2+\cdots+n^2}{n^3}\right)$$

and

$$\beta = \lim_{n\to\infty}\left[\frac{(1^3-1^2)+(2^3-2^2)+(3^3-3^2)+\cdots+(n^3-n^2)}{n^4}\right]$$

Then

(A) $\alpha = 3\beta$ (B) $2\alpha = 3\beta$

(C) $4\alpha = 3\beta$ (D) $3\alpha = 4\beta$

**Solution:** We have

$$\alpha = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}$$

$$\beta = \lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+\cdots+n^3}{n^4}\right) - \lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^4}$$

$$= \lim_{n\to\infty}\frac{n^2(n+1)^2}{4n^4} - \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^4}$$

$$= \frac{1}{4} - 0$$

Therefore $3\alpha = 4\beta$.

**Answer: (D)**

**116.** $\lim_{n\to\infty}\frac{1}{n}\sum_{K=0}^{n-1}\cos\left(\frac{K\pi}{2n}\right)$ is

(A) 1 (B) 0

(C) $\frac{2}{\pi}$ (D) does not exist

**Solution:** First we show that

$$\cos\alpha + \cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+(n-1)\beta)$$
$$= \frac{\cos\dfrac{(\alpha+\alpha+(n-1)\beta)}{2}\sin\left(\dfrac{n\beta}{2}\right)}{\sin\dfrac{\beta}{2}}$$

Let

$$s = \cos\alpha + \cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+(n-1)\beta)$$

Multiply both sides with 2 sin ($\beta/2$). Therefore

$$s\left(2\sin\frac{\beta}{2}\right) = 2\cos\alpha\sin\frac{\beta}{2} + 2\cos(\alpha+\beta)\sin\frac{\beta}{2}$$
$$+\cdots+2\cos(\alpha+(n-1)\beta)\sin\frac{\beta}{2}$$
$$= \left[\sin\left(\alpha+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right)\right]$$
$$+\left[\sin\left(\alpha+\beta+\frac{\beta}{2}\right) - \sin\left(\alpha+\beta-\frac{\beta}{2}\right)\right]$$
$$+\cdots+\left[\begin{array}{l}\sin\left(\alpha+(n-1)\beta+\dfrac{\beta}{2}\right)\\ -\sin\left(\alpha+(n-1)\beta-\dfrac{\beta}{2}\right)\end{array}\right]$$
$$= \sin\left(\alpha+(2n-1)\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right)$$
$$= 2\cos\frac{\left(\alpha+(2n-1)\dfrac{\beta}{2}+\alpha-\dfrac{\beta}{2}\right)}{2}\sin\frac{n\beta}{2}$$

Therefore

$$s = \frac{\cos\left(\dfrac{\alpha+\alpha+(n-1)\beta}{2}\right)\sin\left(\dfrac{n\beta}{2}\right)}{\sin\dfrac{\beta}{2}}$$

In the given problem, $\alpha = 0$ and $\beta = \pi/2n$. Therefore

$$s_n = \cos 0 + \cos\frac{\pi}{2n} + \cos\frac{\pi}{2n} + \cdots + \cos\left(\frac{(n-1)\pi}{2n}\right)$$

$$= \frac{\cos\left(\dfrac{0+\dfrac{(n-1)\pi}{2n}}{2}\right)\sin\left(n\dfrac{\pi}{4n}\right)}{\sin\left(\dfrac{\pi}{4n}\right)}$$

$$= \frac{\cos(n-1)\dfrac{\pi}{4n}\cdot\sin\dfrac{\pi}{4}}{\sin\left(\dfrac{\pi}{4n}\right)}$$

$$= \frac{\dfrac{1}{\sqrt{2}}\cos\left(\dfrac{\pi}{4}-\dfrac{\pi}{4n}\right)}{\sin\left(\dfrac{\pi}{4n}\right)}$$

Therefore

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{n-1}\cos\left(\frac{r\pi}{2n}\right)\right) = \frac{1}{\sqrt{2}}\lim_{n\to\infty}\left(\frac{\cos\left(\dfrac{\pi}{4}-\dfrac{\pi}{4n}\right)}{\dfrac{\left(\sin\dfrac{\pi}{4n}\right)}{\left(\dfrac{\pi}{4n}\right)}}\cdot\frac{4}{\pi}\right)$$

$$= \frac{\dfrac{1}{\sqrt{2}}.\cos\left(\dfrac{\pi}{4}\right)}{1}\cdot\frac{4}{\pi} = \frac{2}{\pi}$$

**Answer: (C)**

**117.** Let

$$P_n = \frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}+\cdots+\frac{n^3-1}{n^3+1};\ n=2,3,4,\ldots.$$

Then $\lim_{n\to\infty} P_n$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{7}{11}$

(C) $\frac{3}{4}$ (D) $\frac{2}{3}$

**Solution:** We have

$$\frac{k^3-1}{k^3+1} = \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}$$

$$= \left(\frac{k-1}{k+1}\right)\left(\frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right)$$

for $k = 2, 3, \ldots, n$. Therefore

$$P_n = \left(\frac{2-1}{2+1}\cdot\frac{3-1}{3+1}\cdot\frac{4-1}{4+1}\cdots\frac{n-2}{n}\cdot\frac{n-1}{n+1}\right)$$

$$\left(\frac{7}{3}\cdot\frac{13}{7}\cdot\frac{21}{13}\cdots\frac{n^2+n+1}{(n-1)^2+(n-1)+1}\right)$$

$$= \left(\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{n-2}{n}\cdot\frac{n-1}{n+1}\right)\left(\frac{7}{3}\cdot\frac{13}{7}\cdot\frac{21}{13}\cdots\frac{n^2+n+1}{(n-1)^2+(n-1)+1}\right)$$

$$= \left(\frac{2}{n(n+1)}\right)\left(\frac{n^2+n+1}{3}\right)$$

$$= \frac{2}{3}\left[1+\frac{1}{n(n+1)}\right]$$

Therefore

$$\lim_{n\to\infty} P_n = \frac{2}{3}(1+0) = \frac{2}{3}$$

**Answer: (D)**

**118.** If $a_n$ and $b_n$ are positive integers and $a_n + \sqrt{2}b_n = (2+\sqrt{2})^n$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$$

(A) 2 (B) $\sqrt{2}$

(C) $e^{\sqrt{2}}$ (D) $e^2$

**Solution:** We have

$$a_n + \sqrt{2}b_n = (2+\sqrt{2})^n$$

$$\Rightarrow a_n - \sqrt{2}b_n = (2-\sqrt{2})^n$$

Therefore

$$a_n = \frac{1}{2}[(2+\sqrt{2})^n + (2-\sqrt{2})^n]$$

and

$$b_n = \frac{[(2+\sqrt{2})^n - (2-\sqrt{2})^n]}{2\sqrt{2}}$$

Therefore

$$\frac{a_n}{b_n} = \sqrt{2}\left[\frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{(2+\sqrt{2})^n - (2-\sqrt{2})^n}\right]$$

$$= \sqrt{2}\,\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{2}\left(\frac{1+0}{1-0}\right) \quad \left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}} < 1\right)$$

$$= \sqrt{2}$$

**Answer: (B)**

**119.** If

$$\lim_{x\to 0}[1+x\log(1+b^2)]^{(1/x)} = 2b\sin^2\theta, b>0$$

and $\theta \in (-\pi, \pi]$ then the value of $\theta$ is

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$

(C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

**(IIT-JEE 2011)**

**Solution:** Let

$$f(x) = [1+x\log(1+b^2)]^{1/x}$$

$$= \left\{[1+x\log(1+b^2)]^{\frac{1}{x\log(1+b^2)}}\right\}^{\log(1+b^2)}$$

Therefore

$$\lim_{x\to 0} f(x) = e^{\log(1+b^2)} = 1+b^2$$

$$\Rightarrow 1+b^2 = 2b\sin^2\theta$$

$$\Rightarrow 2b\sin^2\theta = 1+b^2 \geq 2b\ (\therefore\ b>0)$$

$$\Leftrightarrow b = 1 \text{ and } \sin^2\theta = 1$$

Therefore

$$\theta = \pm\frac{\pi}{2}$$

**Answer: (D)**

## Continuity

**120.** If the function

$$f(x) = \begin{cases} \dfrac{x^2 - 2x + a}{\sin x} & \text{when } x \neq 0 \\ b & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$, then

(A) $a = 0, b = -2$ (B) $a = 1, b = 0$
(C) $a = 1, b = 1$ (D) $a = 0, b = 0$

**Solution:**

$f(x)$ is continuous at $x = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = f(0) = b$

Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2 - 2x + a}{\sin x}$$

exists finitely and the denominator $\sin x \to 0$ as $x \to 0$. Hence, by Corollary 1.3

$$\lim_{x\to 0}(x^2 - 2x + a) = 0 \Rightarrow a = 0$$

Therefore

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2 - 2x}{\sin x}$$

$$= \lim_{x\to 0} \frac{x-2}{\left(\dfrac{\sin x}{x}\right)}$$

$$= \frac{0-2}{1}$$

$$= -2 = f(0) \Rightarrow b = -2$$

Thus $a = 0, b = -2$.

**Answer: (A)**

**121.** The function

$$f(x) = \begin{cases} \dfrac{1 - \cos 4x}{x^2} & \text{if } x < 0 \\ A & \text{if } x = 0 \\ \dfrac{2\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} & \text{if } x > 0 \end{cases}$$

is continuous at $x = 0$ for

(A) $A = 16$ (B) $A = 8$
(C) $A = 4$ (D) no value of $A$

**Solution:** $f$ is continuous at $x = 0$ if

$$\lim_{x\to 0-0} f(x) = \lim_{x\to 0+0} f(x) = f(0) = A$$

(see Theorem 1.24 and Quick Look 6). Now

$$\lim_{x\to 0-0} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(0-h)$$

$$= \lim_{h\to 0} \frac{1 - \cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0} \frac{2\sin^2 2h}{h^2}$$

$$= 2\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2 \times 4$$

$$= 8 \times 1 = 8$$

Also

$$\lim_{x\to 0+0} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(0+h)$$

$$= \lim_{h\to 0} \frac{2\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4}$$

$$= \lim_{h\to 0} \frac{2\sqrt{h}(\sqrt{16+\sqrt{h}} + 4)}{16 + \sqrt{h} - 16}$$

$$= \lim_{h\to 0}(2(\sqrt{16+\sqrt{h}} + 4))$$

$$= 2(4+4) = 16$$

Therefore

$$\lim_{x\to 0-0} f(x) \neq \lim_{x\to 0+0} f(x)$$

So $f(x)$ is not continuous for any value of $A$.

**Answer: (D)**

**122.** $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that the equation $f(x) = x$ has no real solution. Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = f(f(x))$. Then the number of real solutions of the equation $g(x) = x$ is

(A) only one (B) 0
(C) only one in the interval $(0, 1)$ (D) infinite

**Solution:** Let $h(x) = f(x) - x$. Since $h(x) = 0$ has no real solutions, by Corollary 1.10 either $h(x) > 0$ for all real $x$ or $h(x) < 0$ for all real $x$. Therefore,

$$g(x) - x = (f(f(x)) - f(x)) + (f(x) - x)$$

keeps the same sign for all real $x$. That is the graph of $g(x) - x$ can never cross the $x$-axis. Hence $g(x) - x = 0$ has no real solution. Thus $f(f(x)) = x$ has no real solution.

**Answer: (B)**

**123.** Let $a > 0, b > 0$ and

$$f(x) = \begin{cases} \dfrac{a^x - b^x}{x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$

If $f$ is continuous at $x = 0$, then the value of $k$ is

(A) $(\log a)\,(\log b)$ (B) $\log a + \log b$
(C) $\log a - \log b$ (D) 1

**Solution:** We have

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0}\left(\frac{a^x - b^x}{x}\right) \\ &= \lim_{x\to 0}\frac{(a^x-1)-(b^x-1)}{x} \\ &= \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) - \lim_{x\to 0}\left(\frac{b^x-1}{x}\right) \\ &= \log a - \log b\end{aligned}$$

[using part (3) of Important Formulae.]

**Answer: (C)**

**124.** Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that

$$f(x+y) = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}$$

Then, there exists a constant $m$ such that

(A) $f(x) = mx$ only for integer values of $x$
(B) $f(x) = mx$ only for positive integer values of $x$
(C) $f(x) = mx$ only for rational values of $x$
(D) $f(x) = mx$ for all real values of $x$

**Solution:** We have

$$f(x+y) = f(x) + f(y) \Rightarrow f(0) = 0$$

and $$f(-x) = -f(x) \text{ for all } x$$

Let $f(1) = m$. Then

$$f(2) = f(1+1) = f(1) + f(1) = 2m = m(2)$$

$$f(3) = f(1+2) = f(1) + f(2) = m + 2m = 3m = m(3)$$

Hence by induction, $f(n) = mn$ for all positive integers $n$. If $n$ is a negative integer, then

$$\begin{aligned}f(-n) &= -f(n) \\ \Rightarrow f(n) &= -f(-n) = -m(-n) \quad (\because -n > 0) \\ &= mn\end{aligned}$$

That is, $f(x) = mx$ for all integer values of $x$. Now suppose $x = p/q$ is a rational number, $q > 0$ integer. Then

$$f(qx) = f(p) = mp \quad (p \text{ is an integer})$$

Therefore

$$mp = f(qx) \Rightarrow f(x + x + \cdots q \text{ times}) = f(x) + f(x) + \cdots q \text{ times}$$

$$mp = qf(x) \Rightarrow f(x) = m\left(\frac{p}{q}\right) = mx$$

Hence $f(x) = mx$ for all rational numbers $x$.

We now suppose that $x$ irrational.

Let $\{x_n\}$ be a sequence of rational numbers such that $x_n \to x$ as $n\to\infty$. (This is possible due to Theorem 1.49 and the note under it.) Now, by Theorem 1.52 we have

$$f(x_n) \to f(x) \quad \text{as } n\to\infty$$

But $f(x_n) = mx_n$ ($\because x_n$ in rational) and $mx_n \to mx$ as $n\to\infty$. Therefore

$$f(x) = mx$$

Thus $f(x) = mx$ for all irrational $x$. Hence $f(x) = mx$ for all real numbers $x$ where $m = f(1)$.

**Answer: (D)**

**QUICK LOOK**

All straight lines $y = mx$ (except the $y$-axis) are the graphs of the function $f:\mathbb{R}\to\mathbb{R}$ such that $f$ is continuous and

$$f(x+y) = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}$$

**125.** Let $x \neq -1$. For each positive integer $n$, let

$$f_n(x) = \frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \cdots + \frac{x^{2^{n-1}}}{(x+1)(x^2+1)\cdots(x^{2^{n-1}}+1)}$$

for each positive integer $n$. Suppose $f(x) = \lim\limits_{n\to\infty} f(x)$. Then

(A) $f$ is discontinuous at 0
(B) $f$ is discontinuous at 1
(C) $f$ is discontinuous at infinitely many values of $x$
(D) $f$ is continuous for all $x \neq -1$

**Solution:** With reference to Problem 103,

$$f_n(x) = \begin{cases} 1 - \dfrac{x-1}{x^{2^n}-1} & \text{if } x \neq 1 \\ \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} & \text{if } x = 1 \end{cases}$$

**Case I:** $|x| > 1$. That is $x < -1$ or $x > 1$. In this case $x^{2^n} \to \infty$ as $n\to\infty$. Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 1$$

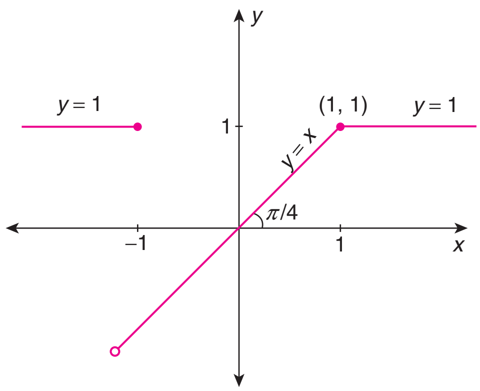


**FIGURE 1.28** Single correct choice type question 125.

**Case II:** $x = 1 \Rightarrow f(x) = \lim_{n\to\infty} f_n(x) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} + \cdots\infty$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

**Case III:** $|x| < 1\,(-1 < x < 1)$. Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 1 - \frac{x-1}{0-1} = x$$

$$f(x) = \begin{cases} 1 & \text{if } x < -1 \\ x & \text{if } -1 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

So $f$ is continuous for all $x \ne -1$. See Fig. 1.28.

**Answer: (D)**

**126.** Suppose $f$ is continuous on the closed interval $[0, 2]$ and $f(0) = f(2)$. Then

(A) $|y-x| = 1$ and $f(x) = f(y)$ for at least one pair $x, y \in [0, 2]$

(B) $|y-x| < 1$ of $f(x) = f(y)$ for same $x, y \in [0, 2]$

(C) $|y-x| < 1$ and $f(x) \ne f(y)$ for infinitely many $x, y \in [0, 2]$

(D) $|y-x| = 1$ and $f(x) = f(y)$ for no pair $x, y \in [0, 2]$

**Solution:** Define $g:[0,1] \to \mathbb{R}$ by

$$g(x) = f(x+1) - f(x)$$

Since $f$ is continuous, $g$ is continuous on $[0, 1]$. Also

$$g(0) = f(1) - f(0),\ g(1) = f(2) - f(1)$$

and $f(0) = f(2) \Rightarrow g(0)$ and $g(1)$ are of opposite sign. Therefore, $g(x) = 0$ for some $x \in (0, 1)$. That is

$$f(x+1) = f(x) \text{ for some } x \in (0, 1)$$

Take $y = x + 1$ so that

$$|y-x| = 1 \quad \text{and} \quad f(y) = f(x)$$

**Answer: (A)**

**127.** Suppose $f$ and $g$ are continuous functions on the closed interval $[a, b]$ such that $f(a) \ge g(a)$ and $f(b) \le g(b)$. Then

(A) $f(x_0) = g(x_0)$ for exactly one $x_0 \in [a, b]$

(B) $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$

(C) $f(x_0) = g(x_0)$ for no value of $x_0 \in [a, b]$

(D) $f(x_0) = g(x_0)$ for infinitely many values of $x_0 \in [a, b]$

**Solution:** Define $Q(x) = g(x) - f(x)$ for $x \in [a, b]$. Therefore $Q$ is continuous on $[a, b]$ and $Q(a) = g(a) - f(a) \le 0$ and $Q(b) = g(b) - f(b) \ge 0$. If either $Q(a) = 0$ or $Q(b) = 0$, we are through. Otherwise $Q(a) < 0$ and $Q(b) > 0$ so that $Q(x) = 0$ for some $x \in (a, b)$. Thus $f(x) = g(x)$ for some $x \in (a, b)$.

**Answer: (B)**

**128.** If the function

$$f(x) = \begin{cases} 2x & \text{for } |x| \le 1 \\ x^2 + ax + b & \text{for } |x| > 1 \end{cases}$$

is continuous, then

(A) $a = 2, b = 1$ (B) $a = -2, b = 1$

(C) $a = 2, b = -1$ (D) $a = 2 = b$

**Solution:** We have

$$f(x) = \begin{cases} x^2 + ax + b & \text{for } x < -1 \\ 2x & \text{for } -1 \le x \le 1 \\ x^2 + ax + b & \text{for } x > 1 \end{cases}$$

Since $f$ is continuous at $x = -1$,

$$\lim_{x\to(-1)-0} f(x) = \lim_{x\to(-1)+0} f(x)$$

Therefore

$$1 - a + b = -2$$

$$a - b = 3 \tag{1.65}$$

Again,

$$\lim_{x\to1-0} f(x) = \lim_{x\to1+0} f(x) \Rightarrow 2 = 1 + a + b$$

So

$$a + b = 1 \tag{1.66}$$

From Eqs. (1.65) and (1.66), $a = 2, b = -1$.

**Answer: (C)**

**129.** If $f$ is a real-valued function satisfying the relation

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x$$

for all real $x \neq 0$, then $\lim_{x\to 0}(\sin x)f(x)$ is equal to

(A) 1 (B) 2
(C) 0 (D) $\infty$

**Solution:** We have

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \tag{1.67}$$

Replacing $x$ with $1/x$, we have

$$2f(x) + f\left(\frac{1}{x}\right) = \frac{3}{x} \tag{1.68}$$

From Eqs. (1.67) and (1.68), we get

$$f(x) = \frac{2}{x} - x$$

Therefore

$$\lim_{x\to 0}(\sin x)f(x) = \lim_{x\to 0}\left(\frac{2\sin x}{x} - x\sin x\right)$$
$$= 2(1) - 0 = 2$$

**Answer: (B)**

**130.** Let

$$A_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

and $\lim_{n\to\infty} A_n = a$. Then

(A) $a = 0$ (B) $0 < a < \frac{1}{2}$

(C) $1.5 < a < 1.7$ (D) $\frac{19}{12} < a < \frac{7}{4}$

**Solution:** We have

$$A_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

Clearly

$$A_n > 1 + \frac{1}{2^2} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{n(n+1)}$$
$$= 1 + \frac{1}{4} + \left[\frac{1}{3} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right] + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 + \frac{1}{4} + \frac{1}{3} - \frac{1}{n+1}$$
$$= \frac{19}{12} - \frac{1}{n+1} \tag{1.69}$$

Also

$$A_n < 1 + \frac{1}{2^2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n}$$
$$= 1 + \frac{1}{4} + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$= 1 + \frac{1}{4} + \frac{1}{2} - \frac{1}{n}$$
$$= \frac{7}{4} - \frac{1}{n} \tag{1.70}$$

Therefore from Eqs. (1.69) and (1.70), we have

$$\frac{19}{12} - \frac{1}{n+1} < A_n < \frac{7}{4} - \frac{1}{n} \text{ for } n \geq 3$$

Taking limits we get

$$\frac{19}{12} - \lim_{n\to\infty}\frac{1}{n+1} < \lim_{n\to\infty} A_n < \frac{7}{4} - \lim_{n\to\infty}\frac{1}{n}$$

Hence

$$\frac{19}{12} < a < \frac{7}{4}$$

**Answer: (D)**

**131.** $P(x)$ is a polynomial such that $P(x) + P(2x) = 5x^2 - 18$. Then

$$\lim_{x\to 3}\left(\frac{P(x)}{x-3}\right) =$$

(A) 6 (B) 9
(C) 18 (D) 0

**Solution:** Since $5x^2 - 18$ is a quadratic polynomial and $P(x) + P(2x) = 5x^2 - 18$ it follows that $P(x)$ must be a quadratic polynomial. Suppose

$$P(x) = ax^2 + bx + c$$

By hypothesis

$$(ax^2 + bx + c) + (4ax^2 + 2bx + c) = 5x^2 - 18$$

or

$$5ax^2 + 3bx + 2c = 5x^2 - 18$$

This gives

$$a = 1, b = 0, c = -9$$

So

$$P(x) = x^2 - 9$$

Therefore

$$\lim_{x\to 3} \frac{P(x)}{x-3} = \lim_{x\to 3}(x+3)$$
$$= 6$$

**Answer: (A)**

**132.** Let

$$f(x) = \lim_{n\to\infty}\left[\frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}\right]$$

for $x > 0$. Then

(A) $f$ is continuous at $x = 1$
(B) $f$ is not continuous at $x = 1$
(C) $f$ has exactly two points of discontinuities in the open interval $(0, 1)$
(D) $f$ is not defined at $x = 1$

**Solution:** If $0 \le x < 1$, then $x^{2n} \to 0$ as $n \to \infty$ so that

$$f(x) = \log(2+x)$$
$$f(1) = \frac{\log 3 - \sin 1}{2}$$

If $x > 1$, then

$$f(x) = \lim_{x\to\infty}\left[\frac{\dfrac{\log(2+x)}{x^{2n}} + \sin x}{\dfrac{1}{x^{2n}} + 1}\right] = \sin x$$

Therefore

$$f(x) = \begin{cases} \log(2+x) & \text{if } 0 \le x < 1 \\ \dfrac{1}{2}[\log 3 - \sin 1] & \text{if } x = 1 \\ \sin x & \text{if } x > 1 \end{cases}$$

Since $\lim\limits_{x\to 1-0} f(x) = \log 3 \ne f(1)$ and $\lim\limits_{x\to 1+0} f(x) = \sin 1 \ne f(1)$, $f$ is not continuous at $x = 1$.

**Answer: (B)**

**133.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous such that

$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2$$

for all $x \in \mathbb{R}$, then

(A) $f(x) = f(0) + \dfrac{9}{16}x^2$

(B) $f(x) = f(0) + \dfrac{4}{9}x^2$

(C) $f(x) = f(0) + \dfrac{9}{16}x^3$

(D) $f(x) = f(0) + \dfrac{16}{9}x^2$

**Solution:** Given relation can be rewritten as

$$f(x) - f\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) + x^2$$

Replacing $x$ with $\dfrac{x}{2}, \dfrac{x}{2^2}, \dfrac{x}{2^3}, \ldots, \dfrac{x}{2^n}$ on both sides and adding all the equations, we have

$$f(x) - f\left(\frac{x}{2^{n+1}}\right) = f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^{n+2}}\right) + x^2\left[\frac{1-\left(\dfrac{1}{2^2}\right)^{n+1}}{1-\dfrac{1}{2^2}}\right] \tag{1.71}$$

Since $f$ is continuous, taking limits on both sides of Eq. (1.71) as $n \to \infty$ we get

$$f(x) - f(0) = f\left(\frac{x}{2}\right) - f(0) + \frac{4}{3}x^2$$

Therefore

$$f(x) - f\left(\frac{x}{2}\right) = \frac{4}{3}x^2 \tag{1.72}$$

In Eq. (1.72), again replace $x$ with $x/2, x/2^2, \ldots, x/2^n$ and add all of them. Then, we have

$$f(x) - f\left(\frac{x}{2^{n+1}}\right) = \frac{4}{3}x^2\left[\frac{1-\left(\dfrac{1}{2^2}\right)^{n+1}}{1-\dfrac{1}{2^2}}\right]$$

Again taking limits on both sides as $n \to \infty$, we get

$$f(x) - f(0) = \frac{16}{9}x^2$$

Therefore

$$f(x) = f(0) + \frac{16}{9}x^2$$

**Answer: (D)**

**134.** $f:[0,1]\to\mathbb{R}$ is continuous and assumes only rational values and $f(0)=3$. Then, the roots of the equation

$$\left(f\left(\frac{1}{2}\right)\right)x^2+\left(f\left(\frac{1}{3}\right)\right)x+f(1)=0$$

are

(A) rational and unequal (B) irrational

(C) imaginary (D) equal

**Solution:** Since $f$ is continuous and in between any two real numbers, there are infinitely many rational and irrational numbers, $f$ must be a constant function. Since $f(0)=3$, we have

$$f(x)=3 \quad \forall x\in[0,1]$$

Therefore the given quadratic equation is $3x^2+3x+3=0$ which has imaginary roots.

**Answer: (C)**

**135.** The values of $a$ and $b$ so that the function

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x, & 0\le x<\dfrac{\pi}{4}\\ 2x\cot x+b, & \dfrac{\pi}{4}\le x\le\dfrac{\pi}{2}\\ a\cos 2x-b\sin x, & \dfrac{\pi}{2}<x\le\pi\end{cases}$$

is continuous for $0\le x\le\pi$ are, respectively,

(A) $\dfrac{\pi}{3},\dfrac{\pi}{12}$ (B) $-\dfrac{\pi}{3},\dfrac{\pi}{12}$

(C) $\dfrac{\pi}{4},\dfrac{\pi}{12}$ (D) $\dfrac{\pi}{6},\dfrac{\pi}{4}$

**Solution:** $f$ is continuous at $x=\pi/4$ implies

$$\lim_{x\to\frac{\pi}{4}-0}f(x)=\lim_{x\to\frac{\pi}{4}+0}f(x)$$

$$\Rightarrow \frac{\pi}{4}+a=\frac{\pi}{2}+b$$

$$\Rightarrow a-b=\frac{\pi}{4} \tag{1.73}$$

Again $f$ is continuous at $\pi/2$ implies

$$\lim_{x\to\frac{\pi}{2}-0}f(x)=\lim_{x\to\frac{\pi}{2}+0}f(x)$$

$$\Rightarrow b=-a-b$$

$$\Rightarrow a+2b=0 \tag{1.74}$$

Equations (1.73) and (1.74) give

$$a=\frac{\pi}{3},\ b=\frac{\pi}{12}$$

**Answer: (A)**

**136.** Let

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0\end{cases}$$

If $f$ is continuous at $x=0$, then

(A) $a=\dfrac{3}{2}, b\ne 0$ is any real number, $c=\dfrac{1}{2}$

(B) $a=-\dfrac{3}{2}, b\ne 0, c=\dfrac{1}{2}$

(C) $a=-\dfrac{5}{2}, b\ne 0, c=\dfrac{1}{2}$

(D) $a=\dfrac{3}{2}, b\ne 0, c=-\dfrac{1}{2}$

**Solution:** $f$ is continuous at $x=0$. This implies

$$\lim_{x\to 0-0}f(x)=\lim_{x\to 0+0}f(x)=c$$

Now

$$\lim_{x\to 0-0}f(x)=\lim\left[\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right]$$

$$=(a+1)+1=c$$

$$\Rightarrow a-c=-2 \tag{1.75}$$

Again

$$\lim_{x\to 0+0}f(x)=\lim_{x\to 0+0}\frac{[1+bx]^{1/2}-1}{bx}$$

$$=\lim_{x\to 0}\frac{1+bx-1}{bx[\sqrt{1+bx}+1]}=\frac{1}{2}=c \tag{1.76}$$

From Eqs. (1.75) and (1.76), we get

$$a=-\frac{3}{2},\ c=\frac{1}{2} \text{ and } b \text{ is any real} \ne 0$$

**Answer: (B)**

**137.** Consider

$$f(x)=\begin{cases}1+x & \text{if } 0\le x\le 2\\ 3-x & \text{if } 2<x\le 3\end{cases}$$

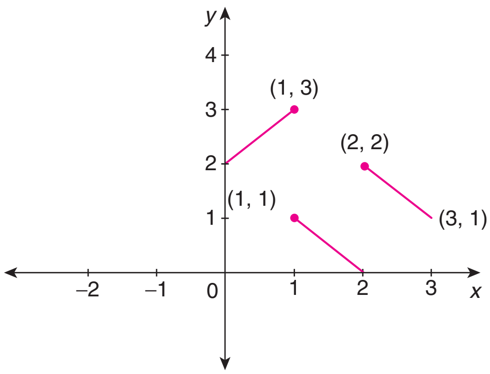


**FIGURE 1.29** Single correct choice type question 137.

Let $g(x) = f(f(x))$. Then the number of values of $x$ at which $f(x)$ is discontinuous is

(A) 0 (B) 1

(C) 2 (D) infinite

**Solution:** We have (see Fig. 1.29)

$$g(x) = f(f(x)) = \begin{cases} 1 + f(x) & \text{if } 0 \le f(x) \le 2 \\ 3 - f(x) & \text{if } 2 < f(x) \le 3 \end{cases}$$

Now

$$\left.\begin{aligned} 0 \le x \le 1 &\Rightarrow 0 \le 1 + x \le 2 \\ \Rightarrow g(x) = f(x+1) &= 1 + (1 + x) = 2 + x \end{aligned}\right\} \quad (1.77)$$

$$1 < x \le 2 \Rightarrow 2 \le 1 + x \le 3$$

$$\Rightarrow g(x) = f(f(x)) = 3 - (1 + x) = 2 - x$$

Therefore

$$1 < x \le 2 \Rightarrow g(x) = 2 - x \quad (1.78)$$

Again

$$2 < x \le 3 \Rightarrow f(x) = 3 - x$$

and $0 \le 3 - x < 1$ so that

$$g(x) = f(3 - x) = 1 + (3 - x) = 4 - x \quad (1.79)$$

From Eqs. (1.77)–(1.79) we have

$$g(x) = \begin{cases} 2 + x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \\ 4 - x & \text{if } 2 < x \le 3 \end{cases}$$

Now

$$\lim_{x \to 1-0} g(x) = 2 + 1 = 3$$

and

$$\lim_{x \to 1-0} g(x) = 2 - 1 = 1$$

Therefore $g$ is discontinuous at $x = 1$. Again

$$\lim_{x \to 2-0} g(x) = 2 - 2 = 0$$

and

$$\lim_{x \to 2+0} g(x) = 4 - 2 = 2$$

So $g(x)$ is discontinuous at $x = 2$ and hence $g$ is discontinuous at $x = 1, 2$.

**Answer: (C)**

**138.** Let $f(x) = [x] + |1 - x|$ for $-1 \le x \le 3$ where $[x]$ is the integral part of $x$. Then, the number of values of $x$ in $[-1, 3]$ at which $f$ is not continuous is

(A) 0 (B) 1

(C) 2 (D) 3

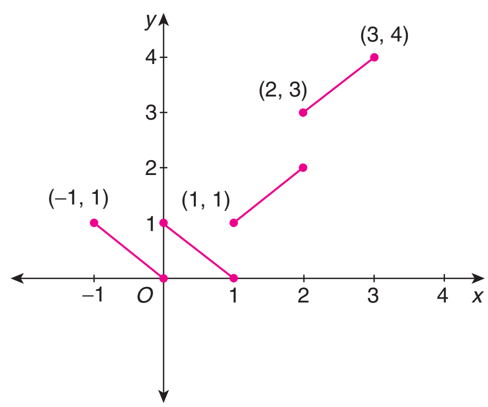


**FIGURE 1.30** Single correct choice type question 138.

**Solution:** We have (see Fig. 1.30)

$$f(x) = \begin{cases} -1 + 1 - x = -x & \text{if } -1 \le x < 0 \\ 1 & \text{if } x = 0 \\ 1 - x & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \\ 1 + x - 1 = x & \text{if } 1 < x < 2 \\ 2 + x - 1 = 1 + x & \text{if } 2 \le x \le 3 \end{cases}$$

More clearly

$$f(x) = \begin{cases} -x & \text{if } -1 \le x < 0 \\ 1 - x & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x < 2 \\ 1 + x & \text{if } 2 \le x \le 3 \end{cases}$$

Clearly $f$ is discontinuous at $x = 0, 1, 2$.

**Answer: (D)**

## Multiple Correct Choice Type Questions

**1.** Let

$$f(x)=[x]\sin\left(\frac{\pi}{[x+1]}\right)$$

where [ ] denotes the greatest integer function. Then

(A) domain of $f$ is $\mathbb{R}-[-1,0)$ (B) $\lim_{x\to 0+0} f(x)=0$

(C) $f$ is continuous on $[0,1)$ (D) $\lim_{x\to 1+0} f(x)=1$

**Solution:**

(A) $f$ is not defined for all those values of $x$ such that

$$[x+1]=0 \Leftrightarrow 0\le x+1<1$$
$$\Leftrightarrow -1\le x<0$$

Therefore domain of $f$ is $\mathbb{R}-[1,0)$. This implies (A) is true.

(B) We have

$$\lim_{x\to 0+0} f(x)=\lim_{\substack{h\to 0\\ h>0}}(f(0+h))$$
$$=\lim_{\substack{h\to 0\\ h>0}}[0+h]\sin\left(\frac{\pi}{[0+h]+1}\right)$$
$$=\lim_{h\to 0}(0\times\sin\pi)$$
$$=0=f(0)$$

Therefore (B) is true.

(C) Now

$$0\le x<1\Rightarrow [x+1]=[x]+1=0+1=1$$

Therefore

$$f(x)=0\times\sin\pi=0$$

for $0\le x<1$ and $f(0)=0$. So $f$ is continuous on $[0,1)$. This means that (C) is true.

(D) Here

$$\lim_{x\to 1+0} f(x)=\lim_{\substack{h\to 0\\ h>0}} f(1+h)$$
$$=\lim_{h\to 0}[1+h]\sin\left(\frac{\pi}{[1+h]+1}\right)$$
$$=1\times\sin\frac{\pi}{2}=1$$

So (D) is also true.

**Answers: (A), (B), (C), (D)**

**2.** Let

$$f(x)=\begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{for } x\ne 0\\ 0 & \text{for } x=0\end{cases}$$

Then

(A) $\lim_{x\to 0} f(x)$ does not exist (B) $f$ is continuous at $x=0$

(C) $\lim_{x\to\infty} f(x)=0$ (D) $\lim_{x\to\infty} f(x)=1$

**Solution:** Since $x\to 0$ and $\sin 1/x$ is a bounded function, by Corollary 1.4

$$x\sin\frac{1}{x}\to 0 \text{ as } x\to 0$$

Therefore

$$\lim_{x\to 0} f(x)=0=f(0)$$

So (B) is true. Now put $x = 1/y$ so that $y\to 0$ as $x\to\infty$. Therefore

$$f(x)=\frac{1}{y}\sin(y)=\frac{\sin y}{y}\to 1 \text{ as } y\to 0$$

So

$$\lim_{x\to\infty} f(x)=1$$

Therefore (D) is also true.

**Answers: (B), (D)**

**3.** Suppose

$$f(x)=\begin{cases}\dfrac{\log(1+2x)-2\log(1+x)}{x^2} & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$$

Then

(A) $\lim_{x\to 0} f(x)$ exists and is equal to 1

(B) $\lim_{x\to 0} f(x)$ exists

(C) $\lim_{x\to 0} f(x)$ exists and is equal to $-1$

(D) $f$ is continuous at $x=0$

**Solution:** Note that $f$ is defined for all $x>-1/2$. Now

$$f(x)=\frac{\log\left(\dfrac{1+2x}{1+2x+x^2}\right)}{x^2} \quad \text{when } x\ne 0$$
$$=\frac{\log\left(1-\left(\dfrac{x}{1+x}\right)^2\right)}{x^2}$$
$$=\left[\frac{\log\left(1-\left(\dfrac{x}{1+x}\right)^2\right)}{-\left(\dfrac{x}{1+x}\right)^2}\right]\times\frac{-1}{(1+x)^2}$$

Therefore, by part (2) of Important Formulae

$$\lim_{x\to 0} f(x) = (1)(-1) = -1$$

**Answers: (B), (C)**

**4.** Let

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then

(A) $\lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = 1$

(B) $\lim_{x\to 0+0} f(x) = 0$

(C) $\lim_{x\to 0-0} f(x) = -1$

(D) $\lim_{x\to 0-0} f(x) = 0$

**Solution:** We have

$$\lim_{x\to 0-0} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(0-h) = \lim_{h\to 0}\left(\frac{-h}{1+e^{-1/h}}\right) = \frac{0}{1+0} = 0$$

$$\lim_{x\to 0+0} f(x) = \lim_{\substack{h\to 0\\ h>0}} f(0+h) = \lim_{h\to 0}\left(\frac{h}{1+e^{1/h}}\right) = 0\times 0 = 0$$

$$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\left(\frac{\dfrac{-h}{1+e^{-1/h}} - 0}{-h}\right)$$

$$= \lim_{h\to 0}\left[\frac{1}{1+\dfrac{1}{e^{1/h}}}\right] = \frac{1}{1+0} = 1$$

**Answers: (A), (B), (D)**

**5.** If $f(x) = 1+|\sin x|$ then

(A) $\lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h}$ does not exist

(B) $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ does not exist

(C) $f$ is continuous for all real $x$

(D) $f$ is continuous for all $x \neq 0$

**Solution:**

(A) We have

$$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{1+|\sin(-h)|-1}{-h}$$

$$= \lim_{h\to 0} \frac{|\sin h|}{-h} = \lim_{h\to 0}\left(\pm\frac{\sin h}{-h}\right) = (\mp)1$$

according as $h \to 0+0$ or $h \to 0-0$. So (A) is true. Similarly

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \pm 1$$

according as $h \to 0+0$ or $h \to 0-0$. So (B) is true. Since $\sin x$ is continuous for all real $x$, $|\sin x|$ is also continuous for all real $x$ and hence (C) is true.

**Answers: (A), (B), (C)**

**6.** Let

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $g(x) = xf(x)$. Then

(A) $f$ is continuous at $x = 0$

(B) $\lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ exists finitely

(C) $g$ is continuous at $x = 0$

(D) $\lim_{h\to 0} \dfrac{g(0+h)-g(0)}{h}$ exists finitely

**Solution:**

(A) We have (by Corollary 1.4)

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0$$

$$= f(0)$$

Therefore $f$ is continuous at $x = 0$. So (A) is true.

(B) We have

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\sin\dfrac{1}{h} - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}$$

This limit does not exist and so (B) is not true.

(C) Since $x$ is continuous at $x = 0$ and $f$ is continuous at $x = 0$, it follows that $g(x)$ is continuous at $x = 0$. So (C) is true.

(D) We have

$$\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{h^2\sin\dfrac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0}\left(h\sin\frac{1}{h}\right)$$

$$= 0 = g(0)$$

Therefore (D) is true.

**Answers: (A), (C), (D)**

**7.** Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \dfrac{1}{n} & \text{if } x = \dfrac{m}{n} \text{ rational, } n > 0 \\ & \text{and } \dfrac{m}{n} \text{ is in lowest terms} \end{cases}$$

Then

(A) $f$ is continuous at $x = 0$

(B) $f$ is discontinuous at all non-zero rationals

(C) $f$ is continuous at all irrational numbers

(D) $f$ is continuous at $x = 0$ only

**Solution:**

**Case I:** $a$ is irrational. Therefore $f(a) = 0$. Suppose $\{x_n\}$ is a sequence of irrational numbers such that $x_n \to a$ as $n \to \infty$, then

$$f(x_n) = 0 = f(a)$$

so that $f(x_n) \to f(a)$ as $n \to \infty$. Now suppose $\{y_n\}$ is a sequence of rational numbers such that $y_n \to a$ as $n \to \infty$. Let $y_n = m/q, q > 0$ and $m/q$ is in lowest terms. Therefore

$$f(y_n) = \frac{1}{q} \to 0 \quad \text{as } n \to \infty$$

because $q \to \infty$ as $n \to \infty$. So

$$f(y_n) \to 0 = f(a) \quad \text{as } n \to \infty$$

Hence by Theorem 1.52, $f$ is continuous at all irrational numbers because any sequence of reals tending to $a$ contains subsequences of rationals and irrationals. So (C) is true.

**Case II:** $a$ is rational. Suppose $a = 0$. Let $\{x_n\}$ and $\{y_n\}$ be sequences of rational and irrational numbers, respectively, such that

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$$

Let $x_n = \{m/q\}$ where $m/q$ is in lowest terms so that

$$f(x_n) = \frac{1}{q} \to 0 \text{ as } n \to \infty$$

Therefore

$$\lim_{n\to\infty} f(x_n) = \lim_{q\to\infty} \frac{1}{q} = 0 = f(0)$$

Also

$$f(y_n) = 0 \ \forall n \Rightarrow \lim_{n\to\infty} f(y_n) = 0 = f(0)$$

Hence $f$ is continuous at 0 because any sequence of reals contains subsequences of rationals as well as irrationals. Suppose $a \neq 0$ be a rational. Assume that $f$ is continuous at $a$. Let $\{z_n\}$ be a sequence of irrational numbers such that $\lim\limits_{n\to\infty} z_n = a$.

Since $f$ is continuous at $a$, by Theorem 1.52

$$\lim_{n\to\infty} f(z_n) = f(a) = \frac{1}{q}$$

where $a = m/q$ is in its lowest terms. But

$$f(z_n) = 0 \ \forall n \Rightarrow \frac{1}{q} = 0$$

which is a contradiction. Therefore $f$ is not continuous at any non-zero rational. Hence (A) and (B) are true.

***Note:*** The above function $f$ is called **Thomae's function.**

**Answers: (A), (B), (C)**

**8.** Let

$$f(x) = \begin{cases} 3x^2 - 1 & \text{if } x < 0 \\ ax + b & \text{if } 0 \le x \le 1 \\ \sqrt{x+3} & \text{if } x > 1 \end{cases}$$

If $f$ is continuous for all real $x$, then

(A) $a = 4$ (B) $b = -1$

(C) $a = 3$ (D) $b = 1$

**Solution:** Since $f$ is continuous for all real $x$, it must be continuous at 0 and 1 also.

$$-1 = \lim_{x\to 0} f(x) = a(0) + b \Rightarrow b = -1$$

Also

$$a(1) - 1 = \lim_{x\to 1} f(x) = \sqrt{1+3} = 2 \Rightarrow a = 3$$

**Answers: (B), (C)**

**9.** Which of the following statement(s) is (are) true?

(A) If $|f|$ is continuous at $a$, then $f$ need not be continuous at $a$.

(B) If $f$ and $g$ are functions such that $f + g$ is continuous in their common domain, then $f$ and $g$ may not be continuous.

(C) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all rational $x$. Then $f(x) = 0$ for all $x$.

(D) Let $f$ and $g$ be two continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $f(x) = g(x)$ for all rationals $x$. Then $f(x) = g(x)$ for all $x$.

**Solution:**

(A) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Let $a$ be any real number. Since in every neighbourhood of "$a$", there are infinitely many rational numbers and infinitely many irrational numbers (See Theorems 0.14 and 0.17), it follows that $f$ is not continuous at $a$. But $|f|(x) = 1$ for all $x$ is continuous at all real numbers. Hence (A) is true.

(B) Define $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x)=\begin{cases}-1 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$$

Then both $f$ and $g$ are discontinuous at every real $x$ whereas $(f+g)(x)=0\ \forall\, x\in\mathbb{R}$ is continuous for all real $x$. So (B) is true.

(C) Let $a\in\mathbb{R}$ be irrational. Choose a sequence $\{x_n\}$ of rational numbers (see Theorem 1.49 and the note under it) such that $x_n\to a$ as $n\to\infty$. Since $f$ is continuous at $x=a$, by Theorem 1.52, we have

$$f(x_n)\to f(a) \text{ as } n\to\infty$$

But

$$\lim_{n\to\infty} f(x_n)=0\Rightarrow f(a)=0$$

Hence $f(a)=0$ for all irrational $a$. Therefore $f(x)=0$ for all real $x$, because any sequence of reals tending to $a$ contains subsequences of rationals and irrationals. Therefore (C) is true.

(D) Take $h(x)=f(x)-g(x)$ so that by the above result [i.e., (C)] $h(x)=0$ for all $x$. Therefore (D) is true.

**Answers: (A), (B), (C), (D)**

**10.** Let

$$f(x)=\begin{cases}\dfrac{2x+2}{x^2+4x+3} & \text{for } x\neq -1,-3\\[2mm] \dfrac{1}{2} & \text{for } x=-1\\[2mm] k & \text{for } x=-3\end{cases}$$

Then

(A) $f$ is not continuous at $x=-1$

(B) $f$ is discontinuous at $x=-3$ for any value of $k$

(C) $f$ is continuous at $x=-1$, if $f(-1)$ is defined to be 1

(D) The number of values of $x$ at which $f$ is discontinuous is 2

**Solution:** We have

$$f(x)=\frac{2(x+1)}{(x+1)(x+3)}=\frac{2}{x+3}\quad \text{if } x\neq -1$$

Now

$$\lim_{x\to -1} f(x)=\frac{2}{-1+3}=1$$

Hence $f$ is discontinuous at $x=-1$ because $f(-1)$ is 1/2. So (A) is true.

Now if $f(-1)$ is defined to be 1, then $f$ is continuous at $x=-1$. So (C) is true.

Again, since $\lim\limits_{x\to -3} f(x)$ does not exist, it follows that $f$ is not continuous at $x=-3$, whatever value of $k$ may be. So (B) is true.

Finally, $f$ is discontinuous at $x=-1$ and $-3$ and at all other values of $x$, $f$ is continuous implies the number values of $x$ at which $f$ is discontinuous is two. Therefore (D) is true.

**Answers: (A), (B), (C), (D)**

**11.** Let

$$f(x)=\begin{cases}x-1 & \text{if } x<1\\ 2x-1 & \text{if } 1\le x\le 2\\ x+1 & \text{if } x>2\end{cases}$$

Then

(A) $f$ is discontinuous at $x=1$

(B) $f$ is continuous at $x=2$

(C) $f$ has intermediate value property on $[0,2]$

(D) $f$ has intermediate value property on $[1,2]$

**Solution:** $f$ is discontinuous at $x=1$, because

$$\lim_{x\to 1-0} f(x)=0 \quad\text{and}\quad \lim_{x\to 1+0} f(x)=1$$

So (A) is true. Now

$$\lim_{x\to 2-0} f(x)=2(2)-1=3$$

and

$$\lim_{x\to 2+0} f(x)=2+1=3$$

imply that $f$ is continuous at $x=2$. So (B) is true.

Since $f$ is discontinuous at $x=1$, $f$ cannot enjoy the intermediate value property on $[0, 2]$ because $[0, 1]$ is contained in $[0,2]$. As $f(x)=2x-1$ is continuous on $[1,2]$, it will have intermediate value property in $[1,2]$. Hence, (C) is not true whereas (D) is true. See Fig. 1.31.

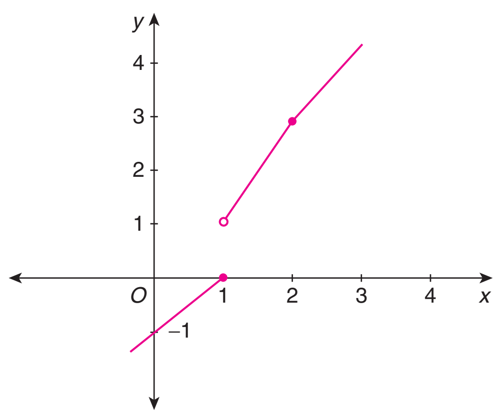


**FIGURE 1.31** Multiple correct choice type question 11.

**Answers: (A), (B), (D)**

**12.** Let $f(x)=[x^2]-[x]^2$ where [ ] is the greatest integer function. Then

(A) $f$ is discontinuous at all integer values of $x$

(B) $f$ is continuous at $x=1$

(C) $f$ is discontinuous at $x = 0$

(D) $f$ is discontinuous at all integer values of $x \neq 1$

**Solution:** We have

$$\lim_{x\to 1-0} f(x) = \lim_{\substack{h\to 0\\ h>0}}([(1-h)^2]-[1-h]^2)$$
$$= \lim_{\substack{h\to 0\\ h>0}}([1-2h+h^2]-1)$$
$$= \lim_{h\to 0}(1+[h^2-2h]-1)$$
$$= 1-1 = 0 \quad (\because [h^2-2h]=0)$$

$$\lim_{x\to 1+0} f(x) = \lim_{\substack{h\to 0\\ h>0}}([(1+h)^2]-[1+h]^2)$$
$$= \lim_{\substack{h\to 0\\ h>0}}([1+2h+h^2]-1)$$
$$= 1-1 \quad (\because 1<1+2h+h^2<2 \text{ for very small positive values of } h)$$
$$= 0$$

Therefore

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1+0} f(x) = 0 = f(1).$$

So $f$ is continuous at $x = 1$. This implies that (B) is true. Now

$$\lim_{x\to 0-0} f(x) = \lim_{\substack{h\to 0\\ h>0}}([(0-h)^2]-[0-h]^2)$$
$$= 0-1 \qquad (\because [0-h]=-1)$$
$$= -1$$

$$\lim_{x\to 0+0} f(x) = \lim_{\substack{h\to 0\\ h\to 0}}([(0+h)^2]-[0+h]^2)$$
$$= 0-0 = 0$$

Therefore $f$ is discontinuous at $x = 0$ and so (C) is true.

Let $n$ be an integer. Then for very small positive $h$, we have

$$n^2 - 1 < (n-h)^2 < n^2$$

so that

$$[(n-h)^2] = n^2 - 1$$

and

$$[n-h]^2 = (n-1)^2$$

Hence

$$\lim_{x\to n-0} f(x) = \lim_{\substack{h\to 0\\ h>0}}([(n-h)^2]-[n-h]^2)$$
$$= (n^2-1)-(n-1)^2$$
$$= 2n-2 \neq 0 \text{ for } n \neq 1$$

Again

$$\lim_{x\to n+0} f(x) = \lim_{\substack{h\to 0\\ h>0}}([(n+h)^2]-[n+h]^2)$$
$$= n^2 - n^2 = 0$$

Because

$$n^2 < (n+h)^2 < n^2+1$$

for small positive values of $h$ implies that $[(n+h)^2] = n^2$. Note that at $n = 1$, and so

$$\lim_{x\to 1-0} f(x) = 2(1) - 2 = 0$$

Therefore (D) is true.

**Answers: (B), (C), (D)**

**QUICK LOOK**

The function $f(x) = [x^2] - [x]^2$ is discontinuous at all integer values of $x \neq 1$ and is continuous at $x = 1$.

**13.** Let

$$f(x) = \lim_{x\to\infty} \frac{1-x^{2n}}{1+x^{2n}}$$

Then

(A) $f$ is continuous for all $x < -1$

(B) $f$ is discontinuous at $x = -1, 1$

(C) $f$ is continuous for $-1 < x < 1$

(D) $f$ is continuous for all $x > 1$

**Solution:** If $|x| > 1$, then $\lim_{n\to\infty} x^{2n} = +\infty$ so that

$$\lim_{n\to\infty} \frac{1}{x^{2n}} = 0$$

Therefore $f(x) = -1$ for $|x| > 1$. By definition $f(\pm 1) = 0$. If $-1 < x < 1$, then $\lim_{n\to\infty}(x^{2n}) = 0$ so that $f(x) = 1$. Therefore

$$f(x) = \begin{cases} -1 & \text{for } x < -1 \\ 0 & \text{for } x = -1 \\ 1 & \text{for } -1 < x < 1 \\ 0 & \text{for } x = 1 \\ -1 & \text{for } x > 1 \end{cases}$$

So $f$ is continuous at all $x \neq -1, 1$. Figure 1.32 shows the graph of $f(x)$.

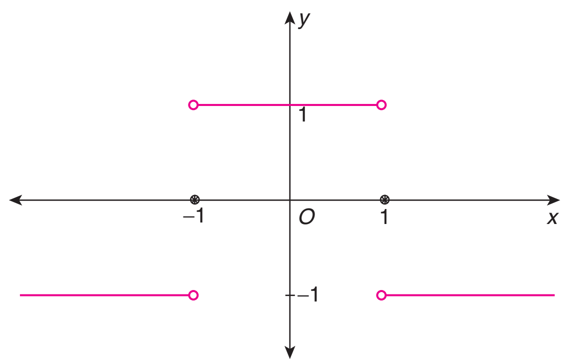


**FIGURE 1.32** Multiple correct choice type question 13.

**Answers: (A), (B), (C), (D)**

**14.** Let $x_1 = 1$ and

$$x_{n+1} = \frac{1}{4}(2x_n + 3)$$

for $n \geq 1$. Then

(A) $x_n < 2$ for all $n$

(B) $\{x_n\}$ is monotonic increasing

(C) $\lim_{n\to\infty}(x_n) = 2$

(D) $\lim_{n\to\infty}(x_n) = 3/2$

**Solution:** We have $x_1 = 1 < 2$ and

$$x_2 = \frac{1}{4}(2x_1 + 3) = \frac{5}{4} < 2$$

Therefore $x_n < 2$ for $n = 1$ and 2. Assume that $x_m < 2$ for same $n = m$. Now

$$x_{m+1} = \frac{1}{4}(2x_m + 3) < \frac{1}{4}(2(2) + 3) = \frac{7}{4} < 2$$

Hence by induction, $x_n < 2$ for all $n$ and so (A) is true. Also

$$x_1 = 1 < x_2 = \frac{5}{4}$$

Assume that $x_m < x_{m+1}$. Now

$$\begin{aligned} x_{m+1} &= \frac{1}{4}(2x_m + 3) \\ &< \frac{1}{4}(2x_{m+1} + 3) \quad (\because x_m < x_{m+1}) \\ &= x_{m+2} \end{aligned}$$

Therefore $\{x_n\}$ is increasing and so (B) is true. Thus $\{x_n\}$ is an increasing sequence and bounded above. Therefore by Theorem 1.43 the sequence $\{x_n\}$ converges to a finite limit, say $l$. Therefore

$$l = \lim_{n\to\infty}(x_{n+1}) = \lim_{n\to\infty}\frac{1}{4}(2x_n + 3) = \frac{1}{4}(2l + 3)$$

Hence $l = 3/2$ and so (D) is true.

**Answers: (A), (B), (D)**

**15.** Let $x_1 \geq 2$ be a fixed number. Define $x_{n+1} = 1 + \sqrt{x_n - 1}$.

(A) Then $\{x_n\}$ is a decreasing sequence

(B) Then $x_n \geq 2$ for all $n$

(C) If $x_1 = 2$, then $\lim_{n\to\infty} x_n = 2$

(D) If $x_1 > 2$, then $\lim_{n\to\infty} x_n = 2$

**Solution:** We have

$$\begin{aligned} x_2 - x_1 &= (1 + \sqrt{x_1 - 1}) - x_1 \\ &= -[(x_1 - 1) - \sqrt{x_1 - 1}] \\ &< 0 \end{aligned}$$

Therefore $x_2 < x_1$. Assume that $x_{n+1} < x_n$. Now

$$\begin{aligned} x_{n+2} - x_{n+1} &= (1 + \sqrt{x_{n+1} - 1}) - (1 + \sqrt{x_n - 1}) \\ &= \sqrt{x_{n+1} - 1} - \sqrt{x_n - 1} < 0 \qquad (\because x_{n+1} < x_n) \end{aligned}$$

Therefore

$$x_{n+2} < x_{n+1}$$

So $\{x_n\}$ is a decreasing sequence. This implies (A) is true. Now

$$x_2 = 1 + \sqrt{x_1 - 1} \quad \text{and} \quad x_1 \geq 2 \Rightarrow x_2 \geq 2$$

Assume $x_n \geq 2$. Now,

$$x_{n+1} = 1 + \sqrt{x_n - 1} \geq 1 + \sqrt{2 - 1} = 2$$

Therefore $x_n \geq 2$ for all $n$. So $\{x_n\}$ is a decreasing sequence which is bounded below by 2. Hence $\{x_n\}$ is convergent by Theorem 1.42. Suppose the limit is $l$. Then

$$\begin{aligned} l &= \lim_{n\to\infty}(x_{n+1}) \\ &= \lim_{n\to\infty}(1 + \sqrt{x_n - 1}) \\ &= 1 + \sqrt{l - 1} \end{aligned}$$

So

$$l^2 - 3l + 2 = 0 \Rightarrow l = 1 \text{ or } 2$$

But $x_n \geq 2$ for all $n$ implies that $l = 2$. So (D) is true. Also when $x_1 = 2$, then $x_n = 2$ for all $x$ so that $\lim_{m\to\infty} x_n = 2$. Hence (C) is true.

**Answers: (A), (B), (C), (D)**

## Matrix-Match Type Questions

**1.** Match the items of Column I with those of Column II.

| Column I | Column II |
|---|---|
| (A) $\lim\limits_{x\to 0}\dfrac{2^x-1}{\sqrt{1+x}-1}$ is | (p) $2\log 2$ |
| (B) $f(x)=\dfrac{\sqrt{1+2x}-\sqrt{3x}}{\sqrt{3+x}-2\sqrt{x}}$. Then $\lim\limits_{x\to 1}f(x)$ equals | (q) 1 |
| (C) $\lim\limits_{x\to 0}\left(\dfrac{2\sin x-\sin 2x}{x^3}\right)$ is | (r) 2 |
| (D) $x_1=1$ and $x_{n+1}=\sqrt{2+x_n}$. Define $y_n=\dfrac{x_1+x_2+\cdots+x_n}{n}$ Then $\lim\limits_{n\to\infty}y_n$ is | (s) $\dfrac{4}{3}(\sqrt{3}-\sqrt{2})$ |
| | (t) $\dfrac{2}{3\sqrt{3}}$ |

**Solution:**

(A)
$$\lim_{x\to 0}\left(\frac{2^x-1}{\sqrt{1+x}-1}\right)=\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\left(\frac{x}{\sqrt{1+x}-1}\right)$$
$$=\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\frac{1}{\left(\dfrac{\sqrt{1+x}-1}{x}\right)}$$
$$=(\log 2)\cdot\frac{1}{\left(\dfrac{1}{2}\right)}$$
$$=2\log 2$$

**Answer: (A) → (p)**

(B)
$$f(x)=\frac{\sqrt{1+2x}-\sqrt{3x}}{\sqrt{3+x}-2\sqrt{x}}$$
$$=\left(\frac{1+2x-3x}{3+x-4x}\right)\left(\frac{\sqrt{3+x}+2\sqrt{x}}{\sqrt{1+2x}+\sqrt{3x}}\right)$$
$$=\frac{1}{3}\left(\frac{\sqrt{3+x}+2\sqrt{x}}{\sqrt{1+2x}+\sqrt{3x}}\right)$$

Therefore

$$\lim_{x\to 1}f(x)=\frac{1}{3}\frac{(2+2)}{2\sqrt{3}}=\frac{2}{3\sqrt{3}}$$

**Answer: (B) → (t)**

(C)
$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}=\lim_{x\to 0}\frac{2\sin x}{x}\left(\frac{1-\cos x}{x^2}\right)$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{4\sin^2\dfrac{x}{2}}{x^2}\right)$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2$$
$$=1\times 1=1$$

**Answer: (C) → (q)**

(D) $x_1=1$, $x_2=\sqrt{3}\Rightarrow x_1<x_2$. Assume that $x_n<x_{n+1}$. Therefore

$$x_{n+2}=\sqrt{2+x_{n+1}}$$
$$\Rightarrow x_{n+2}-x_{n+1}=\sqrt{2+x_{n+1}}-\sqrt{2+x_n}$$

which is positive, because $x_n<x_{n+1}$. So $\{x_n\}$ is an increasing sequence and bounded above by 2. By Theorem 1.42, $\{x_n\}$ converges to a finite limit, say $L$. So

$$L=\lim_{n\to\infty}(x_{n+1})=\lim_{n\to\infty}\sqrt{2+x_n}=\sqrt{2+\mathrm{L}}$$

Now

$$L^2-L-2=0$$
$$\Rightarrow(L-2)(L+1)=0$$
$$\Rightarrow L=2\quad(\because x_n<2\ \forall n)$$

By Cauchy's first theorem on limits (Theorem 1.45)

$$\lim_{n\to\infty}y_n=2$$

**Answer: (D) → (r)**

**2.** Match the items of Column I to those of Column II.

| Column I | Column II |
|---|---|
| (A) $\lim\limits_{x\to\infty}\left(\dfrac{\sin x+\cos x}{x^2}\right)$ is | (p) 1 |
| (B) If $l=\lim\limits_{x\to 0}\left(\dfrac{1}{x\cdot\sqrt[3]{8+x}}-\dfrac{1}{2x}\right)$, then $(48)l$ is | (q) 0 |
| (C) If [ ] denotes greatest integer function, then $\lim\limits_{x\to 2}([x-2]+[2-x]+x)=$ | (r) –1 |
| (D) $\lim\limits_{x\to 1-0}([x]+\lvert x\rvert)$ ($[x]$ is integral part of $x$) is equal to | (s) 2 |

**Solution:**

(A) $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$ (i.e., sin $x$ + cos $x$ is bounded) and $1/x^2 \to 0$ as $x \to \infty$. This implies

$$\lim_{x\to\infty} \frac{\sin x + \cos x}{x^2} = 0$$

**Answer: (A) → (q)**

(B) We have

$$l = \lim_{x\to 2} \frac{2-\sqrt[3]{8+x}}{2x \times \sqrt[3]{8+3}}$$

$$= -\frac{1}{2}\lim_{x\to 2}\left(\frac{\sqrt[3]{8-x}-\sqrt[3]{8}}{x}\right)\frac{1}{\sqrt[3]{8+x}}$$

$$= -\frac{1}{2} \times \frac{1}{3} 8^{-2/3} \times \frac{1}{2}$$

$$= -\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{2}$$

$$= -\frac{1}{48}$$

Therefore (48) $l = -1$.

**Answer: (B) → (r)**

(C) In a neighbourhood (very small) of 2,

$$[x-2]+[2-x] = -1$$

$$\Rightarrow [x-2]+[2-x]+x = -1+x$$

Therefore

$$\lim_{x\to 2}([x-2]+[2-x]+x) = -1+2 = 1$$

**Answer: (C) → (p)**

(D) $[1-h] = 0$ when $h > 0$ is very small. This implies

$$\lim_{x\to 1-0}([x]+|x|) = 0+1 = 1$$

**Answer: (D) → (p)**

**3.** Match the items of Column I to those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $f(x) = 2x^3 - 4x^2 + 5x - 4$ has a zero in the interval | (p) 0 |
| (B) If $f(x) = \sqrt{16-x^2}$ then $f(x)$ assumes the value $\sqrt{7}$ at $x$ equals to | (q) [1, 2] |

(*Continued*)

| *Column I* | *Column II* |
|---|---|
| (C) $f(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1 \\ x-1 & \text{for } x > 1 \end{cases}$ Then $f$ is not continuous at $x$ equal to | (r) [−1, 3] |
| (D) $f(x) = \begin{cases} \dfrac{x^2-a^2}{x-a} & \text{if } x \ne a \\ 0 & \text{if } x = a \end{cases}$ Then $f$ is continuous at $x = a$, if the value of $a$ is | (s) 1 |
| | (t) −3 |

**Solution:**

(A) $f(1) = 2(1) - 4(1) + 5(1) - 4 = -1$

$f(2) = 2(8) - 4(4) + 5(2) - 4 = 6$

$f(1)\ f(2) < 0$ and $f$ is continuous on [1, 2]. This implies $f(x)$ has a zero in $(1, 2)$.

**Answer: (A) → (q)**

(B) $f = \sqrt{16-x^2}$ is continuous on [−4, 0] and $f(0) = 4$, $f(-4) = 0$. Since $0 < \sqrt{7} < 4$, by intermediate value theorem for continuous functions on a closed interval,

$$f(x) = \sqrt{7} \text{ for some } x \in (-4, 0)$$

$$\Rightarrow \sqrt{16-x^2} = \sqrt{7} \text{ for some } x \in (-4, 0)$$

$$\Rightarrow x = \pm 3$$

Therefore $x = -3$.

**Answer: (B) → (t)**

(C) For

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

$$\lim_{x\to 1-0} f(x) = 2$$

and $\displaystyle\lim_{x\to 1+0} f(x) = 1-1 = 0$

implies that $f(x)$ is not continuous at $x = 1$.

**Answer: (C) → (s)**

(D) $\lim_{x\to a} f(x) = 2a$ and $f(a) = 0 \Rightarrow f$ is continuous at $x = a$, only when $a = 0$.

**Answer: (D) → (p)**

**4.** Match the items of Column I to those of Column II.

| Column I | Column II |
|---|---|
| (A) $\lim_{x\to 0}(1+\sin x)^{\operatorname{cosec} x}$ is | (p) 0 |
| (B) $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x}$ is | (q) $e$ |
| (C) $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$ is | (r) 1 |
| (D) $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le -\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \log x, & x>1\end{cases}$ | (s) $e^2$ |
| Then the number of points at which $f$ is not continuous is | (t) $-e$ |

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**Solution:**

(A) $$\lim_{x\to 0}(1+\sin x)^{\operatorname{cosec} x}=\lim_{x\to 0}(1+\sin x)^{1/\sin x}=e$$

**Answer: (A) → (q)**

(B) $$\left(\frac{x+2}{x+1}\right)^{x}=\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)^{x}$$

$$=\frac{\left[\left(1+\frac{2}{x}\right)^{x/2}\right]^{2}}{\left(1+\frac{1}{x}\right)^{x}}$$

Therefore

$$\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x}=\frac{\lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)^{x/2}\right]^{2}}{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}}$$

$$=\frac{e^2}{e}=e$$

**Answer: (B) → (q)**

(C) Let

$$f(x)=\left(\frac{(1+\tan x)}{1+\sin x}\right)^{\operatorname{cosec} x}$$

$$=\frac{\left[(1+\tan x)^{1/\tan x}\right]^{\sec x}}{(1+\sin x)^{1/\sin x}}$$

Therefore

$$\lim_{x\to 0} f(x)=\frac{e}{e}=1$$

**Answer: (C) → (r)**

(D) We have

$$\lim_{x\to\left(-\frac{\pi}{2}\right)-0} f(x)=-\left(-\frac{\pi}{2}\right)-\frac{\pi}{2}$$

$$=0=-\cos\left(-\frac{\pi}{2}\right)$$

$$=\lim_{x\to-\frac{\pi}{2}+0} f(x)$$

Therefore $f$ is continuous at $x=-\pi/2$. Now

$$\lim_{x\to 0-0} f(x)=-\cos 0=-1=0-1=\lim_{x\to 0+0} f(x)$$

Therefore $f$ is continuous at $x=0$. Again

$$\lim_{x\to 1-0} f(x)=1-1=0=\log 1=\lim_{x\to 1+0} f(x)$$

Therefore $f$ is continuous at $x=1$. This implies that $f$ is continuous for all $x$.

**Answer: (D) → (p)**

**5.** Match the items of Column I to those of Column II.

| Column I | Column II |
|---|---|
| (A) Let $f(x)=\begin{vmatrix}1 & \tan x & 1\\ -\tan x & 1 & \tan x\\ -1 & -\tan x & 1\end{vmatrix}$ Then the number of values of $x$ such that $0\le x<\pi/2$ at which $f$ is *not* continuous is | (p) 2 |
| (B) Let $f(x)=\begin{cases}\dfrac{\sin\frac{9x}{2}}{\sin\frac{x}{2}} & \text{if } x\ne 0\\ k & \text{if } x=0\end{cases}$ If $k$ is the value such that $f$ is continuous at $x=0$, then $2k/9$ is | (q) 3 |

(*Continued*)

| *Column I* | *Column II* |
|---|---|
| (C) Let $f(x)=3x-5$. Then $\lim_{x\to 1} f^{-1}(x)$ is equal to | (r) 1 |
| (D) If $f(x)=[x]+\lvert 1-x\rvert$ where $[x]$ is the integral part of $x$, then $\lim_{x\to 2-0} f(x)$ is | (s) −1 |
| | (t) 0 |

**Solution:**

(A) Add $R_3$ to $R_1$. Then

$$f(x)=\begin{vmatrix} 0 & 0 & 2 \\ -\tan x & 1 & \tan x \\ -1 & -\tan x & 1 \end{vmatrix}$$

$$=2(\tan^2 x+1)=2\sec^2 x$$

which is continuous for all $x$ in $[0, \pi/2]$.

**Answer: (A) → (t)**

(B) We have

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left[\left(\frac{\sin\frac{9x}{2}}{\frac{9x}{2}}\right)\frac{9x}{2}\cdot\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)\cdot\frac{2}{x}\right]$$

$$=9=k$$

Therefore

$$\frac{2}{9}(k)=2$$

**Answer: (B) → (p)**

(C) We have

$$f(x)=3x-5$$

$$\Rightarrow f^{-1}(x)=\frac{x+5}{3}$$

Therefore

$$\lim_{x\to 1} f^{-1}(x)=\frac{1+5}{3}=2$$

**Answer: (C) → (p)**

(D) In the left neighbourhood 2, $[2-h]=1$. Therefore

$$\lim_{x\to 2-0} f(x)=\lim_{\substack{h\to 0\\ h>0}}([2-h]+|1-(2-h)|)$$

$$=1+1=2$$

**Answer: (D) → (p)**

**6.** Match the items of Column I to those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)$ equals | (p) 1 |
| (B) Let $s_n=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+n)^2}$. Then $\lim_{n\to\infty}(s_n)$ is equal to | (q) 1/2 |
| (C) Let $x^1=1$ and $x_{n+1}=\frac{4+3x_n}{3+2x_n}$ for $n\ge 1$. Then $\lim_{n\to\infty} x_n$ is equal to | (r) 0 |
| (D) $\lim_{n\to\infty}\left(\frac{n}{2^n}\right)$ is | (s) 2 |
| | (t) $\sqrt{2}$ |

**Solution:**

(A) Let $s_n=\frac{1}{2n-1}$ so that $\lim_{n\to\infty} s_n=0$. Therefore by Cauchy's first theorem on limits (Theorem 1.45)

$$\frac{s_1+s_2+\cdots+s_n}{n}=\frac{1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}}{n}\to 0 \text{ as } n\to\infty$$

**Answer: (A) → (r)**

(B) We have

$$s_n=\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+n)^2}$$

$$\le\frac{1}{n^2}+\frac{1}{n^2}+\cdots\ n \text{ times}$$

$$=\frac{n}{n^2}=\frac{1}{n}$$

Therefore

$$s_n\le\frac{1}{n}\quad\forall n$$

Also

$$s_n\ge\frac{1}{(n+n)^2}+\frac{1}{(n+n)^2}+\cdots n \text{ times}$$

$$=\frac{n}{4n^2}=\frac{1}{4n}$$

That is

$$\frac{1}{4n}\le s_n\le\frac{1}{n}$$

Hence by squeezing theorem $\lim_{n\to\infty} s_n = 0$ because $\lim_{n\to\infty} \frac{1}{n} = 0$.

**Answer: (B) → (r)**

(C) $x_1 = 1,\ x_2 = \frac{7}{5} \Rightarrow x_1 < x_2 < 2$. By induction we can see that $x_n < x_{n+1}$ so that $\{x_n\}$ is an increasing sequence and also $x_n < 2\ \forall n$. That is $\{x_n\}$ is an increasing sequence and bounded above. Hence by Theorem 1.42 $\{x_n\}$ converges to a finite limit say $l$. So

$$l = \lim_{x\to\infty} x_{n+1}$$

$$= \lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)$$

$$= \frac{4+3l}{3+2l}$$

This gives

$$3l + 2l^2 = 4 + 3l$$

$$2l^2 = 4 \quad \text{or} \quad l = \pm\sqrt{2}$$

Hence $x_n > 1\ \forall n \Rightarrow l = \sqrt{2}$.

**Answer: (C) → (t)**

(D) Let $x_n = n/2^n$. Then

$$\frac{x_{n+1}}{x_n} = \frac{1}{2}\left(\frac{n+1}{n}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right) \to 1/2 \text{ as } n \to \infty$$

Therefore by **Ratio test** (Theorem 1.47)

$$x_n \to 0 \text{ as } n \to \infty$$

**Answer: (D) → (r)**

## *Comprehension-Type Questions*

**1. Passage:** It is known that

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1 = \lim_{x\to 0}\frac{\tan x}{x}$$

Based on this information, answer the following questions.

**(i)** $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ is

(A) 1 (B) 2

(C) $\frac{1}{2}$ (D) 0

**(ii)** $\lim_{x\to 0}\frac{\sin^3 2x}{x\sin^2 3x}$ is equal to

(A) $\frac{3}{2}$ (B) $\frac{4}{9}$

(C) $\frac{2}{3}$ (D) $\frac{8}{9}$

**(iii)** $\lim_{x\to 0}\left(\frac{1-2\cos x+\cos 2x}{x^2}\right)$ is

(A) 1 (B) −1

(C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

**Solution:**

**(i)** $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \cdot \frac{1}{2}$$

$$= 1 \times \frac{1}{2} = \frac{1}{2}$$

**Answer: (C)**

**(ii)** $$\lim_{x\to 0}\frac{\sin^3 2x}{x\sin^2 3x} = \lim_{x\to 0}\left[\frac{\left(\frac{\sin 2x}{2x}\right)^3 \cdot 8x^3}{x\left(\frac{\sin 3x}{3x}\right)^2 \cdot 9x^2}\right] = \frac{8}{9}$$

**Answer: (D)**

**(iii)** $$\lim_{x\to 0}\left(\frac{1-2\cos x+\cos 2x}{x^2}\right) = \lim_{x\to 0}\frac{2\cos^2 x - 2\cos x}{x^2}$$

$$= \lim_{x\to 0}\frac{2\cos x(\cos x - 1)}{x^2}$$

$$= -4\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}$$

$$= -\lim\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = -1$$

**Answer: (B)**

**2. Passage:** Using $\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) = na^{n-1}$, answer the following questions.

**(i)** $\lim_{x\to 1}\left(\frac{x^{1/3}+x^{1/2}+x^{3/2}-3}{x^3-1}\right)$ is

(A) $\frac{2}{3}$ (B) $\frac{7}{9}$

(C) $\frac{1}{3}$ (D) $\frac{1}{9}$

**(ii)** If $n$ is a positive integer then

$$\lim_{x\to 1}\left(\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}\right)=$$

(A) $\frac{n}{2}$ (B) $\frac{n+1}{2}$

(C) $\frac{n(n+1)}{2}$ (D) $\frac{n(n+1)(2n-1)}{2}$

**(iii)** $\lim_{x\to 0}\left(\frac{\sqrt{1+2x}-1}{x}\right)$ is

(A) 0 (B) $\frac{1}{2}$

(C) $-\frac{1}{2}$ (D) 1

**Solution:**

**(i)** Let

$$f(x)=\frac{x^{1/3}+x^{1/2}+x^{3/2}-3}{x^3-1}$$

Therefore

$$f(x)=\frac{x^{1/3}+x^{1/2}+x^{3/2}-3}{(x-1)(x^2+x+1)}$$

$$=\left[\left(\frac{x^{1/3}-1}{x-1}\right)+\left(\frac{x^{1/2}-1}{x-1}\right)+\left(\frac{x^{3/2}-1}{x-1}\right)\right]\frac{1}{x^2+x+1}$$

Taking limit we get

$$\lim_{x\to 1} f(x)=\left(\frac{1}{3}+\frac{1}{2}+\frac{3}{2}\right)\frac{1}{1+1+1}$$

$$=\frac{14}{6}\times\frac{1}{3}=\frac{7}{9}$$

**Answer: (B)**

**(ii)** Let

$$f(x)=\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}$$

$$=\frac{x(x^n-1)-n(x-1)}{(x-1)^2}$$

$$=\frac{\dfrac{x(x^n-1)}{x-1}-n}{x-1}$$

$$=\frac{x(x^{n-1}+x^{n-2}+\cdots+x+1)-n}{x-1}$$

$$=\frac{(x^n+x^{n-1}+x^{n-2}+\cdots+x)-n}{x-1}$$

$$=\frac{(x^n-1)+(x^{n-1}-1)+\cdots+(x^2-1)+(x-1)}{x-1}$$

$$=\frac{x^n-1}{x-1}+\frac{x^{n-1}-1}{x-1}+\cdots+\frac{x^2-1}{x-1}+\frac{x-1}{x-1}$$

Therefore

$$\lim_{x\to 1} f(x)=n+(n-1)+(n-2)+\cdots+2+1$$

$$=\frac{n(n+1)}{2}$$

**Answer: (C)**

**(iii)** We have

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-1}{x}=\lim_{x\to 0} 2\left[\frac{\sqrt{1+2x}-1}{(1+2x)-1}\right]$$

$$=2\lim_{y\to 1}\left(\frac{\sqrt{y}-1}{y-1}\right),\text{ where } y=1+2x$$

$$=2\times\frac{1}{2}=1$$

**Answer: (D)**

**3. Passage:** A function $f$ is continuous at a point $a$ if and only if

$$\lim_{x\to a-0} f(x)=\lim_{x\to a+0} f(x)=f(a)$$

Equivalently $\lim_{x\to a} f(x)=f(a)$. Based on this information, answer the following questions.

**(i)** Let

$$f(x)=\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\frac{x^2}{(1+x^2)^3}+\frac{x^2}{(1+x^2)^4}+\cdots\infty$$

Then $\lim_{x\to 0} f(x)$

(A) is 0
(B) is 1
(C) is continuous at $x=0$
(D) is discontinuous at $x=0$

**(ii)** Let

$$f(x)=\begin{cases}1+x & \text{if } x\le 2\\ 5-x & \text{if } x>2\end{cases}$$

$$g(x)=\begin{cases}x^2 & \text{if } |x|>2\\ 4 & \text{if } |x|\le 2\end{cases}$$

Then

(A) $f$ is discontinuous at $x=2$ and $g$ is continuous at $x=2$
(B) $f$ is continuous at 2 and $g$ is discontinuous at 2

(C) both $f$ and $g$ gave continuous at $\pm 2$

(D) both $f$ and $g$ are discontinuous at $\pm 2$

**(iii)** Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and $g(x) = 1 + x - [x]$

where $[x]$ is the integral part of $x$. Define $h(x) = f(g(x))$. Then

(A) $h(x)$ is not continuous at integer values of $x$

(B) $h(x)$ is continuous at $x = 0$ only

(C) $h(x)$ is continuous for all real $x \neq 0, \pm 1$

(D) $h(x)$ is continuous for all real $x$

**Solution:**

**(i)** $f(x)$ is a geometric series with common ratio $1/(1 + x^2)$ which is positive and less then unity when $x \neq 0$. Thus

$$f(x) = \frac{\left(\dfrac{x^2}{1+x^2}\right)}{1 - \dfrac{1}{1+x^2}} = 1$$

So

$$f(x) = \begin{cases} 1 & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

Therefore $f$ is not continuous at $x = 0$.

**Answer: (D)**

**(ii)** We have

$$\lim_{x \to 2-0} f(x) = 1 + 2$$

and

$$\lim_{x \to 2+0} f(x) = 5 - 2 = 3$$

Also

$$\lim_{x \to 2-0} g(x) = 2^2 = 4 = \lim_{x \to 2+0} g(x)$$

Thus both $f(x)$ and $g(x)$ are continuous at $x = 2$. Also both $f$ and $g$ are continuous at $x = -2$. Therefore both $f$ and $g$ are continuous at $x = \pm 2$.

**Answer: (C)**

**(iii)** Since

$$x - [x] \geq 0 \Rightarrow h(x) = f(g(x)) = 1$$

for all $x$, therefore $h$ is continuous for all real $x$.

**Answer: (D)**

**4. Passage:** Let $f : \mathbb{R} \to \mathbb{R}$ be such that

(a) $f(x + y) = f(x) + f(y) \ \forall x, y \in \mathbb{R}$

(b) $f(1) = 2$

(c) $f$ is continuous at origin.

Answer the following questions.

**(i)** Which of the following statements is true?

(A) $f$ is discontinuous at all rational values of $x$

(B) $f$ is discontinuous at all irrational values of $x$

(C) $f$ is continuous for all real $x$

(D) $f(x) = 2^x$

**(ii)** $\lim_{x \to 0} \left( \dfrac{2^{f(x)} - 1}{x} \right)$ is equal to

(A) $\log 4$ (B) $\dfrac{1}{2} \log 2$

(C) $\log 2$ (D) $\dfrac{1}{8} \log 4$

**(iii)** $\lim_{x \to \infty} (1 + f(x))^{1/x}$ is

(A) $e^2$ (B) $e$

(C) $\sqrt{e}$ (D) $e^4$

**(IIT-JEE 2011)**

**Solution:**

**(i)** We shall prove that $f$ is continuous at all real $x$. Let $a \in \mathbb{R}$. Now

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}$$
$$\Rightarrow 0 = f(0 + 0) = f(0) + f(0)$$

Therefore

$$f(0) = 0 \tag{1.80}$$

Also,

$$\begin{aligned} 0 &= f(0) \\ &= f(x + (-x)) = f(x) + f(-x) \end{aligned}$$

$$\Rightarrow f(-x) = -f(x) \tag{1.81}$$

Now

$$\begin{aligned} \lim_{x \to a-0} f(x) &= \lim_{h \to 0} f(a - h) \\ &= \lim_{h \to 0} f(a + (-h)) \\ &= \lim_{h \to 0} (f(a) + f(-h)) \\ &= \lim_{h \to 0} (f(a) - f(h)) \\ &= f(a) - \lim_{h \to 0} f(h) \\ &= f(a) - f(0) \qquad \text{[By (c)]} \\ &= f(a) - 0 \qquad \text{[By Eq. (1.80)]} \\ &= f(a) \end{aligned}$$

Thus $\lim\limits_{x\to a-0} f(x) = f(a)$. Similarly $\lim\limits_{x\to a+0} f(x) = f(a)$. Therefore

$$\lim_{x\to a} f(x) = f(a)$$

so that $f$ is continuous at $a \in \mathbb{R}$.

**Answer: (C)**

Now, we prove that $f(x) = 2x$.

**Case I:** Suppose $x$ is a positive integer. Then

$$\begin{aligned} f(x) &= f(1+1+\cdots+x \text{ times}) \\ &= f(1)+f(1)+\cdots+x \text{ times} \\ &= 2+2+\cdots+2\ (x \text{ times}) \qquad [\because f(1)=2] \\ &= 2x \end{aligned}$$

**Case II:** $x$ is a negative integer, say $x = -y$ where $y$ is a positive integer. Then

$$\begin{aligned} f(x) &= f(-y) \\ &= -f(y) \qquad &&[\text{By Eq. (1.81)}] \\ &= -(2y) &&(\text{By Case I}) \\ &= 2x &&(\because x=-y) \end{aligned}$$

**Case III:** Suppose $x = m/n$ where $m$ is an integer and $n$ is a positive integer. Then

$$\begin{aligned} f(m) = f(nx) &= f(x+x+\cdots \text{ upto } n \text{ times}) \\ &= f(x)+f(x)+\cdots \text{ upto } n \text{ times} \\ &= nf(x) \end{aligned}$$

But $f(m) = 2m$ (by Cases I and II). Therefore

$$2m = nf(x)$$

$$\Rightarrow f(x) = 2\left(\frac{m}{n}\right) = 2x$$

**Case IV:** Suppose $x$ is irrational. According to Theorem 1.49 together with the Note under it, we can always construct a sequence $\{x_n\}$ of **rational numbers** such that $x_n \to x$ as $n \to \infty$. Since $f$ is continuous at $x$, by Theorem 1.52

$$f(x_n) \to f(x) \quad \text{as } n \to \infty$$

But $f(x_n) = 2x_n$ (by Case III). Therefore

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f(x_n) \\ &= \lim_{n\to\infty} (2x_n) \\ &= 2\lim_{n\to\infty} x_n \\ &= 2x \end{aligned}$$

Thus $f(x) = 2x\ \forall\, x \in \mathbb{R}$. Note that all the functions satisfying the conditions $f(x+y) = f(x)+f(y)$ and $f$ is continuous on $\mathbb{R}$ will be determined by the value of $f(1)$. In fact if $f(1) = k$, then $f(x) = kx$.

**(ii)** We have

$$\begin{aligned} \lim_{x\to 0} \frac{(2^{f(x)}-1)}{x} &= \lim_{x\to 0}\left(\frac{2^{2x}-1}{x}\right) \qquad (\because f(x)=2x) \\ &= \lim_{x\to 0}\left(\frac{4^x-1}{x}\right) \\ &= \log 4 = 2\log 2 \end{aligned}$$

**Answer: (A)**

**(iii)** We have

$$\begin{aligned} \lim_{x\to\infty} (1+f(x))^{1/x} &= \lim_{x\to\infty} (1+2x)^{1/x} \\ &= \lim_{x\to\infty} (1+2x)^{(1/2x)\cdot 2} \\ &= e^2 \end{aligned}$$

**Answer: (A)**

**5. Passage:** The function $y = f(x)$ is defined by the parametric equations

$$x = 2t - |t-1|$$

and $$y = 2t^2 + t|t|$$

Answer the following questions.

**(i)** The value of $f(2)$ is

(A) 3 (B) 6

(C) 9 (D) 0

**(ii)** $f(x)$ consumes the value 1/2 at $x$ equal to

(A) $\dfrac{\sqrt{2}-1}{\sqrt{2}}$ (B) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$

(C) $\dfrac{\sqrt{2}+1}{2}$ (D) $\dfrac{\sqrt{3}+1}{2}$

**(iii)** The number of values of $x$ at which $f$ is discontinuous is

(A) 0 (B) 1

(C) 2 (D) 4

**Solution:** First, we express $f$ explicitly which will help us in answering the questions.

**Case I:** $t < 0$. In such case for the given equations we have

$$x = 2t - (1-t) = 3t - 1$$

and $$y = 2t^2 - t^2 = t^2 = \left(\frac{x+1}{3}\right)^2$$

Also $t < 0 \Rightarrow x < -1$.

**Case II:** $0 \le t \le 1$. In this case we have

$$x = 2t - (1-t) = 3t - 1$$

and $$y = 2t^2 + t^2 = 3t^2 = 3\left(\frac{x+1}{3}\right)^2$$

Also $0 \le t \le 1 \Rightarrow -1 \le x \le 2$.

**Case III:** $t > 1$. In this case we have

$$x = 2t - (t-1) = t+1$$

and $$y = 2t^2 + t^2 = 3t^2 = 3(x-1)^2$$

Also $x > 2$. Therefore

$$f(x) = \begin{cases} \frac{1}{9}(x+1)^2 & \text{for } x < -1 \\ \frac{1}{3}(x+1)^2 & \text{for } -1 \le x \le 2 \\ 3(x-1)^2 & \text{for } x > 2 \end{cases}$$

**(i)** We have

$$f(2) = \frac{1}{3}(2+1)^2 = 3$$

**Answer: (A)**

**(ii)** Since $f$ is continuous on $[-1, 2]$ and

$$f(-1) = 0 < \frac{1}{2} < 3 = f(2)$$

by the intermediate value theorem for continuous functions, we have $f(x) = 1/2$ for some $x \in [-1, 2]$. Therefore

$$\frac{1}{3}(x+1)^2 = \frac{1}{2}$$

$$\Rightarrow x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}} \in (-1, 2)$$

**Answer: (B)**

**(iii)** Clearly $f$ is continuous at $x = -1$ and 2 and hence $f$ is continuous for all $x$.

**Answer: (A)**

***Note:*** Draw the graph of $y = f(x)$ of the above function to have a clear picture.

**6. Passage:** Let

$$f(x) = \begin{cases} x+a & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$$

and

$$g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x+1)^2 + b & \text{if } x \ge 0 \end{cases}$$

where $a$ and $b$ are non-negative real constants. For the function $h(x) = (g \circ f)(x)$, answer the following questions.

**(i)** If $h(x)$ is continuous for all real $x$, then

(A) $a = -1, b = 1$ (B) $a = -1, b = 0$
(C) $a = 0, b = 1$ (D) $a = 1, b = 0$

**(ii)** If $h(x)$ is continuous for all $x$, then the number of solutions of the equation $h(x) = 0$ is

(A) 3 (B) 2
(C) 4 (D) 0

**(iii)** The number of values of $x$ at while $f$ is discontinuous is

(A) infinite (B) 2
(C) 0 (D) 4

**(IIT-JEE 2002 mains)**

**Solution:**

**(i)** We will first find $g \circ f$.

$$(g \circ f)(x) = g(f(x))$$

$$= \begin{cases} x+a+1 & \text{for } x < -a \\ (x+a-1)^2 + b & \text{for } -a \le x < 0 \\ (|x-1|-1)^2 + b = x^2 + b & \text{for } 0 \le x < 1 \\ (x-2)^2 + b & \text{for } x \ge 1 \end{cases}$$

Since $g \circ f$ is continuous at $x = -a$, we have that left limit at $x = -a$ is same as right limit at $x = -a$. Therefore

$$1 = 1 + b \Rightarrow b = 0$$

Also at $x = 0$, $g \circ f$ is continuous. Therefore left and right limits at $x = 0$ are equal. So

$$(a-1)^2 + b = 0 + b$$

$$\Rightarrow a = 1$$

Thus $a = 1, b = 0$.

**Answer: (D)**

**(ii)** For these values of $a$ and $b$,

$$h = g \circ f = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1 \\ (x-2)^2 & \text{if } x \ge 1 \end{cases}$$

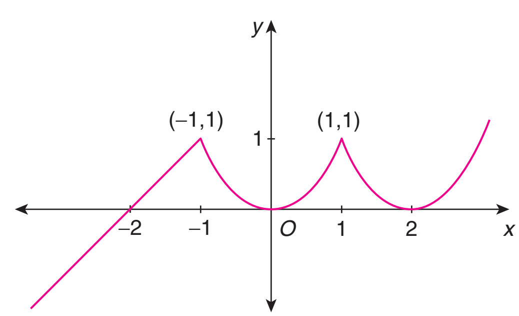



**FIGURE 1.33** Comprehension-type question 6.

Now $h(-2)=0$, $h(0)=0$ and $h(2)=0$ implies that the equation $h(x)=0$ has three solutions.

**Answer: (A)**

**(iii)** Since there is no break in the curve (see Fig. 1.33) $h$ is continuous for all real $x$.

**Answer: (C)**

## *Assertion–Reasoning Type Questions*

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation for Statement I.

(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.

(C) Statement I is true and Statement II is false.

(D) Statement I is false and Statement II is true.

**1. Statement I:** The sum of the infinite series

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots$$

is 1/2.

**Statement II:** The $n$th term of the series is equal to $\frac{1}{n+1}-\frac{1}{n+2}$.

**Solution:** The $n$th term of the series is

$$\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$

Thus, Statement II is true. Now let

$$u_k=\frac{1}{(k+1)(k+2)}=\frac{1}{k+1}-\frac{1}{k+2}$$

Put $k=1,2,\ldots,n$ and add. Then

$$s_n=u_1+u_2+u_3+\cdots+u_n=\frac{1}{2}-\frac{1}{n+2}$$

Thus

$$\lim_{n\to\infty} s_n=\frac{1}{2}-0=\frac{1}{2}$$

Therefore Statement I is also true and Statement II is a correct explanation of Statement I.

**Answer: (A)**

**2. Statement I:** $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$ is equal to $e^{-5}$

**Statement II:** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Clearly Statement II is true [see part (1) of Important Formulae]. Now

$$\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x=\lim_{x\to\infty}\left(\frac{1-\frac{3}{x}}{1+\frac{2}{x}}\right)^x$$

$$=\frac{\lim_{x\to\infty}\left[\left(1-\frac{3}{x}\right)^{-x/3}\right]^{-3}}{\lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)^{x/2}\right]^{2}}$$

$$=\frac{e^{-3}}{e^2}=e^{-5}$$

Statement I is also true and Statement II is a correct explanation of Statement I.

**Answer: (A)**

**3.** Let $[x]$ denote the integral part of $x$.

**Statement I:** Let

$$f(x)=\begin{cases}\dfrac{\sin[x]}{[x]} & \text{if } [x]\neq 0\\ 0 & \text{if } [x]=0\end{cases}$$

Then $\lim_{x\to 0} f(x)$ does not exist.

**Statement II:** $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$

**Solution:** Statement II is true (see Theorem 1.27). Now

$$\lim_{x\to 0-0} f(x)=\lim_{\substack{h\to 0\\ h>0}}\frac{\sin[0-h]}{[0-h]}$$

$$=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}$$

$$=\sin 1$$

Also $\lim_{x\to 0+0} f(x)=0$ because $[0+h]=0$. Therefore

$$\lim_{x\to 0-0} f(x)\neq \lim_{x\to 0+0} f(x) \text{ does not exist}$$

$$\Rightarrow \lim_{x\to 0} f(x) \text{ does not exist}$$

Statement I is also true. That is, both statements are true and Statement II is not a correct explanation of Statement I.

**Answer: (B)**

4. **Statement I:** Let

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{if } x<0\\ ax+b & \text{if } x>0\end{cases}$$

If $\lim_{x\to 0} f(x)$ exists, then $a=1$ and $b=0$.

**Statement II:** $\lim_{x\to 0}(px+q)=q$ where $p$ and $q$ are any real constants.

**Solution:** Statement II is clearly true. Since $\lim_{x\to 0} f(x)$ exists, the left and right limits of $f(x)$ at $x = 0$ must be equal. So

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$\Rightarrow \lim_{x\to 0-0} f(x)=1$$

Now

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}(ax+b)=b$$

Therefore $b = 1$ whereas $a$ may be any real number. Hence Statement I is false.

**Answer: (D)**

5. **Statement I:** If $\lim_{x\to 0}\left(\dfrac{\sin 2x+a\sin x}{x^3}\right)$ exists finitely, then the value of $a$ is $-2$.

**Statement II:** If $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ exists finitely and $\lim_{x\to a} g(x)=0$ then $\lim_{x\to a} f(x)=0$.

**Solution:** According to Corollary 1.3, Statement II is true. Now

$$\lim_{x\to 0}\left(\frac{\sin 2x+a\sin x}{x^3}\right)=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{(2\cos x+a)}{x^2}$$

$$=\lim_{x\to 0}\frac{2\cos x+a}{x^2} \text{ exists finitely}$$

Therefore

$$\lim_{x\to 0}(2\cos x+a)=0$$

So $a=-2$.

**Answer: (A)**

6. **Statement I:** Let

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & \text{if } x<\dfrac{\pi}{2}\\ a & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & \text{if } x>\dfrac{\pi}{2}\end{cases}$$

If $f$ is continuous at $x=\pi/2$ then the values of $a$ and $b$ are, respectively, 1/2 and 4.

**Statement II:** A function $f$ is continuous at $x = c$, if and only if

$$\lim_{x\to c-0} f(x)=\lim_{x\to c+0} f(x)=f(c)$$

**Solution:** Statement II is true (Theorem 1.24). So

$$\lim_{x\to \frac{\pi}{2}-0} f(x)=\lim_{\substack{h\to 0\\ h>0}}\frac{1-\sin^3\left(\dfrac{\pi}{2}-h\right)}{3\cos^2\left(\dfrac{\pi}{2}-h\right)}$$

$$=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3(1-\cos h)(1+\cos h)}$$

$$=\lim_{h\to 0}\frac{1+\cos h+\cos^2 h}{3(1+\cos h)}$$

$$=\frac{3}{3(2)}=\frac{1}{2}=f\left(\frac{\pi}{2}\right)=a$$

Therefore $a=1/2$. Also

$$\lim_{x\to 0+0} f(x)=\lim_{\substack{h\to 0\\ h>0}}\left[\frac{b\left[1-\sin\left(\dfrac{\pi}{2}+h\right)\right]}{\left(\pi-2\left(\dfrac{\pi}{2}+h\right)\right)^2}\right]$$

$$=\lim_{h\to 0}\frac{b(1-\cos h)}{(-2h)^2}$$

$$=\lim_{h\to 0}\frac{2b\sin^2\dfrac{h}{2}}{4h^2}$$

$$=\lim_{h\to 0}\frac{b\sin^2\dfrac{h}{2}}{2h^2}$$

$$=\lim_{h\to 0}\frac{b}{8}\left(\frac{\sin\dfrac{h}{2}}{\dfrac{h}{2}}\right)^2$$

$$= \frac{b}{8}$$

Now

$$\frac{b}{8} = a = \frac{1}{2} \Rightarrow b = 4$$

Therefore

$$a = \frac{1}{2} \quad \text{and} \quad b = 4$$

Hence Statement I is also correct and Statement II is a correct explanation of Statement I.

**Answer: (A)**

## *Integer Answer Type Questions*

**1.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $0<\alpha<1$. If

$$|f(x)-f(y)| \le \alpha|x-y| \ \forall x, y \in \mathbb{R}$$

Then the number of points of $x\in\mathbb{R}$ such that the graph of $y=f(x)$ intersects the line $y=x$ is _______.

**Solution:** Let $x_0 \in \mathbb{R}$. Now define

$$x_1 = f(x_0), x_2 = f(x_1), \ldots, x_{n+1} = f(x_n) \ldots$$

Consider the sequence $\{x_n\}$. By the construction of the sequence $\{x_n\}$ and the property that $f$ is satisfying, we get that

$$|x_n - x_{n+1}| \le \alpha^n |x_0 - x_1| \qquad (1.82)$$

Therefore for $m>n$, we have

$$\begin{aligned}|x_n - x_m| &= |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| \\ &\quad + \cdots + |x_{n+m-(n+1)} - x_{n+m-n}| \\ &\le \alpha^n |x_0 - x_1| + \alpha^{n+1}|x_0 - x_1| \\ &\quad + \cdots + \alpha^{n+m-(n+1)}|x_0 - x_1| \\ &= \alpha^n |x_0 - x_1|(1+\alpha+\alpha^2+\cdots+\alpha^{m-(n+1)}) \\ &\le \alpha^n |x_0 - x_1| \frac{1}{1-\alpha} \\ &\left(\because \ 1+\alpha+\alpha^2+\cdots+\infty = \frac{1}{1-\alpha}\right)\end{aligned}$$

Since $0<\alpha<1$, $\alpha^n \to 0$ as $n\to\infty$ Hence $\{x_n\}$ is a Cauchy sequence so that $\lim\limits_{n\to\infty} x_n$ exists. Suppose $\lim\limits_{n\to\infty} x_n = x$. Also for any two $y$ and $z$ in $\mathbb{R}$

$$|f(y)-f(z)| \le \alpha|y-z| < |y-z| \quad (\because 0<\alpha<1)$$

implies that $f$ is continuous on $\mathbb{R}$. Therefore

$$x_n \to x \Rightarrow f(x_n) \to f(x) \quad \text{(By Corollary 1.12)}$$

But $f(x_n) = x_{n+1} \Rightarrow \{f(x_n)\}$ is a subsequence of $\{x_n\}$. Hence (by Theorem 1.44)

$$x_n \to x \text{ as } n\to\infty \Rightarrow f(x_n) \to x \text{ as } n \to \infty$$

Therefore $f(x) = x$. To prove that the point is unique, suppose $f(x)=x$ and $f(y)=y$. Therefore

$$|x-y| = |f(x)-f(y)| \le \alpha|x-y|$$

$$\Rightarrow (1-\alpha)|x-y| \le 0$$

$$\Rightarrow x = y \quad (\because 1-\alpha>0 \text{ and } |x-y|\ge 0)$$

**Answer: 1**

**2.** Let

$$f(x) = \begin{cases} \dfrac{3x+4\tan x}{x} & \text{if } x \ne 0 \\ k & \text{if } x = 0 \end{cases}$$

If $f$ is continuous at $x=0$, then the value of $k$ is _______.

**Solution:** We have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(3 + \frac{4\tan x}{x}\right)$$

$$= 3+4 = 7$$

Therefore $k=7$.

**Answer: 7**

**3.** Let $f$: $(-1, 1) \to (-1, 1)$ be continuous and $f(x) = f(x^2)$ for all $x \in (-1, 1)$ and $f(0) = \frac{1}{2}$, then the value of $4f\left(\frac{1}{4}\right)$ is _______.

**Solution:** By hypothesis

$$f(0) = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = f\left(\frac{1}{4}\right) = f\left(\frac{1}{16}\right) = \cdots = f\left(\frac{1}{2^n}\right) \text{ for } n \ge 1$$

Since $f$ is continuous at $x=0$, by Theorem 1.52 we have

$$f\left(\frac{1}{2^n}\right) \to f(0) \text{ as } n\to\infty$$

Therefore

$$\lim_{n\to\infty} f\left(\frac{1}{2^n}\right) = f(0) = \frac{1}{2}$$

$$f\left(\frac{1}{4}\right) = \frac{1}{2}$$

$$4f\left(\frac{1}{4}\right) = 2$$

**Answer: 2**

**4.** For $x \in (0, 1)$, define

$$f(x) = \begin{cases} x(1-x) & \text{if } x \text{ is rational} \\ \dfrac{1}{4} - x(1-x) & \text{if } x \text{ is not rational} \end{cases}$$

Then, the number of points in $(0, 1)$ at which $f$ is continuous is ________.

**Solution:** Let $0 < a < 1$. Then $f$ is continuous at $a$ if and only if,

$$a(1-a) = \frac{1}{4} - a(1-a)$$

$$\Leftrightarrow a = \frac{2 \pm \sqrt{2}}{4} \in (0, 1)$$

because in every neighbourhood of $a$, there are infinitely many rationals tending to $a$ and infinitely many irrationals tending to $a$.

**Answer: 2**

**5.** If $m$ and $n$ stand for the number of positive and negative roots, respectively, of the equation $e^x = x$, then $m + n$ is equal to _______.

**Solution:** We have $e^x > 0 \;\forall x \in \mathbb{R}$. Also $e^x \to 0$ as $x \to -\infty$ and $e^x \to +\infty$ as $x \to +\infty$. The function $f(x) = x$ is the graph of the line $y = x$ (see Fig. 1.34). Also $e^x > x$ for $x \geq 0$. Hence $m = 0$ and $n = 0$.

**Answer: 0**

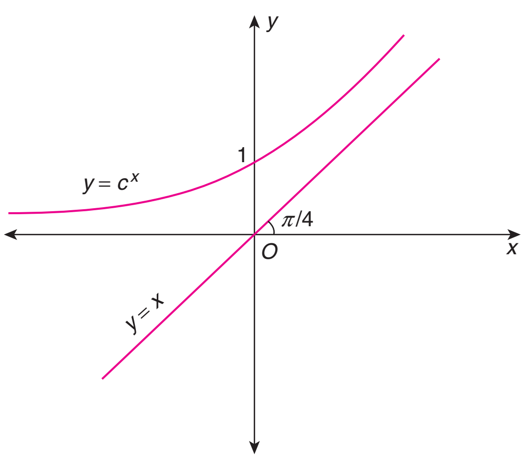



**FIGURE 1.34** Integer type question 5.

**6.** If

$$f(x) = \frac{(2^x - 1)^3}{\sin(x \log 2)\log(1 + x^2 \log 4)}$$

is continuous everywhere, then the integral part of $f(0)$ is ______.

**Solution:** We have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\left(\dfrac{2^x - 1}{x}\right)^3}{(\log 2)\left(\dfrac{\sin(x\log 2)}{x \log 2}\right)\dfrac{\log(1 + x^2 \log 4)}{x^2 \log 4} \cdot \log 4}$$

$$= \frac{(\log 2)^3}{(\log 2)(2 \log 2)}$$

$$= \frac{1}{2}\log 2$$

Therefore $f(0)$ is defined to be $(1/2)\log 2$ so that $[f(0)] = 0$.

**Answer: 0**

**7.** Let

$$f(x) = \begin{cases} \dfrac{x}{1-|x|} & \text{if } |x| < 1 \\ \dfrac{x}{1+|x|} & \text{if } |x| \geq 1 \end{cases}$$

The number of points at which $f$ is not continuous is _______.

**Solution:** We have

$$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \leq -1 \\ \dfrac{x}{1+x} & \text{if } -1 < x < 0 \\ \dfrac{x}{1-x} & \text{if } 0 \leq x < 1 \\ \dfrac{x}{1+x} & \text{if } x \geq 1 \end{cases}$$

Clearly

$$\lim_{x\to 0-0} f(x) = \lim_{x\to 0+0} f(x) = 0 = f(0)$$

Hence $f$ is continuous at $x = 0$. Also

$$\lim_{x\to(-1)-0} f(x) = -\frac{1}{2}$$

but $\lim\limits_{x\to(-1)+0} f(x)$ does not exist and

$$\lim_{x\to 1+0} f(x) = \frac{1}{2}$$

but $\lim\limits_{x\to 1-0} f(x)$ does not exist. Therefore at $x = \pm 1$, $f$ is not continuous.

**Answer: 2**

**8.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the relation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

and $f(1) = 1$, then $f(3)$ is equal to________.

**Solution:** In the given relation, taking $x = y = 0$, we have $f(0) = 0$. Also $x = 0$ implies

$$f(y) + f(-y) = 0 + 2f(y)$$
$$\Rightarrow f(-y) = f(y)$$

Again if we put $y = x$ in the given relation we get

$$f(2x) = 4f(x) = 2^2 f(x) \quad (1.83)$$

Now replacing $y$ with $2x$ in the given relation we obtain

$$f(3x) + f(-x) = 2f(x) + 2f(2x)$$

Therefore $[\because f(-x) = f(x)]$

$$\begin{aligned} f(3x) &= f(x) + 2f(2x) \\ &= f(x) + 2\cdot 2^2 f(x) \quad \text{[By Eq. (1.83)]} \\ &= 3^2 f(x) \end{aligned}$$

Therefore by induction, we have $f(nx) = n^2 f(x)$ for all positive integers $n$. Replacing $n$ with $-n$ and observing that $f(-x) = f(x)\ \forall x$, we have

$$f(-nx) = f(nx) = n^2 f(x) = (-n)^2 f(x)$$

Therefore $f(nx) = n^2 f(x)$ for all integers $x$. Also

$$f(n) = n^2 \qquad (\because f(1) = 1)$$

If $x = p/q$ is rational, then

$$q^2 f(x) = f(qx) = f(p) = p^2 f(1) = p^2 \ (\because f(1) = 1)$$

Therefore

$$f(x) = \frac{p^2}{q^2} = x^2 \text{ for all rational}$$

If $x$ is irrational, then let $\{x_n\}$ be a sequence of rational numbers such that $x_n \to x$ as $n \to \infty$. Since $f$ is continuous, by Theorem 1.52 we have

$$f(x_n) \to f(x) \text{ as } n \to \infty$$

But

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n^2) = x^2$$

Therefore $f(x) = x^2$ when $x$ is irrational. Also $f(x) = x^2$ for all real $x$. Hence

$$f(3) = 3^2 = 9$$

**Answer: 9**

**9.** If $f$ is a real-valued function defined for all $x \neq 0, 1$ and satisfying the relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x}$$

Then $\lim_{x\to 2} f(x)$ is ________.

**Solution:** Given relation is

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x} \quad (1.84)$$

Replacing $x$ with $\frac{1}{1-x}$ in Eq. (1.84) we have

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = 2(1-x) + \frac{2(1-x)}{x} \quad (1.85)$$

Again replacing $x$ with $\frac{1}{1-x}$ in Eq. (1.84), we get

$$f\left(\frac{x-1}{x}\right) + f(x) = -2x - \frac{2x}{1-x} \quad (1.86)$$

Now adding Eqs. (1.84) and (1.86) and subtracting Eq. (1.85) gives

$$\begin{aligned} 2f(x) &= \left(\frac{2}{x} - \frac{2}{1-x} - 2x - \frac{2x}{1-x}\right) - 2(1-x) - \frac{2(1-x)}{x} \\ &= \left(\frac{2}{x} - \frac{2(1-x)}{x}\right) + \left(\frac{-2}{1-x} - \frac{2x}{1-x}\right) - 2x - 2(1-x) \\ &= \frac{2x}{x} - \frac{2(1+x)}{(1-x)} - 2 \\ &= 2 + \frac{2(x+1)}{x-1} - 2 \\ &= \frac{2(x+1)}{x-1} \end{aligned}$$

Therefore

$$f(x) = \frac{x+1}{x-1}$$

Taking limit we get

$$\lim_{x\to 2}\left(\frac{x+1}{x-1}\right) = \frac{2+1}{2-1} = 3$$

**Answer: 3**

**10.** The number of solutions of the equation $\cos x = x$ in the interval $[\pi/6, \pi/4]$ is ________.

**Solution:** Let $f(x)$ cos $x - x$ Clearly $f(x)$ is continuous on $[\pi/6, \pi/4]$. Also

$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6} > 0$$

and

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{\pi}{4} < 0$$

Hence by Corollary 1.10, $f(x) = 0$ for same $x \in (\pi/6, \pi/4)$. Since $x - \cos x$ is strictly increasing in $[0, \pi/2]$, it follows that the equation $x - \cos x = 0$ has unique solution.

**Answer: 1**

# EXERCISES

## Single Correct Choice Type Questions

**1.** Let $f(x) = \log_{10}(3x^2 - 4x + 5)$ where $x$ is real. Then, the domain and range of $f$ are, respectively

(A) $\mathbb{R} - \{0\}$ and $\mathbb{R}^+$

(B) $\mathbb{R}$ and $[\log_{10}(11/3), +\infty)$

(C) $\mathbb{R} - \{0, 1\}$ and $[\log_{10}(13/3), +\infty)$

(D) $\mathbb{R}$ and $\mathbb{R}^+$

**2.** Let $f(x) = \dfrac{x-2}{x^2 - 2x + 3}$. Then the range of $f$ is

(A) $\left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$ (B) $\left[\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$

(C) $\left[0, \dfrac{\sqrt{3}-1}{4}\right]$ (D) $\left[-\dfrac{(\sqrt{3}+1)}{4}, \dfrac{\sqrt{3}-1}{4}\right]$

**3.** The range of $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$ is

(A) $[1, \infty)$ (B) $[-1, \infty)$

(C) $[1, 7/5)$ (D) $[1, 7/3)$

**4.** The domain and range of $f(x) = \log(\sqrt{x^2 + 6x + 10})$ are respectively

(A) $\mathbb{R}$ and $[0, \infty)$ (B) $\mathbb{R}^+$ and $[0, 1]$

(C) $\mathbb{R}$ and $\mathbb{R} - \{0\}$ (D) $\mathbb{R}^+$ and $\mathbb{R}^+ - \{1\}$

**5.** The domain of the function $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$ is

(A) $\mathbb{R} - \{-1, -2\}$ (B) $(-2, \infty)$

(C) $\mathbb{R} - \{-1, -2, -3\}$ (D) $(-3, \infty) - \{-1, -2\}$

**6.** Let

$$f(x) = \frac{9^x}{9^x + 3}, x \in \mathbb{R}$$

and

$$\sum_{k=1}^{2010} f\left(\frac{k}{2011}\right) = n$$

Then $\lim_{x \to 1}\left(\dfrac{x^n - 1}{x - 1}\right)$ is equal to

(A) 2010 (B) 1004

(C) 1005 (D) 1006

**7.** Let $p(x)$ be a polynomial satisfying the relation $p(x) + p(2x) = 5x^2 - 18$. If $n = p(4)$, then

$$\lim_{x \to 1}\left(\frac{x^n - 1}{x - 1}\right) =$$

(A) 6 (B) 7

(C) 8 (D) 9

**8.** Let $A$ be the set of all non-negative integer and for real number $t$, $[t]$ denotes the greatest integer not exceeding $t$. Define $f: A \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left(x - 10\left[\dfrac{x}{10}\right]\right)10^{[\log_{10} x]} + f\left(\left[\dfrac{x}{10}\right]\right) & \text{if } x > 0 \end{cases}$$

If $a = f(7752)$, then $\lim_{x \to 1}\left(\dfrac{x^a - 1}{x - 1}\right)$ is

(A) 2577 (B) 7572

(C) 7275 (D) 2757

**9.** Let $f$ be a real-valued function satisfying the relation

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \text{ for all real } x \neq 0$$

If $n$ is the number of real solutions of the equation $f(x) = f(-x)$, then

$$\lim_{x \to 2}\left(\frac{x^n - 2^n}{x - 2}\right) =$$

(A) 4 (B) 6

(C) 8 (D) 16

**10.** The range of the function $f(x) = \dfrac{1}{\sqrt{4 + 3\cos x}}$ is

(A) $[0, 1]$ (B) $\left[\dfrac{1}{\sqrt{7}}, \dfrac{1}{2}\right]$

(C) $\left[\dfrac{1}{\sqrt{7}}, 1\right]$ (D) $[1, \sqrt{7}]$

**11.** If $[t]$ denotes the greatest integer not exceeding $t$, then the range of the function $\text{Sin}^{-1}\left[\dfrac{1}{2} + x^2\right]$ is

(A) $[0.1]$ (B) $\left\{0, \dfrac{\pi}{2}\right\}$

(C) $\left\{0, \dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$ (D) $\left\{\dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\pi}{2}\right\}$

**12.** Let

$$f(x) = \begin{cases} \dfrac{1}{1-x} & \text{if } x \neq 1 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \end{cases}$$

$$g(x) = (f \circ f \circ f)(x)$$

Then $\lim\limits_{x \to 1} g(x)$ is equal to

(A) 4 (B) 3
(C) does not exist (D) 1

**13.** Let

$$f(x) = \begin{cases} \sin x & \text{if } x \neq n\pi,\ n = 0, \pm 1, \pm 2, \ldots \\ 2 & \text{if } x = n\pi \end{cases}$$

and $g(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

Then $\lim\limits_{x \to 0} (g \circ f)(x)$ is

(A) 0 (B) 1
(C) 3 (D) does not exist

**14.** Let $f(x) = 2x^4 - x^2 + x - 5$. Then $\lim\limits_{x \to -\infty} f(x)$ is

(A) $+\infty$ (B) $-\infty$
(C) 2 (D) 0

**15.** For real $x$, let $\{x\}$ denote the fractional part of $x$. Then

$$\lim_{x \to 1} \frac{x \sin(\{x\})}{x - 1} =$$

(A) 0 (B) $-1$
(C) does not exist (D) 1

**16.** $\lim\limits_{x \to -\infty} \left( \dfrac{3x^3 + 1}{\sqrt{x^4 - x^2}} \right) =$

(A) 1 (B) 0
(C) $\infty$ (D) $-\infty$

**17.** $\lim\limits_{x \to 2} \left( \dfrac{\sqrt{x^2 + 5} - 3}{x^2 - 2x} \right) =$

(A) 1/2 (B) 1/3
(C) 2 (D) 3

**18.** Let $f(x) = 3x^2 - x$. Then

$$\lim_{h \to 0} \left( \frac{f(x+h) - f(x)}{h} \right) =$$

(A) $3x - 1$ (B) $6x - 1$
(C) $9x - 1$ (D) $6x$

**19.** $\lim\limits_{x \to \infty} \left( \dfrac{x^4 + 100x^2}{x^6 + 2x} \right) =$

(A) 0 (B) $-\infty$
(C) $\infty$ (D) 50

**20.** Let $f(x) = 17x^7 - 19x^5 - 1$. Let $P$ and $Q$ be the following statements.

$P$: $f$ is continuous for all real $x$.

$Q$: $f(x) = 0$ has a solution in the interval $(-1, 0)$.

Then
(A) both $P$ and $Q$ are true
(B) $P$ is true whereas $Q$ is false
(C) $P$ is false and $Q$ is true
(D) both $P$ and $Q$ are false

**21.** Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 & \text{if } x < 1 \\ 5 & \text{if } x \geq 1 \end{cases}$$

Consider the following two statements $S_1$ and $S_2$.

$S_1$: $f$ is continuous on the closed interval $[0, 2]$.

$S_2$: $f$ assumes the value 4 in $[0, 2]$.

Then
(A) both $S_1$ and $S_2$ are true
(B) $S_1$ is false and $S_2$ is true
(C) $S_1$ is true and $S_2$ is false
(D) both $S_1$ and $S_2$ are false

**22.** Consider the following two functions defined on the closed interval $[0, 1]$:

$$f(x) = \frac{2x + 3}{2x - 5}; \quad g(x) = \frac{x^2 + 4}{x^2 - 4}$$

Then
(A) both $f$ and $g$ are continuous
(B) $f$ is continuous whereas $g$ is not continuous
(C) $f$ is not continuous whereas $g$ is continuous
(D) both $f$ and $g$ are not continuous

**23.** Consider the function $f(x) = \sqrt{25 - x^2}$ on the interval $[-5, 0]$.
(A) $f$ is continuous on $[-5, 0]$ and $f(x) = \sqrt{7}$ for some $x \in (-5, 0)$
(B) $f$ is continuous on $[-5, 0]$ and $f(x) \neq \sqrt{7}$ for any $x \in [-5, 0]$

(C) $f$ is not continuous on $[-5, 0]$

(D) $f(x) = 7$ for some $x \in [0, 5]$

**24.** $\lim_{x\to 4} \dfrac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}} =$

(A) $\dfrac{\sqrt{2}}{3}$ (B) $\dfrac{2\sqrt{2}}{3}$

(C) $\dfrac{4}{\sqrt{3}}$ (D) $\sqrt{\dfrac{2}{3}}$

**25.** $\lim_{x\to a} \dfrac{x^{(1/m)} - a^{(1/m)}}{x-a} =$

(A) $\dfrac{\sqrt[m]{a}}{ma}$ (B) $ma^{1/m}$

(C) $ma^{(1-m)/m}$ (D) $ma^{(m-1)/m}$

**26.** $\lim_{x\to 0} \dfrac{e^{\alpha x} - e^{\beta x}}{\sin \alpha x - \sin \beta x} =$

(A) $\dfrac{\alpha-\beta}{\alpha+\beta}$ (B) 0

(C) 1 (D) $\dfrac{\alpha+\beta}{\alpha-\beta}$

**27.** The number of points of discontinuity of the function tan $(1/x)$ in the interval $[0, 100]$ is

(A) 100 (B) 101

(C) 50 (D) 51

**28.** The number of points of discontinuities of

$$f(x) = \begin{cases} 1 + 2^{1/x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is

(A) 0 (B) 1

(C) 2 (D) infinite

**29.** $\lim_{x\to \frac{\pi}{3}} \left( \dfrac{1 - 2\cos x}{\sin[x - (\pi/3)]} \right) =$ is

(A) $\dfrac{1}{\sqrt{3}}$ (B) 1

(C) $\sqrt{3}$ (D) $\dfrac{1}{3}$

**30.** $\lim_{x\to 0} (1 + 3\tan^2 x)^{\cot^2 x} =$

(A) $e^2$ (B) $e^3$

(C) $e^{-3}$ (D) 1

**31.** $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^{n+5} =$

(A) $e$ (B) $e^5$

(C) 1 (D) $+\infty$

**32.** If $\lim_{x\to -\infty} (\sqrt{x^2 - x + 1} - ax - b) = 0$ then

(A) $a = 1, b = \dfrac{1}{2}$ (B) $a = 1, b = -\dfrac{1}{2}$

(C) $a = \dfrac{1}{2}, b = 1$ (D) $a = -1, b = \dfrac{1}{2}$

**33.** If $x \neq (\pi/2) + n\pi$, then $\lim_{n\to\infty} (\sin^{2n} x)$ is equal to

(A) 1 (B) 0

(C) $\infty$ (D) does not exist

**34.** Let

$$f(x) = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})$$

where $|x| < 1$, the $\lim_{n\to\infty} f(x)$ is

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{1}{1+x}$

(C) $\dfrac{x}{1-x}$ (D) $\dfrac{x}{1+x}$

**35.** Let

$$f(x) = \begin{cases} (x+1)2^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then, the number of points of discontinuities of $f$ in the interval $[-2, 2]$ is

(A) 1 (B) 0

(C) 2 (D) infinite

**36.** If $k$ is a positive integer, then

$$\lim_{n\to\infty} \left( \frac{1^k + 2^k + 3^k + \cdots + n^k}{n^{k+1}} \right) =$$

(A) $1/k$ (B) $1/(k+1)$

(C) $k/(k+1)$ (D) 1

**Hint:** Use part (i) of Theorem 1.21 and show that

$$\frac{1}{k+1} < \frac{1^k + 2^k + 3^k + \cdots + n^k}{n^{k+1}} < \left\{ \frac{1}{k+1}\left(1 + \frac{1}{n}\right)^{k+1} - \frac{1}{n^{k+1}} \right\}$$

and take limit as $n \to \infty$.

**37.** $\{a_n\}$ is a sequence of non-zero real numbers which are in AP with common difference $d$. Then

$$\lim_{n\to\infty} \left( \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_n a_{n+1}} \right) =$$

(A) $1/d$ (B) $a_1/d$
(C) $d/a_1$ (D) $1/da_1$

**38.** $\lim_{n\to\infty}(n^{1/n}) =$

(A) 1 (B) 0
(C) does not exist (D) finite positive number less than 1

**39.** $\lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right]^{1/n} =$

(A) 1 (B) $1/e$
(C) $e$ (D) $+\infty$

**Hint:** Use Cauchy's second theorem on limits.

**40.** $f:\mathbb{R}\to\mathbb{R}$ is a function satisfying the relation $f(f(x)) - f(x) = ax + b\ \forall\, x \in \mathbb{R}$ where $a \neq 0$. Then the number of solutions of the equation $f(x) = x$ is

(A) 1 (B) 0
(C) 2 (D) infinite

**41.** $\lim_{x\to\infty}(n!)^{-1/n} =$

(A) 0 (B) 1
(C) $+\infty$ (D) cannot be determined

**42.** If $s_n = \sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}}-1\right)$ then $\lim_{x\to\infty}(s_n)$ is

(A) 1 (B) 1/2
(C) 1/4 (D) $+\infty$

**Hint:** $\frac{x}{2+x} < \sqrt{1+x-1} < \frac{x}{2}$ for $x > -1$. Put $x = \frac{k}{n^2}$ and use squeezing theorem.

**43.** $\lim_{x\to 1}\sec\left(\frac{\pi}{2^x}\right)\log x =$

(A) $\frac{1}{\pi\log 2}$ (B) $\frac{2}{\pi\log 2}$
(C) $\frac{\log 2}{2\pi}$ (D) $\frac{\pi}{2\log 2}$

**44.** Let $p$ be a real number and

$$a_n = e^{i(2n\cot^{-1}p)}\left(\frac{pi+1}{pi-1}\right)$$

where $i = \sqrt{-1}$. Then $\lim_{n\to\infty}(a_n)$ is equal to

(A) $p$ (B) $p^2$
(C) $p/2$ (D) 1

**Hint:** Put $\text{Cot}^{-1}p = \theta$ and use De Moivre's theorem.

**45.** Let

$$f(x) = \frac{x^2\tan(1/x)}{\sqrt{8x^2+7x+1}} \text{ for } x > 0$$

Then $\lim_{x\to\infty} f(x)$ is

(A) 1/2 (B) $1/\sqrt{2}$
(C) $1/2\sqrt{2}$ (D) $2\sqrt{2}$

**46.** Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$$g(x) = \sqrt{f(x)+1}$$

Then the number of values of $x$ at which $g(x)$ is discontinuous is

(A) 0 (B) 1
(C) 2 (D) 3

**47.** Let $A$ and $B$ be real constants and

$$f(x) = \begin{cases} Ax - B & \text{if } x \le 1 \\ 3x + 2 & \text{if } 1 < x < 2 \\ Bx^2 - A & \text{if } x \ge 2 \end{cases}$$

If $f$ is continuous at $x = 1$ and 2 then

(A) $A = 6, B = 3$ (B) $A = 3, B = 6$
(C) $A = -3, B = -6$ (D) $A = -3, B = 6$

**48.** Let

$$f(x) = \begin{cases} \frac{1}{x-a}\sin\left(\frac{1}{x-a}\right) & \text{if } x \ne a \\ 0 & \text{if } x = a \end{cases}$$

Then

(A) left limit at $a$ exists and is equal to zero
(B) right limit at $a$ exists and is equal to 1
(C) at $a$, left limit exists finitely, but right limit does not exist
(D) both left and right limits at $a$ do not exist

**49.** Let

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \text{ for } n \ge 0 \text{ integer} \\ 0 & \text{if } x = 0 \end{cases}$$

Then

(A) $f$ is discontinuous in the open interval $\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$

(B) $f$ is not continuous at $x=1$

(C) $f$ is discontinuous at $x=1/2$

(D) $f$ is continuous exactly at $x=\dfrac{1}{2^n}$ for $n=1, 2, 3$

**50.** Function $f$ is defined on [0, 1] by

$$f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

Then

(A) $f$ assumes every value in [0, 1] and is also continuous on [0, 1]

(B) $f$ is continuous exactly at $x = 1/2$ and assumes every value in [0, 1]

(C) $f$ is not continuous at $x=1/2$, but assumes every value between 0 and 1

(D) $f$ is not continuous on [0, 1] and hence it cannot assume all the values between 0 and 1

**51.** If

$$f(x)=\frac{\frac{1}{2}|x-1|+x|x-1|+2-\frac{2}{x}}{\sqrt{x-2+\frac{1}{x}}}$$

then $\lim\limits_{x\to 1/2} f(x)$ is equal to

(A) $-\dfrac{3}{2\sqrt{2}}$ (B) $\dfrac{3}{2\sqrt{2}}$

(C) $3\sqrt{2}$ (D) $-3\sqrt{2}$

**52.** $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\right)$ is

(A) $\infty$ (B) 0

(C) 1 (D) 1/2

**53.** Let $f(x)=\begin{cases} \big||x-1|-1\big| & \text{if } x<1 \\ [x] & \text{if } x\geq 1 \end{cases}$

where [ ] is the greatest integer function. Then the set of points of discontinuities of $f$ consists precisely

(A) all integers $\geq 1$

(B) the integer 1

(C) all integers greater than 1

(D) all negative integers

**54.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying $f(x)+2f(1-x)=x^2+2$ for all $x\in\mathbb{R}$. Then $\lim\limits_{x\to 3} f(x)$ is

(A) 1/3 (B) 1/2

(C) 2/3 (D) 3/2

**55.** $\lim\limits_{x\to\infty}\left(\dfrac{x^4\sin(1/x)+x^2}{1+|x|^3}\right)=$

(A) 0 (B) $-1$

(C) 1/2 (D) does not exist

**56.** $\lim\limits_{x\to\infty}\dfrac{(2+x)^{40}(4+x)^5}{(2-x)^{45}}=$

(A) 1 (B) $-1$

(C) $+\infty$ (D) $-\infty$

**57.** $\lim\limits_{x\to 0}\dfrac{x10^x-x}{1-\cos x}=$

(A) log 10 (B) 2 log 10

(C) 3 log 10 (D) 4 log 10

**58.** $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{1+3x}-\sqrt{1+2x}}{x+2x^2}\right)=$

(A) 1 (B) 2

(C) 1/2 (D) $-1/2$

**59.** Let $f(x)=\begin{cases} (e^{1/x}+1)^{-1} & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$

Then

(A) $\lim\limits_{x\to 0+0} f(x)=1$

(B) $\lim\limits_{x\to 0-0} f(x)=1$

(C) $\lim\limits_{x\to 0} f(x)=1$

(D) $\lim\limits_{x\to 0} f(x)=0$

**60.** Let $s_n=\sum\limits_{k=1}^{n}\text{Tan}^{-1}\left(\dfrac{1}{1+k+k^2}\right)$.

Then $\lim\limits_{n\to\infty}(s_n)$ is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$

(C) 0 (D) 1

**61.** Let $s_n=\sum\limits_{k=1}^{n}\text{Cot}^{-1}(2k^2)$. Then $\lim\limits_{n\to\infty}(s_n)$ is

(A) $\dfrac{\pi}{2}$ (B) 0

(C) 1 (D) $\dfrac{\pi}{4}$

**62.** Let $s_n=\sin\theta+\sin 2\theta+\sin 3\theta+\cdots+\sin(n\theta)$. Then

$$\lim_{n\to\infty}\left(\frac{s_1+s_2+s_3+\cdots+s_n}{n}\right)=$$

(A) $\frac{1}{2}\cot\frac{\theta}{2}$ (B) $\frac{1}{2}\tan\frac{\theta}{2}$

(C) 1 (D) $\frac{1}{4}\tan\frac{\theta}{2}$

**63.** $\lim_{n\to\infty}\left(\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}\right)=$

(A) $(\sin x)/x$ (B) 0

(C) $\sin x$ (D) does not exist

**64.** $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}=$

(A) $e$ (B) $1/e$

(C) $-e$ (D) $\sqrt{e}$

**65.** $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}=$

(A) 1/4 (B) 4

(C) 8 (D) 1/8

## Multiple Correct Choice Type Questions

**1.** Let

$$f(x)=\begin{cases}\text{Cos}^{-1}(\cot(x-[x])) & \text{for } x<\frac{\pi}{2}\\ \pi[x]-1 & \text{for } x\geq\frac{\pi}{2}\end{cases}$$

where $[x]$ is the integer part of $x$. Then

(A) $\lim_{x\to\frac{\pi}{2}-0} f(x)=\frac{\pi}{2}$ (B) $\lim_{x\to\frac{\pi}{2}-0} f(x)=\frac{\pi}{2}-1$

(C) $\lim_{x\to\frac{\pi}{2}+0} f(x)=\frac{\pi}{2}+1$ (D) $\lim_{x\to\frac{\pi}{2}+0} f(x)=\frac{\pi}{2}-1$

**2.** Let

$$f(x)=\begin{cases}x & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

$$g(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x & \text{if } x \text{ is irrational}\end{cases}$$

Then the function $f-g$ is

(A) continuous for all real $x$

(B) is one-one

(C) is onto

(D) is continuous exactly at $x=0$

**3.** Let

$$f(x)=\begin{cases}(\cos x-\sin x)^{\text{cosec}\,x} & \text{if } -\frac{\pi}{2}<x<0\\ a & \text{if } x=0\\ \dfrac{e^{1/x}+e^{2/x}+e^{3/x}}{ae^{2/x}+be^{3/x}} & \text{if } 0<x<\frac{\pi}{2}\end{cases}$$

Then

(A) $a=1/e$ (B) $b=1/e$

(C) $a=e$ (D) $b=e$

**4.** Let

$$f(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ 1 & \text{if } x>0\end{cases}$$

Then

(A) $\lim_{x\to 0} f(x)$ does not exist

(B) $\lim_{x\to 0}(f(x))^2=0$

(C) $\lim_{x\to 0}(f(x))^2=1$

(D) $\lim_{x\to 0}(f(x))^2$ does not exist

**5.** $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function satisfying the relation

$$f(x+2y, x-2y)=xy$$

for all $x, y\in\mathbb{R}$. Then

(A) $f(x, y)=\frac{x^2-y^2}{8}$

(B) $f(x, y)=\frac{x^2+y^2}{8}$

(C) $f(2\sqrt{2}, 0)=1$

(D) $f(3\sqrt{2}, 0)=\frac{9}{4}$

**6.** Let

$$A=\left(-\frac{1}{2}, 1\right), B=\left(-\frac{1}{2}, \frac{1}{2}\right), C=\left(\frac{1}{2}, 1\right), D=\left(\frac{3}{2}, \infty\right)$$

and $$f(x)=\log_{\left(x+\frac{1}{2}\right)}\left(\frac{x^2+2x-3}{4x^2-4x-3}\right)$$

Then $f$ is defined as $x$ belongs to

(A) $A$ (B) $B$

(C) $B\cup C$ (D) $B\cup C\cup D$

**7.** Let

$$f(x)=\begin{cases}0 & \text{if } x=0\\ x^2\sin\left(\dfrac{\pi}{2x}\right) & \text{if } |x|<1\\ x^2 & \text{if } |x|\ge 1\end{cases}$$

Then

(A) $f$ is even for $x \ge 1$

(B) $f$ is odd in $(-1, 1)$

(C) $f$ is continuous in $(-1, 1)$

(D) $f$ is discontinuous at $x = \pm 1$

**8.** Which of the following are true?

(A) $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x-11}\right)^{x+4} = e^{17}$

(B) $\lim\limits_{x\to\frac{\pi}{3}}\left(\dfrac{\tan^3 x - 3\tan x}{\cos\left(x+\dfrac{\pi}{6}\right)}\right) = -24$

(C) $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$ does not exist

(D) If $f:\mathbb{R}\to\mathbb{R}$ is continuous at origin and satisfies the relation

$$f(x+y)=f(x)+f(y)$$

for all $x$, $y$ in $\mathbb{R}$ and $f(1) = 2$, then $f(x) = 2x$ for $x \in \mathbb{R}$

**9.** For each $x \in [0, 1]$, let

$$f(x) = \text{Max}\,\{x^2, (x-1)^2, 2x(1-x)\}$$

Then

(A) $f$ is continuous at $x = 1/3$

(B) $f$ is continuous at $x = 2/3$

(C) $f$ is discontinuous at $x = 1/2$

(D) $f$ is continuous on $[0, 1]$

**10.** Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = x - [x]$ where $[x]$ is the integral part of $x$. Then

(A) $f$ is discontinuous at all integer values of $x$

(B) $f$ is continuous at all non-integer values of $x$

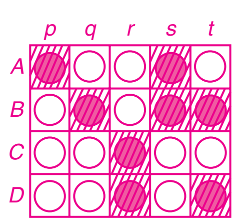
(C) $0 \le f(x) < 1 \;\forall x \in \mathbb{R}$

(D) $f(x)$ assumes 0 but never 1

## *Matrix-Match Type Questions*

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

**Example:** If the correct matches are (A) → (p), (s), (B) → (q), (s), (t),(C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t), then the correct darkening of bubbles will look as follows:




**1.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\tan^2 x+\tan x-2}{\sin x-\cos x}\right)=$ | (p) 2 |
| (B) $\lim\limits_{x\to\frac{\pi}{2}}(1-\sin x)\tan^2 x$ | (q) 1 |
| (C) $f(x)=\begin{cases}ax-b & \text{if } x\le 2\\ \lvert x^2-5x+6\rvert & \text{if } x>2\end{cases}$ | (r) $3\sqrt{2}$ |
| If $f$ is to be continuous on $\mathbb{R}$ then $b/a$ is equal to | (s) $2\sqrt{2}$ |
| (D) $f(x)=\begin{cases}2^{1-x} & \text{if } x<1\\ ax^2+bx+1 & \text{if } x\ge 1\end{cases}$ If $f$ is continuous at $x=1$ then the value of $a+b$ is | (t) 0 |

**2.** In Column I, functions are given and against them values of $x$ are mentioned at which $f$ is to be defined such that $f$ becomes continuous at the mentioned points. In Column II, the values of $f$ at these points are given. Match these.

| Column I | Column II |
|---|---|
| (A) $f(x)=\dfrac{x-3}{\sqrt[3]{x^2-1}-2}$ at $x=3$ | (p) 2 |
| (B) $f(x)=\dfrac{\cos^2 x-\sin^2 x-1}{\sqrt{x^2+1}-1}$ at $x=0$ | (q) 1/2 |
| (C) $f(x)=\dfrac{1}{x}\tan\left(\dfrac{x}{2}\right)$ at $x=0$ | (r) −1/2 |
| (D) $f(x)=\dfrac{\tan^2 x-2\tan x-3}{\tan^2 x-4\tan x+3}$ at $x=\mathrm{Tan}^{-1}(3)$ | (s) −4 |
| | (t) 0 |

**3.** Match the items of Column I with those of Column II.

| Column I | Column II |
|---|---|
| (A) $\lim\limits_{x\to 0}\left(\dfrac{\tan 3x-\tan^3 x}{\tan x}\right)$ is | (p) 1/2 |
| (B) $\lim\limits_{x\to 3} 4\left(\dfrac{\sqrt{x^2+7}-4}{x^2-5x+6}\right)$ is | (q) $-\sqrt{2}$ |
| (C) $\lim\limits_{x\to 2}\left(\dfrac{2^{x+2}-16}{4^x-16}\right)$ is | (r) 3 |
| (D) $\lim\limits_{x\to \frac{\pi}{4}}\left[(\sin x-\cos x)\tan\left(\dfrac{\pi}{4}+x\right)\right]$ is | (s) $\sqrt{2}$ |
| | (t) −1/2 |

**4.** In Column I, $n$th terms of a sequence are given. In Column II, their respective limits as $n\to\infty$ are given. Match them.

| Column I | Column II |
|---|---|
| (A) $\dfrac{\sqrt[3]{n^2+n}}{n+1}$ | (p) 1 |
| (B) $\dfrac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[3]{n^4+2}-\sqrt{n^3+1}}$ | (q) 4/3 |
| (C) $\dfrac{(n+2)!+(n+1)!}{(n+3)!}$ | (r) 3/4 |
| (D) $\dfrac{1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}}$ | (s) 0 |
| | (t) 2/3 |

**5.** Match the items of Column I with those of Column II.

| Column I | Column II |
|---|---|
| (A) $\lim\limits_{x\to 0}\left(\dfrac{e^{x^2}-\cos x}{x^2}\right)=$ | (p) 1 |
| (B) $\lim\limits_{x\to 0}\left(\dfrac{e^{\sin 2x}-e^{\sin x}}{x}\right)=$ | (q) 3 |
| (C) $\lim\limits_{x\to\infty}(x(e^{1/x}-1))$ | (r) 2 |
| (D) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{\cos 2x}}{\tan^2(x/2)}=$ | (s) $e$ |
| | (t) 3/2 |

## Comprehension-Type Questions

**1. Passage:** $f:\mathbb{R}\to\mathbb{R}$ is a function satisfying the following three conditions:

(a) $f(-x)=-f(x)\ \forall x\in\mathbb{R}$

(b) $f(x+1)=f(x)+1\ \forall x\in\mathbb{R}$

(c) $f\left(\dfrac{1}{x}\right)=\dfrac{f(x)}{x^2}\ \forall x\neq 0$

Answer the following questions.

**(i)** $\lim\limits_{x\to 2}\dfrac{2^{f(x+1)}-2^{f(x)}}{x}$ is

(A) 2 (B) log 2

(C) 2 log 2 (D) 2/(log 2)

**(ii)** $\lim\limits_{x\to 1}(f^{-1}(x))$ is

(A) 1 (B) 0

(C) does not exist (D) $e$

**(iii)** The number of common points of the graph of $y=f(x)$ with the line $y=(x)$ is

(A) 2 (B) 4

(C) 8 (D) infinite

**Hint:** $f(x)=x\ \forall x\in R$.

**2. Passage:** Let $f(x)$ be a function defined in a neighbourhood of $a\in\mathbb{R}$. Then $\lim\limits_{x\to a}f(x)$ exists finitely if and only if $\lim\limits_{x\to a-0} f(x)$, $\lim\limits_{x\to a+0} f(x)$ exist finitely and are equal. Further if $f(a)$ is defined and $\lim\limits_{x\to a} f(x) = f(a)$, then $f$ is said to be continuous at $x = a$. Answer the following questions.

**(i)** Let $f(x) = \begin{cases} 1+\dfrac{2x}{a} & \text{for } 0 \le x < 1 \\ ax & \text{for } 1 \le x < 2 \text{ where } a > 0. \end{cases}$

If $\lim\limits_{x\to 1} f(x)$ exist, then $a$ is equal to

(A) $-1$ (B) 1
(C) $-2$ (D) 2

**(ii)** Let

$$f(x) = \begin{cases} x^2+x+1 & \text{if } x \ge -1 \\ \sin(a-1)\pi & \text{if } x < -1 \end{cases}$$

If $f$ is to be continuous at $x = -1$, then

(A) $a = \dfrac{3}{2} + 2n$ (B) $a = \dfrac{1}{2} + 2n$
(C) $a = \dfrac{1}{2} + 2n\pi$ (D) $a = \dfrac{3}{2} + 2n\pi$

**(iii)** Let

$$f(x) = \begin{cases} ax+1, & x \le \dfrac{\pi}{2} \\ \sin x + b, & x > \dfrac{\pi}{2} \end{cases}$$

If $f$ is continuous at $x = \pi/2$ then the $a:b$ is equal to

(A) $\pi:2$ (B) $2:\pi$
(C) $1:\pi$ (D) $\pi:1$

**3.** $f(x)$ is real-valued function satisfying the functional relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x}$$

for all $x \ne 0$ and 1. $g(x)$ is a polynomial of degree $n$ satisfying the relation

$$g(x) + g(y) + g(xy) = 2 + g(x)g(y)$$

for all real $x$ and $y$ and $g(4) = 17$. Answer the following questions.

**(i)** $\lim\limits_{x\to 1} f(x) =$

(A) 0 (B) 1
(C) does not exist (D) 2

**(ii)** The number of real solutions of the equation $(g \circ f)(x) = 2$ is

(A) 1 (B) 2
(C) 0 (D) infinite

**(iii)** Number of discontinuities of $g \circ f$ is

(A) 2 (B) 4
(C) 3 (D) 1

**Hint:** Show that $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = x^2 + 1$

## Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation Statement I.
(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.
(C) Statement I is true and Statement II is false.
(D) Statement I is false and Statement II is true.

**1.** **Statement I:** $f:(0,1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is a bounded function.

**Statement II:** Every continuous function defined on a closed interval is bounded.

**2.** **Statement I:** If $-1 < x^1$, than $\lim\limits_{x\to 0} x^n = 0$.

**Statement II:** For $n \ge 3$, $-x^2 \le x^n \le x^2$ where $-1 < x < 1$.

**3.** **Statement I:** Let $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Then $f$ is continuous at $x = 0$.

**Statement II:** Let $a \in \mathbb{R}$. In $a$ neighbourhood of $a$ two functions $f$ and $g$ are defined such that $\lim\limits_{x\to a} f(x) = 0$ and $g(x)$ is bounded. Then $\lim\limits_{x\to a} f(x)g(x) = 0$.

**Hint:** See Corollary 1.3.

**4.** **Statement I:** If $n$ is a positive integer, then $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$.

**Statement II:**

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin\left(\dfrac{n+1}{2}\right)\theta \sin\dfrac{n\theta}{2}}{\sin(\theta/2)}$$

and $\lim\limits_{\theta\to 0}\left(\dfrac{\sin\theta}{2\theta}\right) = 1$

5. **Statement I:** If $f$ and $g$ are continuous at $x = a$, then $h(x) = \text{Max}\,(f(x), g(x))$ is also continuous at $x = a$.

**Statement II:** $\text{Max}(f, g) = \frac{1}{2}(f+g)+\frac{1}{2}|f-g|$

6. **Statement I:** Let $f(x) = \begin{cases} \sin\frac{1}{x} & \text{for} \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Then $f$ is discontinuous at $x = 0$.

**Statement II:** $f$ assumes all values between –1 and 1 both inclusive.

7. **Statement I:** Suppose $f$ is continuous on the closed interval $[0, 2]$ and that $f(0) = f(2)$. Then there exist $x$, $y$ in $[0, 2]$ such that $f(x) = f(y)$ and $|y-x| = 1$.

**Statement II:** If $g$ is continuous on a closed interval $[a, b]$ and $g(a)g(b) < 0$, then $g(x)$ varies for some value in $[a, b]$.

**Hint:** Consider $g(x) = f(x+1) - f(x)$ on $[0, 1]$ to prove Statement I.

8. **Statement I:** Let $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

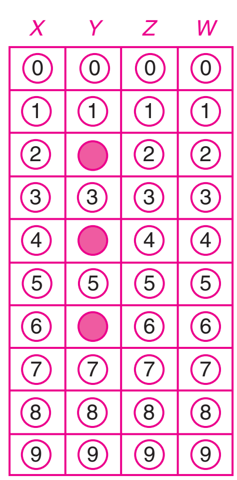
Then $f$ is continuous at $x = 0$ and discontinuous at all $x = 0$.

**Statement II:** A function $g$ is continuous at $x_0 \in [a, b]$ if and only if for any sequence $\{x_n\}$ of real numbers in $[a, b]$, $x_n \to x_0$ as $n \to \infty$ implies $f(x_n) \to f(x_0)$ as $n \to \infty$.

**Hint:** If $x_0 \in \mathbb{R}$, then there exist sequences of rational and irrationals tending to $x_0$.

## *Integer Answer Type Questions*

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number $Y$ is 246, then the bubbles under $Y$ labeled as 2, 4, 6 are to be darkened.




1. $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)} = e^k$ where $k$ is _______.

2. If $\dfrac{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)\lim_{x\to\infty}\left(1-\frac{2}{x}\right)}{\lim_{x\to\infty}\left(\frac{3}{x^2}\right)-\lim_{x\to\infty}\left(\frac{3}{x}\right)+2} = l$, then $\frac{1}{l}$ is _______.

3. If $f : (a, \infty) \to \mathbb{R}$ is a function such that $\lim_{x\to\infty}(x\,f(x) = l, l \in IR)$, then $\lim_{x\to\infty} f(x)$ is equal to _______.

4. Let $f(x) = 2^{-x}$ and $g(x) = e^x$ for all $x \in IR$. Then $\lim_{x\to\infty}(f \circ g)(x)$ equals_______.

5. $\lim_{x\to\infty} 2x^2\left(1-\cos\frac{1}{x}\right)$ is equal to _______.

**Hint:** Put $x = \frac{1}{\theta}$.

6. If $a = \lim_{x\to 0}(\cos x + \sin x)^{1/x}$, then integer part of $a$ is _______.

**Hint:** Show that $a = e$.

7. Let $f(x) = a_0 + a_1x + a_2x^3 + a_4x^4 + a_5x^5 + a_6x^6$ where $a_0$, $a_1, \ldots, a_6$ are real and $a_6 \neq 0$. If $\lim_{x\to 0}\left(\frac{1+f(x)}{x^3}\right)^{1/x} = e^2$, then $\sum_{i=0}^{4} a_i$ is equal to _______.

8. $\lim_{x\to 2}\left(\dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}\right)$ is equal to _______.

9. Let $x_1 = 3$ and $x_{n+1} = \sqrt{2+x_n}$ for $n \ge 1$. Then $\lim\limits_{x\to\infty} x_n$ exists and is equal to _______.

10. $\lim\limits_{x\to\infty}\left(\dfrac{3e^x + 2e^x}{e^x + 3e^{-x}}\right)$ is _______.

11. $\lim\limits_{x\to\infty}\left(\dfrac{2^{1/x} + \left(\frac{3}{2}\right)^{1/x} + 3^{1/x}}{3}\right)^{3x}$ is _______.

12. Let $f(x) = \text{Max}\{x, x^2\}$. Then the number of points at which $f$ is not continuous is _______.

## ANSWERS

### Single Correct Choice Type Questions

1. (B)
2. (D)
3. (D)
4. (A)
5. (D)
6. (C)
7. (B)
8. (A)
9. (A)
10. (C)
11. (B)
12. (D)
13. (B)
14. (A)
15. (C)
16. (D)
17. (B)
18. (B)
19. (A)
20. (A)
21. (D)
22. (A)
23. (A)
24. (B)
25. (A)
26. (C)
27. (D)
28. (B)
29. (C)
30. (B)
31. (A)
32. (B)
33. (B)
34. (A)
35. (A)
36. (B)
37. (D)
38. (A)
39. (C)
40. (A)
41. (A)
42. (C)
43. (B)
44. (D)
45. (C)
46. (B)
47. (A)
48. (D)
49. (C)
50. (B)
51. (A)
52. (C)
53. (C)
54. (A)
55. (B)
56. (B)
57. (B)
58. (C)
59. (B)
60. (B)
61. (D)
62. (A)
63. (A)
64. (B)
65. (D)

### Multiple Correct Choice Type Questions

1. (A), (D)
2. (B), (C), (D)
3. (A), (D)
4. (A), (C)
5. (A), (C), (D)
6. (B), (C), (D)
7. (A), (B), (C)
8. (A), (B), (C), (D)
9. (A), (B), (D)
10. (A), (B), (C), (D)

## Matrix-Match Type Questions

1. (A) $\to$ (r); (B) $\to$ (t); (C) $\to$ (p); (D) $\to$ (p)
2. (A) $\to$ (p); (B) $\to$ (s); (C) $\to$ (q); (D) $\to$ (p)
3. (A) $\to$ (r); (B) $\to$ (r); (C) $\to$ (p); (D) $\to$ (q)
4. (A) $\to$ (s); (B) $\to$ (s); (C) $\to$ (s); (D) $\to$ (q)
5. (A) $\to$ (t); (B) $\to$ (p); (C) $\to$ (p); (D) $\to$ (q)

## Comprehension Type Questions

1. (i) (A); (ii) (A); (iii) (D)
2. (i) (D); (ii) (A); (iii) (B)
3. (i) (C); (ii) (A); (iii) (D)

## Assertion–Reasoning Type Questions

1. (D)
2. (A)
3. (A)
4. (A)
5. (A)
6. (B)
7. (A)
8. (A)

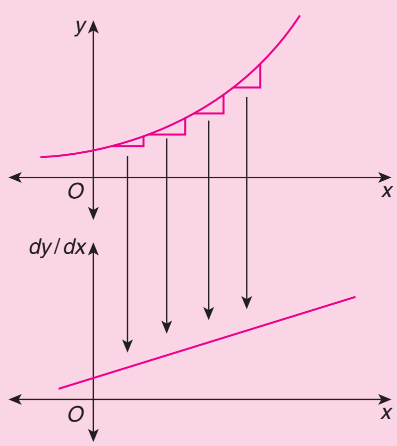
## Integer Answer Type Questions

1. 2
2. 2
3. 0
4. 0
5. 1
6. 2
7. 2
8. 8
9. 2
10. 3
11. 9
12. 0

# Derivative and Differentiability

# 2

Derivative and Differentiability




Contents



The **derivative** is a measure of how a function changes as its input changes. In other words, a derivative can be thought of as how much one quantity is changing in response to changes in some other quantity. The process of finding a derivative is called **differentiation**.

## 2.1 | Derivatives: An Introduction

Until the 17th century, a curve was described as a locus of points satisfying certain geometric conditions. It seems that mathematicians realized that all the curves cannot be described using geometrical conditions. To overcome this, Analytical Geometry was created and developed by Rene' Descartes (1596–1650) and Pierre De Fermat (1601–1665). In this new found idea, geometric problems were re-described in terms of algebraic equations and new class of curves defined algebraically rather than using geometric conditions. The concept of *derivative* evolved in this new context which led to its geometrical interpretation as a slope of a tangent to a curve, velocity of a particle and rate measure. The notion of derivate of a function was initiated by Newton and Leibniz in 1680s. Wherever we consider rate of change of a function, naturally differentiation steps in. Let us begin with the formal definition of derivative.

**DEFINITION 2.1** Suppose $f$ is a function defined in a neighbourhood of a point $c$ (i.e., there exist real numbers $a, b$ such that $a < c < b$ and $f:[a, b] \to \mathbb{R}$). If

$$\lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \right)$$

exists then we say that $f$ is differentiable at $c$ and then the limit,

$$\lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \right)$$

denoted by $f'(c)$, is called the derivative of $f$ at $c$.

We also say that $f'(c)$ is the differential coefficient of $f$ at $c$. Usually if we write $y = f(x)$, we denote $f'(c)$ by $(df/dx)_{x=c}$ or $(dy/dx)_{x=c}$. If $f$ is differentiable at every point $c$ of $(a, b)$, then we say that $f$ is differentiable in the open interval $(a, b)$. If $f$ is differentiable at each point in its domain, we obtain a new function on the domain of $f$ called the *derived function* which is denoted by $f'(x)$ or $df/dx$ or $dy/dx$, where $y = f(x)$.

**DEFINITION 2.2** **Left and Right Derivatives** Suppose $f$ is *a* function defined in *a* neighbourhood of $c$ as described in Definition 2.1. If

$$\lim_{x \to c-0} \left( \frac{f(x) - f(c)}{x - c} \right)$$

exists, then it is denoted by $f'(c-0)$ and is called the *left derivative* of $f$ at $c$. Similarly, if

$$\lim_{x \to c+0} \left( \frac{f(x) - f(c)}{x - c} \right)$$

exists, then it is denoted by $f'(c+0)$ and is called the *right derivative* of $f$ at $c$.

**QUICK LOOK 1**

1. $f'(c-0) = \lim\limits_{\substack{h \to 0 \\ h > 0}} \left( \dfrac{f(c-h) - f(c)}{-h} \right)$
2. $f'(c+0) = \lim\limits_{\substack{h \to 0 \\ h > 0}} \left( \dfrac{f(c+h) - f(c)}{h} \right)$
3. $f'(c) = \lim\limits_{h \to 0} \left( \dfrac{f(c+h) - f(c)}{h} \right)$

### Example 2.1

Let $f(x) = x^2$ in $(0, 2)$. Show that $f$ is differentiable at 1 and find $f'(1)$.

***Solution:*** Take $c = 1$. Then

$$\lim_{x\to 1}\left(\frac{f(x)-f(1)}{x-1}\right)=\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$$
$$=\lim_{x\to 1}(x+1)=2$$

Hence $f$ is differentiable at 1 and $f'(1)=2$.

## Example 2.2

Let $f(x)=x^3$ in $(-1, 1)$. Show that $f$ is differentiable at 0 and find $f'(0)$.

***Solution:*** Take $c=0$. Then

$$\lim_{x\to 0}\left(\frac{f(x)-f(0)}{x-0}\right)=\lim_{x\to 0}\left(\frac{x^3-0}{x}\right)$$
$$=\lim_{x\to 0}(x^2)=0$$

Hence $f$ is differentiable at 0 and $f'(0)=0$.

## Example 2.3

Let $f(x)=|x|$ in $(-1, 1)$. Is $f$ differentiable at 0?

***Solution:*** Take $c=0$. Then

$$\lim_{x\to 0}\left(\frac{f(x)-f(0)}{x-0}\right)=\lim_{x\to 0}\left(\frac{|x|}{x}\right)$$

Hence

$$\lim_{x\to 0}\left(\frac{f(x)-f(0)}{x-0}\right)$$

does not exist. Therefore $f(x)=|x|$ is not differentiable at 0. Note that $|x|$ is continuous at 0.

## Example 2.4

Let $f(x)=K$ for all $x\in[a, b]$ (i.e., $f$ is a constant function on $[a, b]$). Show that $f$ is differentiable at 0 and find $f'(c)$ where $c\in(a, b)$.

***Solution:*** Let $c\in(a, b)$. Then

$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)=\lim_{x\to c}\left(\frac{K-K}{x-c}\right)=0$$

Hence, $f$ in differentiable at $c$ and $f'(c)=0$.

More usually, if $f$ is a constant function on an interval $J$, then $f'(x)=0\ \forall\, x\in J$.

## Example 2.5

Let

$$f(x)=\begin{cases} x\sin\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

Show that $f$ is not differentiable at 0.

***Solution:*** For the given function we have that

$$\lim_{x\to 0}\left(\frac{f(x)-f(0)}{x-0}\right)=\lim_{x\to 0}\frac{x\sin(1/x)}{x}$$
$$=\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$$

does not exist. Therefore $f$ is not differentiable at 0. Also, we know that $f$ is continuous at 0.

In the following theorem, we prove that a necessary condition for a function to be differentiable at a point is that the function is continuous at that point. Theorem 2.1 gives a more precise description.

**THEOREM 2.1** If $f:[a, b]\to\mathbb{R}$ is differentiable at $c\in(a, b)$, then $f$ is continuous at $c$.

**PROOF** By hypothesis

$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)=f'(c)$$

Let $\varepsilon>0$. Then there exists $\delta$ such that

$$0<\delta<\frac{\varepsilon}{|f'(c)|+\varepsilon},\ (c-\delta, c+\delta)\subset(a, b)\ \text{and}\ x\in(c-\delta, c+\delta)$$

$x\neq c$ implies

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\varepsilon$$

Now,

$$\begin{aligned} x\in(c-\delta, c+\delta)\Rightarrow|f(x)-f(c)|&<(|f'(c)|+\varepsilon)|x-c|\\ &<|f'(c)+\varepsilon|\delta<\varepsilon\end{aligned}$$

Hence $f$ is continuous at $c$. ■

***Note:*** The converse of the above theorem is not true. See Example 2.3. However, *there are functions which are everywhere continuous on* $\mathbb{R}$ *but nowhere differentiable*. One such example is as follows.

## Example

Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}\cos(3^n x),\ x\in\mathbb{R}$$

Then $f$ is continuous on $\mathbb{R}$ but not differentiable at any point of $\mathbb{R}$. The proof of this result is beyond the scope of this book.

In the following theorem, we discuss the differentiability of sum, the product, and the quotient of two differentiable functions.

**THEOREM 2.2** Suppose $f$ and $g$ are differentiable at $c$ and $\lambda, \mu$ are any two real numbers. Then

**(i)** $\lambda f+\mu g$ is differentiable at $c$ and $(\lambda f+\mu g)'(c)=\lambda f'(c)+\mu g'(c)$.

**(ii)** $fg$ is differentiable at $c$ and

$$(fg)'(c)=f'(c)g(c)+f(c)g'(c)\ \text{(Product Rule)}$$

**(iii)** If $g(c)\neq 0$, then $f/g$ is differentiable at $c$ and

$$\left(\frac{f}{g}\right)'(c)=\frac{f'(c)g(c)-g'(c)f(c)}{(g(c))^2}\ \text{(Quotient Rule)}$$

**PROOF** **(i)** We write, when $x\neq c$,

$$p(x)=\frac{f(x)-f(c)}{x-c}\quad\text{and}\quad q(x)=\frac{g(x)-g(c)}{x-c}$$

By hypothesis $p(x)\to f'(c)$ and $q(x)\to g'(c)$ as $x\to c$. Hence

$$(\lambda p+\mu q)(x)\to\lambda f'(c)+\mu g'(c)\ \text{as}\ x\to c$$

That is

$$\frac{(\lambda f+\mu g)(x)-(\lambda f+\mu g)(c)}{x-c}=\lambda\left(\frac{f(x)-f(c)}{x-c}\right)+\mu\left(\frac{g(x)-g(c)}{x-c}\right)$$
$$=\lambda p(x)+\mu q(x)$$
$$\to \lambda f'(c)+\mu g'(c) \quad \text{as } x\to c$$

Therefore $(\lambda f+\mu g)$ is differentiable at $c$ and

$$(\lambda f+\mu g)'(c)=\lambda f'(c)+\mu g'(c)$$

**(ii)** We have

$$(fg)(x)-(fg)(c)=f(x)g(x)-f(c)g(c)$$
$$=(f(x)-f(c))g(x)+f(c)(g(x)-g(c))$$

Therefore

$$\lim_{x\to c}\left(\frac{(fg)(x)-(fg)(c)}{x-c}\right)=\lim_{x\to c}\frac{(f(x)-f(c))g(x)}{x-c}+\lim_{x\to c} f(c)\frac{(g(x)-g(c))}{x-c}$$
$$=f'(c)g(c)+f(c)g'(c) \quad [\because\ g'(c) \text{ exists} \Rightarrow \lim_{x\to c} g(x)=g(c)]$$

Therefore, $fg$ is differentiable at $c$ and

$$(fg)'(c)=f'(c)g(c)+f(c)g'(c)$$

**(iii)** Now $f$ is continuous at $c$ implies that $g$ is continuous at $c$. Also since $g$ is continuous and $g(c)\neq 0$, by Theorem 1.8 $g(x)\neq 0$ in a neighbourhood $(c-\delta, c+\delta)$ of $c$. Hence $f/g$ is well defined in $(c-\delta, c+\delta)$. Now

$$\frac{1}{x-c}\left[\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(c)\right]=\frac{1}{x-c}\left(\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)}\right)$$
$$=\frac{1}{x-c}\,\frac{[\{f(x)-f(c)\}g(c)-f(c)\{g(x)-g(c)\}]}{g(x)g(c)}$$
$$=\frac{\left(\frac{f(x)-f(c)}{x-c}\right)g(c)-f(c)\left(\frac{g(x)-g(c)}{x-c}\right)}{g(x)g(c)}$$
$$\to \frac{f'(c)g(c)-f(c)g'(c)}{g(c)g(c)}$$

($\because\ g(c)\neq 0$ and $g$ is differentiable at $c$ implies $g(x)\to g(c)$ as $x\to c$). Therefore $f/g$ is differentiable at $c$ and

$$\left(\frac{f}{g}\right)'(c)=\frac{f'(c)\,g(c)-f(c)\,g'(c)}{(g(c))^2}$$

■

The following theorem is on differentiability of composite function and is also called **chain rule**.

**THEOREM 2.3 (CHAIN RULE)** If $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$, then $g\circ f$ is differentiable at $c$ and $(g\circ f)'(c)$ is equal to $g'(f(c))\cdot f'(c)$.

**PROOF** Write $u = f(c)$ and $y = f(x)$. Let

$$R(x) = \frac{f(x) - f(c)}{x - c} - f'(c)$$

for $x \neq c$ so that $R(x) \to 0$ as $x \to c$. Now, let

$$S(y) = \frac{g(y) - g(u)}{y - u} - g'(u)$$

for $y \neq u$ so that $S(y) \to 0$ as $y \to u$. [Since $g$ is differentiable at $u = f(c)$.] Now,

$$\begin{aligned}
\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} &= \frac{g(f(x)) - g(f(c))}{x - c} \\
&= \frac{g(y) - g(u)}{x - c} \\
&= \left(\frac{g(y) - g(u)}{y - u}\right) \cdot \left(\frac{y - u}{x - c}\right) \\
&= [g'(u) + S(y)]\left(\frac{y - u}{x - c}\right) \\
&= [g'(u) + S(y)]\left(\frac{f(x) - f(c)}{x - c}\right) \\
&\to [g'(u) + 0] f'(c) \quad \text{as } x \to c
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{x \to c} \frac{[(g \circ f)(x) - (g \circ f)(c)]}{x - c} &= g'(u) f'(c) \\
&= g'(f(c)) f'(c)
\end{aligned}$$

Hence $g \circ f$ is differentiable at $c$ and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$. ■

***Note:*** The chain rule is also called as *differentiation by substitution*.

The following theorem, known as Carathéodory Theorem (asked in IIT-JEE 2001 mains), gives us a necessary and sufficient condition for the differentiability of a function at a point.

**THEOREM 2.4 (CARATHÉODORY THEOREM)** Suppose $f : [a, b] \to \mathbb{R}$ is a function and $c \in (a, b)$. Then $f$ is differentiable at $c$ if and only if

**(i)** $f$ is continuous at $c$.

**(ii)** There exists a function $g : [a, b] \to \mathbb{R}$ such that $g$ is continuous at $c$ and $f(x) - f(c) = g(x)(x-c)$ for all $x \in [a, b]$. In this case, $g(c) = f'(c)$.

**PROOF** Suppose $f$ is differentiable at $c$. Then by Theorem 2.1, $f$ is continuous at $c$. Thus (i) holds. Now define

$$g(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \neq c, x \in [a, b] \\ f'(c) & \text{if } x = c \end{cases} \tag{2.1}$$

Then

$$\begin{aligned}
\lim_{x \to c} g(x) &= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}\right) \\
&= f'(c) \quad (\because f \text{ is differentiable at } c) \\
&= g(c)
\end{aligned}$$

Hence $g$ is continuous at $c$. Further, by Eq. (2.1)

$$f(x) - f(c) = g(x)(x-c)\ \forall x \in [a, b]$$

Hence (ii) holds and by definition $g(c) = f'(c)$.
Conversely, suppose that both (i) and (ii) hold. Then

$$\lim_{x \to c}\left(\frac{f(x)-f(c)}{x-c}\right) = \lim_{x \to c} g(x)$$
$$= g(c) \quad (\because g \text{ is continuous at } c)$$

Hence $f$ is differentiable at $c$ and $f'(c) = g(c)$. ■

Here is an example to illustrate the above theorem.

## Example

Let $f(x) = x^3$ for $x \in \mathbb{R}$ and $a \in \mathbb{R}$. We know that

$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$

Here $g(x) = x^2 + ax + a^2$ satisfies the conditions of Carathéodory's theorem and $f'(a) = g(a) = 3a^2$.

**Theorem 2.5 (Differentiability of the Reciprocal of a Function)** Let $f:[a, b] \to \mathbb{R}$ be a function and $f(x) \neq 0\ \forall\ x \in [a, b]$. Let $c \in (a, b)$ and $f$ be differentiable at $c$. Then the function

$$g = \frac{1}{f} \text{ (i.e. } g(x) = \left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}\ \forall\ x \in [a, b])$$

is differentiable at $c$ and

$$g'(c) = \frac{-f'(c)}{(f(c))^2}$$

**Proof** We can consider $g$ as a quotient of two functions by considering

$$g(x) = \frac{h(x)}{f(x)}\ \forall\ x \in [a, b]$$

where $h(x) = 1$ is the constant function on $[a, b]$. Then by Quotient Rule

$$g'(c) = \frac{h'(c)f(c) - h(c)f'(c)}{(f(c))^2}$$
$$= \frac{0 \cdot f(c) - 1 \cdot f'(c)}{(f(c))^2}$$
$$= \frac{-f'(c)}{(f(c))^2}$$

In the second step we have used the fact that since $h$ is a constant function, by Example 2.4, $h'(c) = 0$. ■

***Advice:*** It is better for the reader to prove Theorem 2.5 using the concept of the differentiable coefficient and the property that if $f$ is differentiable at $c$, then $f$ is continuous at $c$.

**Theorem 2.6** Let $f:[a,b]\to[c,d]$ be a bijection and $g$ be the inverse of $f$. If $f$ is differentiable at $x_0\in(a,b)$ and $f'(x_0)\neq 0$ and $g$ is continuous at $f(x_0)$, then $g$ is differentiable at $f(x_0)$ and

$$g'(f(x_0))=\frac{1}{f'(x_0)}$$

**Proof** Write $y=f(x)$ and $y_0=f(x_0)$. Let $k$ be a non-zero real number such that $y_0+k\in[c,d]$. Write

$$g(y_0+k)-g(y_0)=h$$

so that

$$g(y_0+k)=g(y_0)+h=x_0+h \tag{2.2}$$

and hence

$$f(x_0+h)=f(g(y_0+k))=y_0+k \quad (\because g=f^{-1}) \tag{2.3}$$

Since $f'(x_0)\neq 0$ and

$$\frac{f(x_0+h)-f(x_0)}{h}\to f'(x_0) \text{ as } h\to 0$$

we have

$$\lim_{h\to 0}\frac{h}{f(x_0+h)-f(x_0)}=\frac{1}{f'(x_0)}$$

Now, by Eqs. (2.2) and (2.3) we obtain

$$\begin{aligned}\frac{g(y_0+k)-g(y_0)}{k}&=\frac{h}{f(x_0+h)-y_0}\\&=\frac{h}{f(x_0+h)-f(x_0)}\to\frac{1}{f'(x_0)} \text{ as } h\to 0\end{aligned}$$

Since $h\to 0$ implies $k\to 0$, we have that

$$\lim_{k\to 0}\frac{g(y_0+k)-g(y_0)}{k}=\frac{1}{f'(x_0)}$$

Hence $g=f^{-1}$ is differentiable at $f(x_0)$ and

$$g'(f(x_0))=\frac{1}{f'(x_0)}$$

■

***Note:*** In the lines immediately succeeding Definition 2.1, we mentioned that if $y=f(x)$ is differentiable at each point of its domain then we denote $f'(x)$ by $dy/dx$. Accordingly, if $y=f(x)$ admits $f^{-1}$, then $x=f^{-1}(y)$ so that

$$(f^{-1})'(y)=\frac{1}{dy/dx}$$

which we write as $dx/dy$ and hence

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=f'(x)(f^{-1})'(y)=\frac{f'(x)}{f'(x)}=1$$

It is in this sense, we write

$$\frac{dx}{dy}=\frac{1}{dy/dx}$$

We now recall the concepts of increasing and decreasing functions (see Definition 1.40, p. 52, Vol. 1).

**DEFINITION 2.3** Let $A$ be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Then

1. $f$ is called *increasing* or *monotonically increasing* if $f(x) \le f(y)$ whenever $x \le y$ belong to $A$. If $f(x) < f(y)$ whenever $x < y$, then $f$ is called *strictly increasing function* or *strictly monotonic increasing*.
2. $f$ is called *decreasing* or *monotonically decreasing* if $f(x) \ge f(y)$ whenever $x < y$. If $f(x) > f(y)$ whenever $x < y$, then $f$ is called *strictly decreasing* or *strictly monotonic decreasing*.

The following is a corollary to Theorem 2.6.

**COROLLARY 2.1** Suppose $f : [a, b] \to \mathbb{R}$ is continuous and strictly monotonic increasing. Write $\alpha = f(a)$ and $\beta = f(b)$ so that $f^{-1}$ exists, is continuous and strictly monotonic increasing on $[\alpha, \beta]$. If $a < c < b$ and $f$ is differentiable at $c$ and $f^{-1}(c) \ne 0$, then $f^{-1}$ is differentiable at $f(c)$ and

$$(f^{-1})'(c) = \frac{1}{f'(c)}$$

**PROOF** Since $f$ is strictly increasing on $[a, b]$, $f$ is bijection from $[a, b]$ to $[\alpha, \beta]$ and hence $f^{-1}$ exists on $[\alpha, \beta]$. Since $f'(c) \ne 0$, by Theorem 2.6, $f^{-1}$ is differentiable at $c$ and

$$(f^{-1})'(c) = \frac{1}{f'(c)}$$

The following proof is based on Carathéodory's theorem and hence we can say that this is also corollary to Carathéodory's theorem.

**ALITER** Since $f$ is differentiable at $c$, by Carathéodory's theorem there exists a function $h : [a, b] \to \mathbb{R}$ such that $h$ is continuous at $c$ and $h(c) = f'(c)$ and $f(x) - f(c) = h(x)(x - c)$ for all $x \in [a, b]$. Since $h$ is continuous at $c$ and $h(c) \ne 0$, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and $h(x) \ne 0$ in $(c - \delta, c + \delta)$. Write $I = (c - \delta, c + \delta)$. If $y \in f(\mathrm{I})$, then

$$\begin{aligned} f(f^{-1}(y)) - f(c) &= h(f^{-1}(y))(f^{-1}(y) - c) \\ &= h(f^{-1}(y))(f^{-1}(y) - f^{-1}(f(c))) \end{aligned}$$

Hence

$$\begin{aligned} f^{-1}(y) - f^{-1}(f(c)) &= \frac{f(f^{-1}(y)) - f(c)}{h(f^{-1}(y))} \\ &= \frac{y - f(c)}{h(f^{-1}(y))} \\ &= \frac{y - f(c)}{(h \circ f^{-1})(y)} \end{aligned}$$

By Carathéodory's theorem, $f^{-1}$ is differentiable at $f(c)$ and

$$\begin{aligned} (f^{-1})'(f(c)) &= \frac{1}{(h \circ f^{-1})(f(c))} \\ &= \frac{1}{h(c)} \\ &= \frac{1}{f'(c)} \end{aligned}$$

Hence

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}$$

■

***Note:***

1. The theorem is equally valid if $f$ is continuous and strictly decreasing on $[a, b]$.
2. If $f'(c) = 0$, the above result is ***not*** valid. For, let $f(x) = x^3$, then $f$ is continuous and strictly increasing on $\mathbb{R}$ and $f$ is differentiable at 0. But $f^{-1}(x) = x^{1/3}$ is not differentiable at 0 because $f^{-1}(0) = 0$, $1/f'(0)$ is not defined.

## 2.2 | Derivatives of Some Standard Functions

In this section we derive the derivatives of the standard functions like $x^\alpha$, $\log x$, $e^x$, $\sin x$, $\cos x$, etc. and later we will use them in solving problems.

### Example 2.6

Let $a > 0$ and $\alpha$ is real. Then show that the function $f(x) = x^\alpha$, $x > 0$ is differentiable at $a$ and $f'(a) = \alpha\, a^{\alpha-1}$. In particular, if $n$ is rational and $x > 0$, then $f'(x) = n\, x^{n-1}$.

***Solution:*** We have (by Theorem 1.23)

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\left(\frac{x^\alpha - a^\alpha}{x-a}\right) = \alpha\, a^{\alpha-1}$$

Therefore

$$f'(a) = \alpha\, a^{\alpha-1}$$

If we replace $a$ by $x$ and $\alpha$ by $n$, then $f'(x) = nx^{n-1}$.

### Example 2.7

If $a > 0$, show that the function $f(x) = \log x$ (with base $e$) is differentiable at $a$ and $f'(a) = 1/a$. In general, $f'(x) = 1/x$ where $x > 0$.

***Solution:*** We have

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\left(\frac{\log x - \log a}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{\log(x/a)}{x-a}\right)$$

$$= \lim_{x\to a} \frac{\log\left(1+\left(\frac{x}{a}-1\right)\right)}{\left(\frac{x}{a}-1\right)}\cdot\frac{1}{a}$$

$$= \lim_{y\to 0} \frac{\log(1+y)}{ay} \qquad \left(\text{where } y = \frac{x}{a}-1\right)$$

$$= \frac{1}{a}\cdot 1 \text{ [By (3) of Important Formulae in Chapter 1]}$$

$$= \frac{1}{a}$$

Therefore

$$f'(a) = \frac{1}{a}$$

### Example 2.8

Let $a > 0$ and $f(x) = a^x$ for $x \in \mathbb{R}$. Then show that $f$ is differentiable at any $c \in \mathbb{R}$ and $f'(c) = a^c \log a$. In general, $f'(x) = a^x \log a$. In particular, if $f(x) = e^x$, then $f'(x) = e^x$.

***Solution:*** We have

$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) = \lim_{x\to c} \frac{a^x - a^c}{x-c}$$

$$= \lim_{x\to c} a^c \frac{(a^{x-c}-1)}{x-c}$$

$$= a^c \lim_{y\to 0}\left(\frac{a^y - 1}{y}\right) \quad (\text{where } y = x - c \to 0 \text{ as } x \to c)$$

$$= a^c \log a \quad \text{[By (3) of Important Formulae in Chapter 1]}$$

## Example 2.9

Show that the function $f(x) = \sin x$ is differentiable at any $a \in \mathbb{R}$ and $f'(a) = \cos a$. In general, $f'(x) = \cos x$ for $x \in \mathbb{R}$.

***Solution:*** We have

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\left(\frac{\sin x - \sin a}{x-a}\right)$$

$$= \lim_{x\to a}\frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a}$$

$$= \lim_{x\to a}\left(\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right)\cos\left(\frac{x+a}{2}\right)$$

$$= 1\times\cos a \quad (\because \cos x \text{ is continuous})$$

$$= \cos a$$

## Example 2.10

Show that the function $g(x) = \cos x$ is differentiable at any $a \in \mathbb{R}$ and $g'(a) = -\sin a$. In general, $g'(x) = -\sin x$.

***Solution:*** We have

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\left(\frac{\cos x - \cos a}{x-a}\right)$$

$$= \lim_{x\to a}\frac{(-2)\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a}$$

$$= -\lim_{x\to 0}\left(\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\right)\sin\left(\frac{x+a}{2}\right)$$

$$= -(1)\sin a$$

$$= -\sin a$$

## Example 2.11

Show that the function $h(x) = \tan x$ is differentiable at any $a \in \mathbb{R}$ and $a$ is not an odd multiple of $\pi/2$ and $f'(a) = \sec^2 a$. In general, if $x$ is not an odd multiple of $\pi/2$, then $h'(x) = \sec^2 x$.

***Solution:*** We have

$$\lim_{x\to a}\frac{h(x)-h(a)}{x-a} = \lim_{x\to a}\left(\frac{\tan x - \tan a}{x-a}\right)$$

$$= \lim_{x\to a}\frac{\sin(x-a)}{(x-a)\cos x\cos a}$$

$$= \lim_{x\to a}\left(\frac{\sin(x-a)}{x-a}\right)\times\frac{1}{\cos x}\times\frac{1}{\cos a}$$

$$= 1\times\frac{1}{\cos a}\times\frac{1}{\cos a}$$

$$= \sec^2 a$$

## Example 2.12

If $a \in \mathbb{R}$ and is not a multiple of $\pi$, then show that the function $f(x) = \cot x$ is differentiable at $a$ and $f(a) = -\operatorname{cosec}^2 a$. In general, $f'(x) = -\operatorname{cosec}^2 x$ for all real $x \neq n\pi$, $n \in \mathbb{Z}$.

***Solution:*** We have

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\cot x - \cot a}{x-a}$$

$$= \lim_{x\to a}\left(\frac{-\sin(x-a)}{(x-a)\sin x\sin a}\right)$$

$$= -\lim_{x\to a}\left(\frac{\sin(x-a)}{x-a}\times\frac{1}{\sin x}\times\frac{1}{\sin a}\right)$$

$$= -1\times\frac{1}{\sin a}\times\frac{1}{\sin a}$$

$$= -\operatorname{cosec}^2 a$$

## Example 2.13

If $a$ is not a multiple of $\pi$, then show that the function $g(x) = \operatorname{cosec} x$ is differentiable at $a$ and $g'(a) = -\operatorname{cosec} a \cot a$. In general, $g'(x) = -\operatorname{cosec} x \cot x$ for all $x \neq n\pi$, $n \in \mathbb{Z}$.

***Solution:*** We have

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a}\frac{\sin a - \sin x}{(x-a)\sin x \sin a}$$

$$= -\lim_{x\to a}\frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{(x-a)\sin x\sin a}$$

$$= \lim_{x\to a}\left(\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right)\cdot\frac{\cos\left(\frac{x+a}{2}\right)}{\sin x\sin a}$$

$$= -\frac{(1)\cos a}{\sin^2 a}$$

$$= -\operatorname{cosec} a \cot a$$

## Example 2.14

If $a \neq (2n+1)(\pi/2)$, $n \in \mathbb{Z}$, then show that the function $h(x) = \sec x$ is differentiable at $a$ and $h'(a) = \sec a \tan a$. In general, $h'(x) = \sec x \tan x$ for all $x \neq (2n + 1)\ (\pi/2)$, $n \in \mathbb{Z}$.

***Solution:*** We have

$$\lim_{x\to a}\frac{h(x)-h(a)}{x-a} = \lim_{x\to a}\frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a}\frac{\cos a - \cos x}{(x-a)\cos x\cos a}$$

$$= \lim_{x\to a}\frac{2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right)}{(x-a)\cos x\cos a}$$

$$= \lim_{x\to a}\left(\frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}\right)\cdot\frac{\sin\left(\frac{a+x}{2}\right)}{\cos x\cos a}$$

$$= 1\cdot\frac{\sin a}{\cos^2 a}$$

$$= \sec a \tan a$$

In the coming set of examples, we find the derivatives of inverse trigonometric functions, hyperbolic functions and inverse hyperbolic functions. Here afterwards we write $y = f(x)$ (function) and write $dy/dx$ for $f'(x)$.

## Example 2.15

Let $y = \operatorname{Sin}^{-1} x$ whose domain is $[-1, 1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Find its derivative.

***Solution:*** We have

$$y = \operatorname{Sin}^{-1} x \Leftrightarrow x = \sin y$$

Therefore

$$\frac{dx}{dy} = \cos y$$

Also $-\pi/2 < y < \pi/2$ for $-1 < x < 1 \Rightarrow \cos y > 0$. So

$$\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

Using the Note under Theorem 2.6 we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Hence

$$\frac{d(\operatorname{Sin}^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$

## Example 2.16

Let $y = \mathrm{Cos}^{-1}x$, whose domain is [–1, 1] and range is $[0, \pi]$. Find its derivative.

***Solution:*** We have

$$y = \mathrm{Cos}^{-1}x \Leftrightarrow x = \cos y$$

Therefore

$$\frac{dx}{dy} = -\sin y$$

where $y \in (0, \pi)$ whenever $-1 < x < 1$ and $\sin y > 0$. So

$$\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 x} = -\sqrt{1-x^2}$$

Hence

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$

This implies

$$\frac{d}{dx}(\mathrm{Cos}^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

***Note:*** The derivative of $\mathrm{Cos}^{-1}x$ can also be realized from the relation

$$\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = \frac{\pi}{2} \quad \text{for } -1 \le x \le 1$$

## Example 2.17

Let $y = \mathrm{Tan}^{-1}x$ where $x \in \mathbb{R}$ and the range is $(-\pi/2, \pi/2)$. Find its derivative.

***Solution:*** We have

$$y = \mathrm{Tan}^{-1}x \Leftrightarrow x = \tan y$$

Therefore

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

So

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

This implies

$$\frac{d}{dx}(\mathrm{Tan}^{-1}x) = \frac{1}{1+x^2} \ \forall\, x \in \mathbb{R}$$

## Example 2.18

Find the derivative of $\mathrm{Cot}^{-1}x$ for all $x \in \mathbb{R}$.

***Solution:*** We know that

$$\mathrm{Tan}^{-1}x + \mathrm{Cot}^{-1}x = \frac{\pi}{2} \ \forall\, x \in \mathbb{R}$$

Therefore

$$\frac{d}{dx}(\mathrm{Cot}^{-1}x) = 0 - \frac{d}{dx}(\mathrm{Tan}^{-1}x) = \frac{-1}{1+x^2} \ \forall\, x \in \mathbb{R}$$

Observe that $\mathrm{Cot}^{-1}x \in (0, \pi)$.

## Example 2.19

Let $y = \mathrm{Sec}^{-1}x, x \in \mathbb{R} - [-1, 1]$ and $y \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$. Find out $dy/dx$.

***Solution:*** We have

$$x = \sec y \Rightarrow \frac{dx}{dy} = \sec y \tan y = \sec^2 y \sin y > 0$$

Now

1. $x < -1 \Rightarrow \sec y < 0$ and hence $\tan y < 0$.
2. $x > 1 \Rightarrow \sec y > 0$ and hence $\tan y > 0$.

Therefore

$$\tan y = \begin{cases} -\sqrt{\sec^2 y - 1} = -\sqrt{x^2-1} & \text{if } x < 1 \\ \sqrt{x^2-1} & \text{if } x > 1 \end{cases}$$

So

$$\frac{dy}{dx} = \begin{cases} \dfrac{-1}{x\sqrt{x^2-1}} & \text{if } x < -1 \\ \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \end{cases}$$

$$= \frac{1}{|x|\sqrt{x^2-1}} \quad \text{for } -1 < x < 1$$

## Example 2.20

Let $y = \text{Cosec}^{-1}x,\ x \in \mathbb{R} - [-1, 1]$ and $y \in \left(-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right)$. Find out $dy/dx$.

***Solution:*** Working along the same lines as in Example 2.19, we get

$$\frac{d}{dx}(\text{Cosec}^{-1}x) = \begin{cases} \dfrac{-1}{x\sqrt{x^2-1}} & \text{if } x < -1 \\ \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \end{cases}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}$$

When $x$ is real, then $\dfrac{e^x + e^{-x}}{2}$ is called cosine hyperbolic and is denoted by $\cosh x$. Also $\dfrac{e^x - e^{-x}}{2}$ is called sine hyperbolic and is denoted by $\sinh x$. Also observe that $\cosh^2 x - \sinh^2 x = 1$. The functions $\tanh x, \coth x, \text{cosech}\, x$ and $\text{sech}\, x$ are defined similar to $\tan x, \cot x, \text{cosec}\, x$ and $\sec x$.

## Example 2.21

Let $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$. Find $dy/dx$.

***Solution:*** By part (i) of Theorems 2.2 and 2.3 we have

$$\frac{dy}{dx} = \frac{e^x - e^{-x}(-1)}{2}$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

## Example 2.22

Let $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$. Find $dy/dx$.

***Solution:*** We have

$$\frac{dy}{dx} = \frac{e^x + (-1)e^{-x}}{2}$$

$$= \frac{e^x - e^{-x}}{2}$$

$$= \sinh x$$

## Example 2.23

Let $y = \tanh x = \dfrac{\sinh x}{\cosh x}$. Find $dy/dx$.

***Solution:*** We have

$$\frac{dy}{dx} = \frac{(\cosh x)\cosh x - (\sinh x)(\sinh x)}{\cosh^2 x} \quad \text{(Quotient Rule)}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}$$

$$= \text{sech}^2 x \quad \left(\because \text{sech}\, x \text{ is defined as } \frac{1}{\cosh x}\right)$$

**Try it out** Show that

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

## Example 2.24

Let $y = \operatorname{cosech} x, x \in \mathbb{R} - \{0\}$. Find $dy/dx$.

***Solution:*** We have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sinh x}\right)$$

$$= \frac{(-1)\dfrac{d}{dx}(\sinh x)}{(\sinh x)^2}$$

$$= \frac{-\cosh x}{\sinh^2 x} \quad (\because \text{ The derivative of } \sinh x \text{ is } \cosh x \text{ and by Quotient Rule})$$

$$= -\operatorname{cosech} x \coth x$$

Therefore

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

## Example 2.25

Let $y = \operatorname{sech} x, x \in \mathbb{R}$. Find $dy/dx$.

***Solution:*** We have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\cosh x}\right)$$

$$= \frac{(-1)\dfrac{d}{dx}(\cosh x)}{\cosh^2 x} \quad \text{(By Quotient Rule)}$$

$$= \frac{(-1)\sinh x}{\cosh^2 x} \quad (\because \text{ The derivative of } \cosh x \text{ is } \sinh x)$$

$$= -\operatorname{sech} x \tanh x$$

Therefore

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

The following are the derivatives of inverse hyperbolic functions.

## Example 2.26

Let $y = \operatorname{Sinh}^{-1} x$ for all $x \in \mathbb{R}$. Find its derivative.

***Solution:*** We have

$$y = \operatorname{Sinh}^{-1} x \Rightarrow x = \sinh y$$

$$\Rightarrow \frac{dx}{dy} = \cosh y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{1+\sinh^2 y}} \quad (\because \cosh^2 y - \sinh^2 y = 1)$$

$$= \frac{1}{\sqrt{1+x^2}}$$

Therefore

$$\frac{d}{dx}(\operatorname{Sinh}^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

## Example 2.27

$y = \operatorname{Cosh}^{-1} x \Leftrightarrow x = \cosh y, x \in (1, \infty)$. Find its derivative.

***Solution:*** We have

$$y = \operatorname{Cosh}^{-1} x$$

$$\Rightarrow x = \cosh y$$

$$\Rightarrow \frac{dx}{dy} = \sinh y$$

$$= \sqrt{\cosh^2 y - 1}$$
$$= \sqrt{x^2 - 1}$$

For $x > 1$,

$$\frac{d}{dx}(\operatorname{Cosh}^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$$

## Example 2.28

Find the derivative of $y = \operatorname{Tanh}^{-1} x,\, x \in (-1, 1)$.

***Solution:*** We have

$$y = \operatorname{Tanh}^{-1} x \Leftrightarrow x = \tanh y$$

Therefore

$$\frac{dx}{dy} = \operatorname{sech}^2 y$$
$$= \left(\frac{2}{e^y + e^{-y}}\right)^2$$
$$= \frac{4}{(e^y + e^{-y})^2}$$
$$= 1 - \left(\frac{e^y - e^{-y}}{e^y + e^{-y}}\right)^2$$
$$= 1 - \tanh^2 y$$
$$= 1 - x^2 > 0 \quad (\because -1 < x < 1)$$

Therefore

$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

and hence

$$\frac{d}{dx}(\operatorname{Tanh}^{-1} x) = \frac{1}{1 - x^2}$$

## Example 2.29

Find the derivative of $y = \operatorname{Coth}^{-1} x,\; x \in \mathbb{R} - (-1, 1)$.

***Solution:*** We have

$$y = \operatorname{Coth}^{-1} x \Leftrightarrow x = \coth y$$

Therefore

$$\frac{dx}{dy} = -\operatorname{cosech}^2 y$$
$$= \frac{-1}{\sinh^2 y}$$
$$= -\frac{(\cosh^2 y - \sinh^2 y)}{\sinh^2 y}$$
$$= -\coth^2 y + 1$$
$$= 1 - x^2$$

Therefore

$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

So

$$\frac{d}{dx}(\operatorname{Coth}^{-1} x) = \frac{1}{1 - x^2}$$

**QUICK LOOK 2**

Though the domains of $\operatorname{Tanh}^{-1} x$ and $\operatorname{Coth}^{-1} x$ are, respectively, $(-1, 1)$ and $\mathbb{R} - (-1, 1)$, but their derivatives are same.

## Example 2.30

Find the derivative of $y = \operatorname{Sech}^{-1} x,\, x \in (0, 1)$

***Solution:*** We have

$$y = \operatorname{Sech}^{-1} x \Leftrightarrow x = \operatorname{sech} y$$

Therefore

$$\frac{dx}{dy} = -\operatorname{sech} y \tanh y$$

$$= -\text{sech}\, y\sqrt{1-\text{sech}^2 y}$$
$$= -x\sqrt{1-x^2}$$

So

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$$

## Example 2.31

Find the derivative of $y = \text{Cosech}^{-1}x,\ x \neq 0$.

***Solution:*** We have

$$y = \text{Cosech}^{-1}x \Leftrightarrow x = \text{cosech}\, y$$

Now

$$\frac{dx}{dy} = -\text{cosech}\, y\ \coth y$$
$$= -\text{cosech}\, y\ (\pm\sqrt{1+\text{cosech}^2 y})$$

because for any real $x$, $\text{cosech}\, x > 0\, (< 0)$ according as $x > 0\, (< 0)$. Therefore

$$\frac{dx}{dy} = -x(\pm\sqrt{1+x^2})$$

and hence

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{1+x^2}}$$

### QUICK LOOK 3

1. $\text{Sinh}^{-1}x = \log(x+\sqrt{x^2+1})$
2. $\text{Cosh}^{-1}x = \log(x+\sqrt{x^2-1})$
3. $\text{Tanh}^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\ (|x|<1)$

The following are illustrative examples of the rules of differential of sum of two functions, product of two functions, quotient rule, chain rule using the derivative of standard functions, derivatives of inverse trigonometric functions, hyperbolic functions and their inverses.

## Example 2.32

Find the derivate of $xe^x$.

***Solution:*** Let $y = x^2e^x$. Take $f(x) = x^2$ and $g(x) = e^x$. Therefore

$$y = f(x)g(x)$$

Therefore by product rule [see (ii) of Theorem 2.2]

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$
$$= (2x)e^x + x^2(e^x) \quad \text{(See Examples 2.6 and 2.8)}$$
$$= e^x(2x + x^2)$$

## Example 2.33

Find the derivative of $x\log x + \sin x$.

***Solution:*** Let $y = x\log x + \sin x$. Take $f(x) = x,\ g(x) = \log x$ and $h(x) = \sin x$. Therefore using parts (i), (ii) of Theorem 2.2 and referring to the derivatives of standard functions (Examples 2.6–2.14) we get

$$\frac{dy}{dx} = \frac{d}{dx}\ (fg+h)(x)$$
$$= \frac{d}{dx}(fg)(x) + \frac{d}{dx}(h(x))$$
$$= f'(x)g(x) + f(x)g'(x) + h'(x)$$
$$= 1\log x + x\left(\frac{1}{x}\right) + \cos x$$
$$= \log x + 1 + \cos x$$

## Example 2.34

Find the derivative of $y = \log_7(\tan x)$ whenever defined.

***Solution:*** We have

$$y = \log_7(\tan x) = \log_e(\tan x) \cdot \log_7 e$$

Therefore using chain rule, we get

$$\frac{dy}{dx} = (\log_7 e)\left(\frac{1}{\tan x}\right)(\sec^2 x)$$

$$= (\log_7 e)\left(\frac{1}{\sin x \cos x}\right)$$

## Example 2.35

Find the derivative of $y = 5^x + \log x + x \sin x$.

***Solution:*** We have

$$\frac{dy}{dx} = \frac{d(5^x)}{dx} + \frac{d(\log x)}{dx} + \frac{d(x \sin x)}{dx}$$

$$[\because (f+g)' = f' + g']$$

$$= 5^x \log 5 + \frac{1}{x} + (\sin x + x \cos x)$$

$$\left(\because \frac{d(a^x)}{dx} = a^x \log a, a > 0\right)$$

## Example 2.36

Find the derivative of the function $y = (x + \sqrt{1+x^2})^2$.

***Solution:*** If $y = (f(x))^n$ then we know that by chain rule

$$\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$$

Therefore

$$y = (x + \sqrt{1+x^2})^2$$

$$\Rightarrow \frac{dy}{dx} = 2(x + \sqrt{1+x^2})\left(1 + \frac{1}{2\sqrt{1+x^2}}(2x)\right)$$

$$= 2(x + \sqrt{1+x^2})\frac{(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= \frac{2(x + \sqrt{1+x^2})^2}{\sqrt{1+x^2}}$$

$$= \frac{2y}{\sqrt{1+x^2}}$$

So

$$\frac{dy}{dx} = \frac{2y}{\sqrt{1+x^2}}$$

## Example 2.37

If $y = \log(\text{Sin}^{-1} x)$, then find $dy/dx$.

***Solution:*** Note that $\text{Sin}^{-1} x$ must belong to $(0, \pi/2)$ so that $x \in (0, 1)$. Therefore, using chain rule we get

$$\frac{dy}{dx} = \left(\frac{1}{\text{Sin}^{-1} x}\right)\frac{d}{dx}(\text{Sin}^{-1} x)$$

$$= \frac{1}{\text{Sin}^{-1} x} \times \frac{1}{\sqrt{1-x^2}}$$

## Example 2.38

Find the derivative of $e^x x^e$.

***Solution:*** Let $y = e^x x^e$. Then

$$\frac{dy}{dx} = e^x x^e + e^x \cdot e x^{e-1}$$

$$= e^x x^e \left(1 + \frac{e}{x}\right)$$

$$= y\left(1 + \frac{e}{x}\right)$$

## Example 2.39

Find the derivative of

$$y = \frac{x\,\mathrm{Sin}^{-1}x}{\sqrt{1-x^2}}$$

***Solution:*** Let $f(x) = x\,\mathrm{Sin}^{-1}x,\ g(x) = \sqrt{1-x^2}$. Therefore

$$y = \frac{f}{g}$$

Hence

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Now

$$f'(x) = \frac{d}{dx}(x\,\mathrm{Sin}^{-1}x)$$

$$= \mathrm{Sin}^{-1}x + \frac{x}{\sqrt{1-x^2}} \quad \text{(Product Rule)}$$

$$g'(x) = \frac{d}{dx}(\sqrt{1-x^2})$$

$$= \frac{1}{2}\times(1-x^2)^{-1/2}(0-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

Therefore

$$\frac{dy}{dx} = \frac{\left(\mathrm{Sin}^{-1}x + \frac{x}{\sqrt{1-x^2}}\right)\sqrt{1-x^2} - (x\,\mathrm{Sin}^{-1}x)\left(\frac{-x}{\sqrt{1-x^2}}\right)}{(\sqrt{1-x^2})^2}$$

$$= \frac{\sqrt{1-x^2}\,\mathrm{Sin}^{-1}x + x + \frac{x^2\,\mathrm{Sin}^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{(1-x^2)\mathrm{Sin}^{-1}x + x\sqrt{1-x^2} + x^2\,\mathrm{Sin}^{-1}x}{(1-x^2)^{3/2}}$$

$$= \frac{\mathrm{Sin}^{-1}x + x\sqrt{1-x^2}}{(1-x^2)^{3/2}}$$

## Example 2.40

Find the derivative of $y = \log(\cosh 3x)$.

***Solution:*** Put $u = \cosh 3x$ so that $y = \log u$. Therefore

$$\frac{du}{dx} = (\sinh 3x)(3)$$

Now

$$\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx}$$

$$= \frac{1}{\cosh 3x}\times(3\sinh 3x)$$

$$= 3\tanh 3x$$

## Example 2.41

If $y = \log|x|$, then show that $\dfrac{dy}{dx} = \dfrac{1}{x}$.

***Solution:*** Since $|x| > 0$ for all $x \neq 0$, the domain of the function is $\mathbb{R} - \{0\}$.

**Case I:** Suppose $x > 0$. Therefore

$$y = \log|x| = \log x$$

Taking derivative we get

$$\frac{dy}{dx} = \frac{1}{x}$$

**Case II:** Suppose $x < 0$. Then

$$y = \log|x| = \log(-x)$$

Taking derivative and using chain rule we get

$$\frac{dy}{dx} = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{x} \quad \text{for } x \neq 0$$

or

$$\frac{d(\log|x|)}{dx} = \frac{1}{x}$$

## Example 2.42

Find the derivative of $y = \mathrm{Sin}^{-1}(e^x)$.

**Solution:** We have $y = \mathrm{Sin}^{-1}u$ where $u = e^x$. Therefore (using chain rule or substitution) we have

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

## Example 2.43

If $y = \log(\log x)$, then find $\dfrac{dy}{dx}$.

**Solution:** Put $\log x = u$. Then $y = \log u$ so that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot\frac{1}{x} = \frac{1}{x\log x}$$

## Example 2.44

If $y = \mathrm{Sin}^{-1}(\sin x)$, then show that $\dfrac{dy}{dx} = \dfrac{\cos x}{|\cos x|}$.

**Solution:** Put $u = \sin x$ so that $-1 \le u \le 1$ and $du/dx = \cos x$. Now

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-u^2}}\cdot\cos x\\ &= \frac{\cos x}{\sqrt{1-\sin^2 x}}\\ &= \frac{\cos x}{|\cos x|} = \pm 1 \quad \text{(Justify the signs)}\end{aligned}$$

## Example 2.45

Find $dy/dx$ where $y = \mathrm{Tanh}^{-1}(e^x)$.

**Solution:** Put $u = e^x$ so that $du/dx = e^x$. Now,

$$y = \mathrm{Tanh}^{-1}\, u$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{1}{1-u^2}\cdot e^x\\ &= \frac{e^x}{1-e^{2x}} \quad \text{(see Example 2.28)}\end{aligned}$$

## Example 2.46

If $y = \log(\sec x + \tan x)$, then show that $\dfrac{dy}{dx} = \sec x$.

**Solution:** Put $u = \sec x + \tan x$ so that

$$\begin{aligned}\frac{du}{dx} &= \sec x\tan x + \sec^2 x\\ &= \sec x(\sec x + \tan x)\end{aligned}$$

Now

$$\begin{aligned}y &= \log u\\ \Rightarrow \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{1}{u}\cdot\sec x(\sec x + \tan x)\\ &= \frac{1}{u}\cdot(\sec x)u\\ &= \sec x\end{aligned}$$

## 2.3 | Special Methods of Differentiation

Sometimes the usual rules of differentiation may be a cumbersome process for finding derivatives of some typical functions. In such cases we need *special methods* which we are going to discuss in this section.

### 2.3.1 Substitution Methods

The chain rule (Theorem 2.3) is also called substitution method. However, some functions need a suitable substitution to compute the derivative elegantly. This is illustrated in the following examples.

**Example 2.47**

If

$$y = \mathrm{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}$$

where $|x| < 1$, then find $dy/dx$.

***Solution:*** Put $x = \cos\theta, \theta \in (0, \pi)$. Then

$$\frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}$$

This implies

$$\sqrt{\frac{1-x}{1+x}} = \tan\frac{\theta}{2}$$

so that

$$y = \mathrm{Tan}^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}\\ &= \frac{dy}{d\theta} \div \frac{dx}{d\theta}\\ &= \frac{1}{2}\times\frac{1}{(-\sin\theta)}\\ &= -\frac{1}{2}\times\frac{1}{\sqrt{1-\cos^2\theta}}\\ &= \frac{-1}{2\sqrt{1-x^2}}\end{aligned}$$

**Example 2.48**

If $y = \mathrm{Tan}^{-1}\left(\dfrac{3a^2x - x^3}{a(a^2-3x^2)}\right)$, then show that

$$\frac{dy}{dx} = \frac{3a}{a^2+x^2}$$

***Solution:*** Put $x = a\tan\theta$ so that

$$\begin{aligned}\frac{dx}{d\theta} &= a\sec^2\theta\\ &= a\left(1+\frac{x^2}{a^2}\right)\\ &= \frac{a^2+x^2}{a}\end{aligned}$$

Now

$$\begin{aligned}y &= \mathrm{Tan}^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)\\ &= \mathrm{Tan}^{-1}(\tan 3\theta) = 3\theta\end{aligned}$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= 3\cdot\frac{d\theta}{dx}\\ &= \frac{3}{dx/d\theta}\\ &= \frac{3}{(a^2+x^2)/a}\\ &= \frac{3a}{a^2+x^2}\end{aligned}$$

## Example 2.49

If $y = \log\left(\dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}\right)$, then show that

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1+x^2}}$$

***Solution:*** Put $x = \tan\theta$ so that

$$\frac{dx}{d\theta} = \sec^2\theta = 1 + x^2$$

Now

$$y = \log\left(\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}\right)$$

$$= \log(\sec\theta - \tan\theta)^2$$

$$= 2\log(\sec\theta - \tan\theta) \quad (\because \sec\theta > \tan\theta)$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{2(\sec\theta\tan\theta - \sec^2\theta)}{(\sec\theta - \tan\theta)} \div \sec^2\theta$$

$$= \frac{-2}{\sec\theta}$$

$$= \frac{-2}{\sqrt{1+x^2}}$$

### 2.3.2 Logarithmic Differentiation

Use of logarithms will be advantageous in computing the derivatives of functions of the form $(f(x))^{g(x)}$. More precisely, let $f$ and $g$ be defined and differentiable on an interval $I$. Let $h(x) = (f(x))^{g(x)}$ so that

$$\log(h(x)) = g(x)\log(f(x))$$

Differentiating both sides with respect to $x$, we get

$$\frac{h'(x)}{h(x)} = g'(x)\log(f(x)) + g(x)\frac{f'(x)}{f(x)}$$

Hence

$$h'(x) = h(x)\left(g'(x)\log(f(x)) + \frac{g(x)f'(x)}{f(x)}\right)$$

Now replace $h(x)$ with $(f(x))^{g(x)}$.

***Note:*** If $y = (f(x))^{g(x)}$, then

$$\frac{dy}{dx} = y\left(g'(x)\log(f(x)) + \frac{g(x)f'(x)}{f(x)}\right)$$

We illustrate the above-stated theory with some examples now.

## Example 2.50

If $y = x^x$, find $dy/dx$ where $x > 0$.

***Solution:*** Taking logarithm on both sides of $y = x^x$ we get

$$\log y = x\log x$$

Differentiating both sides w.r.t. $x$ we obtain

$$\frac{1}{y}\frac{dy}{dx} = \log x + \frac{x}{x} = \log x + 1$$

Therefore

$$\frac{dy}{dx} = x^x(\log x + 1)$$

### Example 2.51

If $y = (\tan)^{\sin x}$, $x \in (0, \pi/2)$, then find $dy/dx$.

***Solution:*** Taking logarithm on both sides we get

$$\log y = (\sin x)\log(\tan x)$$

Note that both $\tan x$ and $\sin x$ are positive for $x \in (0, \pi/2)$. Differentiating both sides w.r.t. $x$ we get

$$\frac{1}{y}\frac{dy}{dx} = (\cos x)\log(\tan x) + \sin x\left(\frac{\sec^2 x}{\tan x}\right)$$
$$= (\cos x)\log(\tan x) + \sec x$$

Therefore

$$\frac{dy}{dx} = y(\cos x \log(\tan x) + \sec x)$$

### Example 2.52

Let $y = (x+1)^{2/x}$ for $x \neq 0, x > -1$. Then find $dy/dx$.

***Solution:*** Taking logarithm on both sides we get

$$\log y = \frac{2}{x}\log(x+1)$$

Differentiating both sides w.r.t. $x$ we get

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{\left(\frac{x}{x+1} - \log(x+1)\right)}{x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{2y}{x^2}\left(\frac{x}{x+1} - \log(x+1)\right)$$

### 2.3.3 Parametric Differentiation

Suppose $x$ and $y$ are functions of a parameter $t$, say $x = f(t)$ and $y = g(t)$ in any interval. Further assume that $f$ is invertible and $f, g, f^{-1}$ are differentiable in the relevant intervals. Then to find $dy/dx$, we proceed as follows:

$$y = g(t) = g(f^{-1}(x)) = (g \circ f^{-1})(x)$$

Hence

$$\frac{dy}{dx} = g'(f^{-1}(x)(f^{-1})'(x)$$
$$= \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= \frac{dy}{dt} \div f'(t)$$
$$= \frac{g'(t)}{f'(t)}$$

This process is called parametric differentiation or differentiation of one function $g$ w.r.t. another function $f$. We illustrate the above-stated theory with some examples now.

### Example 2.53

If $x = a\cos t$ and $y = a\sin t$, for $0 < t < \pi/2$ and $a \neq 0$, find $dy/dx$.

***Solution:*** Write $x = f(t)$ and $y = g(t)$ in $(0, \pi/2)$ and note that $f, g$ are differentiable in $(0, \pi/2)$ and $f^{-1}$ exists in $(0, 1)$ and is differentiable. Hence using parametric differentiation, we get

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$
$$= \frac{a\cos t}{-a\sin t}$$
$$= \frac{x}{-y} = \frac{-x}{y}$$

***Note:*** In Example 2.53, $x$ and $y$ are functions of $t$ and we computed the derivative of the function $y$ w.r.t. another function $x$.

## Example 2.54

If $f(t) = \log t + \sin t$ and $g(t) = e^t + \cos t$ in $(0, \pi/2)$, then find $dg/df$.

**Solution:** Since

$$\frac{df}{dt} = \frac{1}{t} + \sin t$$

$$\frac{dg}{dt} = e^t - \sin t$$

and

$$\frac{df}{dt} \neq 0$$

we have

$$\frac{dg}{df} = \frac{dg}{dt} \div \frac{df}{dt}$$

$$= \frac{e^t - \sin t}{(1/t) + \cos t}$$

$$= \frac{t(e^t - \sin t)}{1 + t\cos t}$$

## Example 2.55

If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that

$$(x^2 + 4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + 4)$$

**Solution:** We have

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$$

and $$\frac{dy}{d\theta} = n\sec^{n-1}\theta(\sec\theta\tan\theta) - n\cos^{n-1}\theta(-\sin\theta)$$

$$= n(\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta)$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$= \frac{n(\sec^n\theta + \cos^n\theta)}{\sec\theta + \cos\theta}$$

From this we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{n^2[\sec^n\theta + \cos^n\theta]^2}{(\sec\theta + \cos\theta)^2}$$

$$= \frac{n^2[(\sec^n\theta - \cos^n\theta)^2 + 4]}{(\sec\theta - \cos\theta)^2 + 4}$$

$$= \frac{n^2(y^2 + 4)}{x^2 + 4}$$

Hence

$$(x^2 + 4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + 4)$$

### 2.3.4 Differentiation of Implicit Functions

Suppose $y$ is a function of $x$, $y$ is not explicitly in terms of $x$, but $x$ and $y$ are connected through a relation $F$. Write $F(x, y) = 0$. Here $y$ is a function of $x$, say, $y = f(x)$, but $x$ and $y$ are connected through the relation $F(x, y) = 0$, that is $F(x, f(x)) = 0$. If this be the case, we say that $y$ is an *implicit* function of $x$. The following are illustrative examples.

## Example

Suppose $y = f(x), x \in [a, b]$ and $x^2 + y^2 = 1 \ \forall \ x \in [a, b]$. In this case $f(x)$ may not be known explicitly, but we may compute $f(x)$ using the relation

$$x^2 + (f(x))^2 = 1$$

Then

$$f(x) = \sqrt{1 - x^2} \quad \text{or} \quad f(x) = -\sqrt{1 - x^2}$$

If $y$ is an implicit function of $x$ and $x$ and $y$ are connected through a relation $F(x, y)=0$, it is possible, sometimes, to find $y' = f'(x)$, by differentiating $F(x, y)=0$ on both sides w.r.t. $x$. Here, $x^2+y^2=1$. On differentiating both sides w.r.t. $x$, since $y$ is a function of $x$, say $y=f(x)$, we get

$$2x+2yy'=0$$

Here $y' = f'(x)$. Hence

$$y' = \frac{-x}{y} = \frac{-x}{f(x)}$$

## Example 2.56

If $y$ is an implicit function of $x$, and $x$ and $y$ are connected through the relation $x^3+y^3-3xy=0$, find $y'$ or $dy/dx$.

***Solution:*** On differentiating $x^3+y^3-3xy=0$ w.r.t. $x$ we get

$$3x^2+3y^2y'-3(y+xy')=0$$

$$\Rightarrow (y^2-x)y' = y-x^2$$

$$\Rightarrow y' = \frac{y-x^2}{y^2-x}$$

## Example 2.57

If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then find $dy/dx$ where all the coefficients are constants and at least one of $a, h, b$ is not zero.

***Solution:*** Differentiating the given equation w.r.t. $x$ we get

$$2ax+2h\left(y+x\frac{dy}{dx}\right)+2by\frac{dy}{dx}$$

$$+2g+2f\frac{dy}{dx}=0$$

$$\Rightarrow (hx+by+f)\frac{dy}{dx}+ax+hy+g=0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(ax+hy+g)}{hx+by+c}$$

Finally, we conclude this theory part of differentiation with the derivatives of second order, third order, etc.

## 2.4 | Successive Derivatives of a Function

**DEFINITION 2.4** Suppose $f:(a, b)\to\mathbb{R}$ is a function and is differentiable in $(a, b)$. Then the derived function $f'$ can be regarded as a function on $(a, b)$. That is, $f':(a, b)\to\mathbb{R}$ is also a function. Suppose $f'$ is differentiable in $(a, b)$ with derived function $(f')'(x)$, then this function is denoted by $f''(x)$ and is called the *second derivative* of $f$ in $(a, b)$.

If further $f''$ is differentiable in $(a, b)$, then its derived function $(f'')'(x)$ is denoted by $f'''(x)$ and is called the *third derivative* of $f$ in $(a, b)$. In general, the $(n+1)$th derivative of $f$ is the derivative of $f^{(n)}(x)$ (if exists), $n=1, 2, 3, \ldots$.

***Note:*** Even though the $n$th derivative of a function is explained, as per the scope of the syllabus, we restrict to second- and third-order derivatives. If $y=f(x)$, then $f^{(n)}(x)$ is also denoted by $d^n y/dx^n$.

The following are some illustrative examples.

## Examples

**1.** Let $f(x) = x^2$ in $(0, 1)$. For the given function

$$f'(x) = 2x$$

$$f''(x) = 2$$

and $$f^{(n)}(x) = 0 \text{ for } n \geq 3$$

**2.** Let $f(x) = x^n$ where $n$ is a positive integer. Then

$$f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2}$$

$$f'''(x) = n(n-1)(n-2)x^{n-3}$$

and in general

$$f^{(n)}(x) = n!$$

In this case $f^{(n+1)}(x) = 0$, $f^{(n+2)}(x) = 0, \ldots$.

**3.** If $f(x) = \sin x$ in $(0, \pi/2)$, then

$$f'(x) = \cos x, f''(x) = -\sin x$$

$$f'''(x) = -\cos x, f^{(4)}(x) = \sin x$$

and in general

$$f^{(n)}(x) = \begin{cases} \cos x & \text{if } n \text{ is of the form } 4k+1 \\ -\sin x & \text{if } n \text{ is of the form } 4k+2 \\ -\cos x & \text{if } n \text{ is of the form } 4k+3 \\ \sin x & \text{if } n \text{ is of the form } 4k \end{cases}$$

***Note:*** One can see that if $f(x) = \sin x$, then its $n$th derivative is

$$f^{(n)}(x) = \sin\left(\frac{n\pi}{2} + x\right) \quad \text{for } n = 1, 2, 3, \ldots$$

**4.** If $f(x) = e^{\lambda x}$, then $f^{(n)}(x) = \lambda^n e^{\lambda x} = \lambda^n f(x)$.

**5.** If $f(x) = \log x$, then $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$.

## Example 2.58

If $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$, then find $d^2y/dx^2$.

***Solution:*** From the given equations,

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta$$

$$\frac{dy}{d\theta} = \cos\theta - (\cos\theta - \theta\sin\theta) = \theta\sin\theta$$

Now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta$$

Therefore

$$\frac{d^2y}{dx^2} = (\sec^2\theta)\frac{d\theta}{dx} = \frac{\sec^2\theta}{\theta\cos\theta} = \frac{\sec^3\theta}{\theta}$$

## Example 2.59

If $y = \sin(\sin x)$, then show that

$$\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0$$

***Solution:*** Differentiating $y = \sin(\sin x)$ we get

$$\frac{dy}{dx} = \cos(\sin x)(\cos x)$$

Again

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cos^2 x + \cos(\sin x)(-\sin x)$$

$$= -y\cos^2 x + \cos(\sin x)\cos x\left(\frac{-\sin x}{\cos x}\right)$$

$$= -y\cos^2 x - (\tan x)\frac{dy}{dx}$$

Therefore

$$\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0$$

## WORKED-OUT PROBLEMS

***Note:*** Before going to Worked-Out Problems, we recall the concepts of continuity and differentiability of a function at $x = a$.

1. Left limit at $x=a$ is denoted by $f(a-0)$ and is given by
$$f(a-0)=\lim_{\substack{h\to 0\\ h>0}} f(a-h)$$
2. Right limit at $x = a$ is denoted by $f(a + 0)$ and is given by
$$f(a+0)=\lim_{\substack{h\to 0\\ h>0}} f(a+h)$$
3. $f$ is continuous at $x = a$ if and only if $f(a-0)$ and $f(a+0)$ exist finitely and are equal to $f(a)$.
4. Left derivative at $x=a$ is denoted by $f'(a-0)$ and is given by
$$f'(a-0)=\lim_{\substack{h\to 0\\ h>0}} \frac{f(a-h)-f(a)}{-h}$$
5. Right derivative at $x=a$ is denoted by $f'(a+0)$ and is given by
$$f'(a+0)=\lim_{\substack{h\to 0\\ h>0}} \frac{f(a+h)-f(a)}{h}$$
6. Derivative at $x=a$ is denoted by $f'(a)$ and is given by
$$f'(a)=\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$
7. $f(x)$ is differentiable at $x = a$ if and only if both $f'(a-0)$ and $f'(a-0)$ exist finitely and are equal. The common value is $f'(a)$.
8. $f$ is continuous on a set of real numbers (interval) implies that there are no breaks on the graph of $y=f(x)$.
9. $f$ is not differentiable at a point $x = x_0$ implies that there is sharp bend like "∧" or "∨" at $(x_0, y_0)$ where $y_0 = f(x_0)$ on the graph of $y = f(x)$.
10. If $f$ is discontinuous at $x = a$, then $f$ is not differentiable at $x = a$.

### *Single Correct Choice Type Questions*

**1.** Let $f$ and $g$ be differentiable functions such that $f(3)=5, g(3)=7, f'(3)=13, g'(3)=6, f'(7)=2$ and $g'(7)=0$. If $h(x)=(f\circ g)(x)$, then $h'(3)$ is equal to

(A) 14 (B) 6
(C) 12 (D) 10

**Solution:** By chain rule
$$h'(x)=f'(g(x))\cdot g'(x)$$
Therefore
$$\begin{aligned}h'(3)&=f'(g(3))\cdot g'(3)\\&=f'(7)\cdot 6\\&=2\cdot 6\\&=12\end{aligned}$$

**Answer: (C)**

**2.** Let $f(x)=2/(x+1)$ and $g(x)=3x$. It is given that $(f\circ g)(x_0)=(g\circ f)(x_0)$. Then $(g\circ f)'(x_0)$ equals

(A) $-32$ (B) $\dfrac{32}{3}$

(C) $\dfrac{-32}{9}$ (D) $\dfrac{-32}{3}$

**Solution:** We have
$$(f\circ g)(x)=(g\circ f)(x) \Rightarrow \frac{2}{3x+1}=\frac{6}{x+1}$$
$$\Rightarrow x=-\frac{1}{4}$$
Therefore
$$x_0=-\frac{1}{4}$$
Now,
$$f'(x)=\frac{-2}{(x+1)^2} \text{ and } g'(x)=3\ \forall x$$
Therefore, using chain rule we have
$$\begin{aligned}(g\circ f)'(x_0)&=g'(f(x_0))\cdot f'(x_0)\\&=3\times\frac{-2}{\left(\frac{-1}{4}+1\right)^2}\\&=\frac{3\times(-2)(16)}{9}\\&=\frac{-32}{3}\end{aligned}$$

**Answer: (D)**

**3.** If $f(x)$ and $g(x)$ are two functions from $\mathbb{R}$ to $\mathbb{R}$ such that $(f\circ g)(x)=(x^3-x^2+2)^8$, then $f'(1)g'(1)$ is

(A) 8 (B) 16
(C) 12 (D) 24

**Solution:** We have

$$(f\circ g)(x)=(x^3-x^2+2)^8$$

$$\Rightarrow f(x)=x^8 \quad \text{and} \quad g(x)=x^3-x^2+2$$

Therefore

$$f'(1)=8 \quad \text{and} \quad g'(1)=3(1)-2(1)=1$$

$$\Rightarrow f'(1)g'(1)=8\times 1=8$$

**Answer: (A)**

**4.** If $f(x)=\sqrt{x}\ (x\ge 0)$ and $g(x)=1+x^2$, then $(f\circ g)'(1)$ is equal to

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

**Solution:** We have

$$(f\circ g)(x)=\sqrt{1+x^2}$$

$$\Rightarrow (f\circ g)'(x)=\frac{x}{\sqrt{1+x^2}}$$

and hence

$$(f\circ g)'(1)=\frac{1}{\sqrt{2}}$$

**Answer: (C)**

**5.** The number of points at which the function $f(x)=|x+1|+|x-1|$ is not differentiable is

(A) 2 (B) 1
(C) 0 (D) infinite

**Solution:** We have

$$f(x)=\begin{cases} -(x+1)+1-x=-2x & \text{if } x<-1 \\ (x+1)+(1-x)=2 & \text{if } -1\le x\le 1 \\ (x+1)+(x-1)=2x & \text{if } x>1 \end{cases}$$

Now

$$f'(-1-0)=\lim_{\substack{h\to 0\\ h>0}}\frac{f(-1-h)-f(-1)}{-h}$$

$$=\lim_{h\to 0}\frac{-2(-1-h)-2}{-h}=\lim_{h\to 0}\left(\frac{2h}{-h}\right)=-2$$

$$f'(-1+0)=0$$

Therefore

$$f'(-1-0)\ne f'(-1+0)$$

Hence $f$ is not differentiable at $x=-1$. Similarly, $f'(1-0)=0$ and $f'(1+0)=2$. Therefore $f$ is not differentiable at $x=1$. Thus, $f$ is not differentiable at $x=-1, 1$.

**Answer: (A)**

**6.** If $x=2\cos t-\cos 2t$, $y=2\sin t-\sin 2t$, then $dy/dx$ at $t=\pi/6$ is

(A) $\sqrt{3}$ (B) $\frac{1}{\sqrt{3}}$

(C) $\sqrt{2}-1$ (D) 1

**Solution:** We have

$$\frac{dx}{dt}=-2\sin t+2\sin 2t$$

and
$$\frac{dy}{dt}=2\cos t-2\cos 2t$$

Therefore

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

$$=\frac{2(\cos t-\cos 2t)}{2(\sin 2t-\sin t)}$$

$$=\frac{2\sin\frac{3t}{2}\sin\frac{t}{2}}{2\cos\frac{3t}{2}\sin\frac{t}{2}}=\tan\frac{3t}{2}$$

Hence

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{6}}=\tan\left(\frac{3}{2}\left(\frac{\pi}{6}\right)\right)=\tan\frac{\pi}{4}=1$$

**Answer: (D)**

**7.** If $\sqrt{1-x^2}+\sqrt{1-y^2}=\sqrt{3}(x-y)$, then $dy/dx$ is equal to

(A) $\sqrt{(1-x^2)(1-y^2)}$ (B) $\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

(C) $\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$ (D) $\frac{x^2-y^2}{\sqrt{(1-x^2)(1-y^2)}}$

**Solution:** The given relation is

$$\sqrt{1-x^2}+\sqrt{1-y^2}=\sqrt{3}(x-y)$$

(This is differentiation of implicit function.) Put $x=\sin\alpha$ and $y=\sin\beta$. Then

$$\cos\alpha+\cos\beta=\sqrt{3}(\sin\alpha-\sin\beta)$$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=\sqrt{3}\left(2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right)$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right)=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha-\beta = 2\,\mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}$$

$$\Rightarrow \mathrm{Sin}^{-1}x-\mathrm{Sin}^{-1}y = 2\,\mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}$$

Differentiating both sides w.r.t. $x$ we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\left(\frac{dy}{dx}\right)=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

**Answer: (B)**

**8.** If

$$(\sin y)^{\sin(\pi x/2)}-\frac{\sqrt{3}}{2}\mathrm{Sec}^{-1}(2x)+2^x\tan(\log(x+2))=0$$

then $dy/dx$ at $x=-1$ is

(A) $\dfrac{3}{\sqrt{\pi^2-3}}$ (B) $\dfrac{1}{\pi\sqrt{\pi^2-3}}$

(C) $\dfrac{3}{\pi\sqrt{\pi^2-3}}$ (D) $\dfrac{3\pi}{\sqrt{\pi^2-3}}$

**Solution:** This problem also comes under implicit function. On differentiating both sides w.r.t. $x$ we get

$$\left.\begin{aligned}&(\sin y)^{\sin(\pi x/2)}\times\log(\sin y)\times\cos\frac{\pi x}{2}\times\frac{\pi}{2}+\\&\left(\sin\frac{\pi x}{2}\right)(\sin y)^{\left(\sin\frac{\pi x}{2}\right)-1}\times\cos y\times\frac{dy}{dx}-\\&\frac{\sqrt{3}}{2}\left(\frac{2}{(2|x|)\sqrt{4x^2-1}}\right)+\frac{2^x\cdot\sec^2(\log(x+2))}{x+2}+\\&2^x\times\log 2\times\tan(\log(x+2))=0\end{aligned}\right\}\quad(2.4)$$

From the given equation we have

$$x=-1\Rightarrow \sin y=\frac{+\sqrt{3}}{\pi}$$

Therefore substituting $x=-1$ and $\sin y=+\sqrt{3}/\pi$ in Eq. (2.4) we have

$$\left(\frac{dy}{dx}\right)_{x=-1}=\frac{\left(\frac{+\sqrt{3}}{\pi}\right)^2}{\sqrt{1+\left(\frac{+\sqrt{3}}{\pi}\right)^2}}=\frac{3}{\pi\sqrt{\pi^2-3}}$$

**Answer: (C)**

**9.** If $f'(x)=\sin^2 x$ and

$$y=f\left(\frac{2x-1}{x^2+1}\right)$$

then $dy/dx$ at $x=1$ is

(A) $\dfrac{1}{4}\sin\left(\dfrac{1}{2}\right)$ (B) $\dfrac{1}{4}\sin^2\left(\dfrac{1}{2}\right)$

(C) $\sin^2\left(\dfrac{1}{4}\right)$ (D) $\dfrac{1}{2}\sin^2\left(\dfrac{1}{2}\right)$

**Solution:** We have

$$y=f\left(\frac{2x-1}{x^2+1}\right)$$

$$\Rightarrow \frac{dy}{dx}=f'\left(\frac{2x-1}{x^2+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$=\sin^2\left(\frac{2x-1}{x^2+1}\right)\left(\frac{2(x^2+1)-2x(2x-1)}{(x^2+1)^2}\right)$$

$$=\sin^2\left(\frac{2x-1}{x^2+1}\right)\left(\frac{-2x^2+2x+2}{(x^2+1)^2}\right)$$

Therefore

$$\left(\frac{dy}{dx}\right)_{x=1}=\frac{1}{2}\sin^2\left(\frac{1}{2}\right)$$

**Answer: (D)**

**10.** If

$$y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{c}{x-c}+1$$

then $dy/dx$ is equal to

(A) $\dfrac{-y}{x}\left(\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}\right)$

(B) $\dfrac{1}{x}\left(\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}\right)$

(C) $\dfrac{y}{x}\left(\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}\right)$

(D) $-\dfrac{1}{x}\left(\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}\right)$

**Solution:** We have

$$y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{x}{x-c}$$

$$=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{x^2}{(x-b)(x-c)}$$

$$=\frac{x^3}{(x-a)(x-b)(x-c)}$$

Therefore

$$\log y = 3\log x-\log(x-a)-\log(x-b)-\log(x-c)$$

Differentiating both sides w.r.t. $x$ we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}$$
$$= \left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)$$
$$= \frac{-a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}$$

Therefore

$$\frac{dy}{dx} = \frac{-y}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$$

**Answer: (A)**

**11.** If $f:\mathbb{R}\to\mathbb{R}$ is an even function, then

(A) $f'(0)=0$
(B) $f'(x)$ is an even function
(C) $f(0)=0$
(D) nothing can be said about $f'(x)$

**Solution:** $y=f(x)$ is an even function. Therefore

$$f(-x) = f(x)\ \forall x\in\mathbb{R}$$

Differentiating both sides w.r.t. $x$ we get

$$f'(-x)(-1) = f'(x)$$
$$f'(-x) = -f'(x)$$

Hence

$$f'(0) = -f'(0) \Rightarrow f'(0) = 0$$

**Answer: (A)**

**12.** If a function is represented parametrically by the equations

$$x = \frac{1+t}{t^3},\quad y = \frac{3}{2t^2} + \frac{2}{t}$$

then

(A) $x\left(\frac{dy}{dx}\right)^2 = 1 + \frac{dy}{dx}$
(B) $x\left(\frac{dy}{dx}\right)^3 = 1 + \frac{dy}{dx}$
(C) $\left(\frac{dy}{dx}\right)^3 = x + \frac{dy}{dx}$
(D) $x\left(\frac{dy}{dx}\right)^3 = 1 + x\frac{dy}{dx}$

**Solution:** We have

$$x = \frac{1}{t^3} + \frac{1}{t^2},\quad y = \frac{3}{2t^2} + \frac{2}{t}$$

Therefore

$$\frac{dx}{dt} = \frac{-3}{t^4} - \frac{2}{t^3}$$

and

$$\frac{dy}{dt} = \frac{-3}{t^3} - \frac{2}{t^2}$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= \frac{-\frac{1}{t}\left(\frac{3}{t^2} + \frac{2}{t}\right)}{-\frac{1}{t^2}\left(\frac{3}{t^2} + \frac{2}{t}\right)} = t$$

So

$$x\left(\frac{dy}{dx}\right)^3 = xt^3 = 1 + t = 1 + \frac{dy}{dx}$$

**Answer: (B)**

**13.** If $x = 3\cos t$, $y = 4\sin t$, then $dy/dx$ at the point $(x = 3\sqrt{2}/2,\ y = 2\sqrt{2})$ is

(A) $\frac{2}{3}$
(B) $\frac{-2}{3}$
(C) $\frac{4}{3}$
(D) $-\frac{4}{3}$

**Solution:** We have

$$x = \frac{3\sqrt{2}}{2} \Rightarrow \cos t = \frac{1}{\sqrt{2}}$$

and

$$y = 2\sqrt{2} \Rightarrow \sin t = \frac{1}{\sqrt{2}}$$

Now

$$\frac{dx}{dt} = -3\sin t \quad\text{and}\quad \frac{dy}{dt} = 4\cos t$$

implies

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= \frac{4\cos t}{-3\sin t}$$
$$= \frac{4\left(\frac{1}{\sqrt{2}}\right)}{-3\left(\frac{1}{\sqrt{2}}\right)}$$
$$= \frac{-4}{3}$$

**Answer: (D)**

**14.** If $y = \dfrac{1+\log x}{x - x\log x}$, then $dy/dx$ is equal to

(A) $\frac{x^2y^2+1}{x^2}$
(B) $\frac{x^2y^2+1}{2x^2}$
(C) $\frac{xy+1}{2x}$
(D) $\frac{xy+1}{2x^2}$

**Solution:** We have

$$y(x - x\log x) = 1 + \log x$$

Differentiating both sides w.r.t. $x$ we get

$$\frac{dy}{dx}(x - x\log x) + y(1 - \log x - 1) = \frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1+\log x}{y}\right) - y\log x = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} + y^2 \log x}{1+\log x}$$

$$= \frac{y(1+xy\log x)}{x(1+\log x)} \qquad (2.5)$$

But by the hypothesis,

$$\log x = \frac{xy-1}{xy+1}$$

Substituting the value of $\log x$ is Eq. (2.5), we get that

$$\frac{dy}{dx} = y\frac{\left[1+xy\left(\frac{xy-1}{xy+1}\right)\right]}{x\left(1+\frac{xy-1}{xy+1}\right)}$$

$$= \frac{y}{x}\frac{(1+x^2y^2)}{2xy}$$

$$= \frac{1+x^2y^2}{2x^2}$$

**Answer: (B)**

**15.** If $f(x) = 1 - x^{2/3} + \frac{16}{x}$, then $f'(-8)$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$

(C) $\frac{1}{12}$ (D) $\frac{1}{3}$

**Solution:** We have

$$f'(x) = -\frac{2}{3}x^{-1/3} - \frac{16}{x^2}$$

Therefore

$$f'(-8) = -\frac{2}{3}(-8)^{-1/3} - \frac{16}{(-8)^2}$$

$$= -\frac{2}{3}\left(\frac{-1}{2}\right) - \frac{1}{4}$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

**Answer: (C)**

**16.** If $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$, $x > 0$, then $dy/dx$ at $x = 1$ is

(A) $\frac{1}{2\sqrt{1+\sqrt{2}}}\left(\frac{3+4\sqrt{2}}{4\sqrt{2}}\right)$ (B) $\frac{1}{\sqrt{1+\sqrt{2}}}\left(\frac{3+4\sqrt{2}}{4\sqrt{2}}\right)$

(C) $\left(\frac{3+4\sqrt{2}}{8\sqrt{2}}\right)$ (D) $\frac{3+4\sqrt{2}}{\sqrt{1+\sqrt{2}}}$

**Solution:** We have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$$

Therefore

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2\sqrt{1+\sqrt{2}}}\left(1+\frac{1}{2\sqrt{2}}\cdot\frac{3}{2}\right)$$

$$= \frac{1}{2\sqrt{1+\sqrt{2}}}\left(\frac{3+4\sqrt{2}}{4\sqrt{2}}\right)$$

**Answer: (A)**

**17.** The number of values of $x$ at which

$$f(x) = \mathrm{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$$

is not differentiable is

(A) 0 (B) 1

(C) 2 (D) 3

**Solution:** Since

$$\frac{d}{dx}(\mathrm{Sin}^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad (|x|<1)$$

we have

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\frac{[2(1+x^2)-2x(2x)]}{(1+x^2)^2}$$

$$= \frac{(1+x^2)}{\sqrt{(1-x^2)^2}}\cdot\frac{2(1-x^2)}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{|1-x^2|(1+x^2)}$$

$$= \begin{cases} \dfrac{2}{1+x^2} & \text{if } |x|<1 \\[2ex] \dfrac{-2}{1+x^2} & \text{if } |x|>1 \end{cases}$$

Therefore $f'(x)$ does not exist at $|x|=1$ (i.e., $x=\pm 1$) because $f'(-1-0)=-1$ and $f'(-1+0)=1$ and $f'(1-0)=1, f'(1+0)=-1$.

**Answer: (C)**

**18.** If $f(x)=xe^{-x^2/2}$, then $f'(1)$ is

(A) 1 (B) 0

(C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$

**Solution:** We have

$$f'(x)=e^{-x^2/2}+xe^{-x^2/2}\left(\frac{-2x}{2}\right)$$
$$=e^{-x^2/2}(1-x^2)$$

Therefore $f'(1)=0$.

**Answer: (B)**

**19.** The function $f(x)=|\log x|$ is

(A) continuous and differentiable for $0<x\le 1$
(B) differentiable for $0<x<1$, but not differentiable at $x=1$
(C) discontinuous at $x=1$
(D) not differentiable for all $x>0$

**Solution:** We have

$$f(x)=\begin{cases}-\log x & \text{if } 0<x<1\\ \log x & \text{if } x\ge 1\end{cases}$$

and
$$f'(x)=\begin{cases}-\dfrac{1}{x} & \text{if } 0<x<1\\ \dfrac{1}{x} & \text{if } x\ge 1\end{cases}$$

Therefore $f'(1-0)=-1$ and $f'(1+0)=1$ and hence $f'(1)$ does not exist.

**Answer: (B)**

**20.** Let $f(x)=[x]$ where $[t]$ is the greatest integer not exceeding $t$ and $g(x)=|x|$. Then $(f\circ g)(x)$ is

(A) differentiable at $x=-1$ and $(f\circ g)'(-1)=1$
(B) differentiable at $x=-1$ and $(f\circ g)'(-1)=-1$
(C) differentiable at $x=-1$ and $(f\circ g)'(-1)=0$
(D) not differentiable at $x=-1$

**Solution:** Let

$$H(x)=(f\circ g)(x)=f(g(x))=[|x|]$$

Now

$$H'(-1-0)=\lim_{\substack{h\to 0\\ h>0}}\frac{H(-1-h)-H(-1)}{-h}$$
$$=\lim_{h\to 0}\frac{[|-1-h|]-1}{-h}$$
$$=\lim_{h\to 0}\frac{[1+h]-1}{-h}$$
$$=\lim_{h\to 0}\frac{[1-1]}{-h}=0$$
$$H'(-1+0)=\lim_{\substack{h\to 0\\ h>0}}\frac{H(-1+h)-H(-1)}{h}$$
$$=\lim_{h\to 0}\frac{[|-1+h|]-1}{h}$$
$$=\lim_{h\to 0}\frac{[1-h]-1}{h}$$
$$=\lim_{h\to 0}\frac{0-1}{h}=-\infty$$

Therefore $H'(-1-0)$ is finite $H'(-1+0)$ is infinite. Hence $H$ is not differentiable at $x=-1$.

**Answer: (D)**

**21.** If $x=\dfrac{3t}{1+t^2}$, $y=\dfrac{3t^2}{1+t^2}$, then $dy/dx$ at $t=2$ is

(A) $\dfrac{2}{3}$ (B) $-\dfrac{2}{3}$

(C) $-\dfrac{4}{3}$ (D) $\dfrac{4}{3}$

**Solution:** We have

$$\frac{dx}{dt}=\frac{3(1+t^2-2t^2)}{(1+t^2)^2}=\frac{3(1-t^2)}{(1+t^2)^2}$$
$$\frac{dy}{dt}=\frac{3[2t(1+t^2)-t^2(2t)]}{(1+t^2)^2}=\frac{6t}{(1+t^2)^2}$$

Therefore

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{2t}{1-t^2}$$

so that

$$\left(\frac{dy}{dx}\right)_{t=2}=-\frac{4}{3}$$

**Answer: (C)**

**22.** If $x=\dfrac{3t}{1+t^2}$, $y=\dfrac{3t^2}{1+t^2}$, then $h(t)=dy/dx$ is

(A) exactly differentiable at $t=\pm 1$

(B) not differentiable at $t = \pm 1$
(C) not differentiable at $t = 0$
(D) not differentiable at more than two points

**Solution:** From Problem 21,

$$h(t) = \frac{dy}{dx} = \frac{2t}{1-t^2}$$

which is not defined at $t = \pm 1$. Hence $h(t)$ in not differentiable at $t = \pm 1$.

**Answer: (B)**

**23.** Let $f(x)$ and $g(x)$ be functions defined on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$$

Then

(A) $f$ is differentiable at $x = 0$ and $g$ is not differentiable at $x = 0$
(B) $f$ is not differentiable at $x = 0$, whereas $g$ is differentiable at $x = 0$
(C) both $f$ and $g$ are not differentiable at $x = 0$
(D) both $f$ and $g$ are differentiable at $x = 0$

**Solution:** We have

$$f'(0-0) = 0 \quad \text{and} \quad f'(0+0) = 1$$

Therefore $f$ is not differentiable at $x = 0$. Again

$$g'(0-0) = 0 \quad \text{and} \quad g'(0+0) = 2(0) = 0$$

Therefore $g$ is differentiable at $x = 0$.

**Answer: (B)**

**24.** Let $f(x) = |x|$ and $g(x) = |x|^3$. Then

(A) both $f$ and $g$ are not differentiable at $x = 0$
(B) both $f$ and $g$ are differentiable at $x = 0$
(C) $f$ is differentiable at $x = 0$ but $g$ is not
(D) $g$ is differentiable at $x = 0$ but $f$ is not

**Solution:** We have

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

and

$$g(x) = \begin{cases} -x^3 & \text{if } x < 0 \\ x^3 & \text{if } x \ge 0 \end{cases}$$

Clearly

$$f'(0-0) = -1 \quad \text{and} \quad f'(0+0) = 1$$
$$g'(0-0) = -3(0) = 0 \quad \text{and} \quad g'(0+0) = 3(0) = 0$$

Therefore $f$ is not differentiable at $x = 0$ but $g'(0) = 0$.

**Answer: (D)**

**25.** Let $f(x) = a_0|x|^3 + a_1|x|^2 + a_2|x| + a_3$. Then

(A) $f$ is differentiable at $x = 0$ if $a_2 \ne 0$
(B) $f$ is not differentiable at $x = 0$, whatever be $a_0, a_1, a_2, a_3$
(C) $f$ is differentiable at $x = 0$ if and only if $a_2 = 0$
(D) If $f$ is differentiable at $x = 0$, then $a_0 = 0$ and $a_2 = 0$

**Solution:** In Problem 24, we have seen that $|x|$ is not differentiable at $x = 0$, whereas $|x|^3$ is differentiable at $x = 0$. Also $|x|^2 = x^2$ is differentiable for all real $x$. If $a_2 = 0$, then

$$f(x) = a_0|x|^3 + a_1|x|^2 + a_3$$

is differentiable at $x = 0$. Conversely, if $f(x)$ is differentiable at $x = 0$, then

$$a_2|x| = f(x) - a_0|x|^3 - a_1|x|^2 - a_3$$

is differentiable at $x = 0$ which is possible when $a_2 = 0$.

**Answer: (C)**

**26.** Let $f(x) = x[x]$ where $[x]$ denotes the integral part of $x$. If $x$ is not an integer, then $f'(x)$ is equal to

(A) $[x]$ (B) $2[x]$
(C) $2x$ (D) does not exist

**Solution:** Suppose $x > 0$ and $n < x < n+1$ where $n \ge 0$ integer. Therefore

$$f(x) = x[x] = nx$$

so that

$$f'(x) = n = [x]$$

If $x < 0$ and $-(n+1) < x < -n$ where $n \ge 0$ integer, then

$$f(x) = -(n+1)x$$

so that

$$f'(x) = -(n+1) = [x]$$

Therefore when $x$ is not an integer, then $f'(x) = [x]$.

**Answer: (A)**

**27.** Let $f(x) = e^x$, $g(x) = e^{-x}$ and $h(x) = g(f(x))$. Then $h'(0)$ is equal to

(A) $\dfrac{-1}{e}$ (B) $-e$
(C) 1 (D) $-1$

**Solution:** We have

$$h(x) = g(e^x) = e^{-e^x}$$
$$\Rightarrow h'(x) = (e^{-e^x})(-e^x)$$

Therefore

$$h'(0) = (e^{-1})(-1) = \frac{-1}{e}$$

**Answer: (A)**

**28.** Consider the function $f(x) = |\sin x| + |\cos x|$ for $0 < x < 2\pi$. Then

(A) $f(x)$ is differentiable $\forall x \in (0, 2\pi)$
(B) $f(x)$ is not differentiable at $x = \pi/2, \pi$ and $3\pi/2$ and differentiable at all other values in $(0, 2\pi)$
(C) $f(x)$ is not differentiable at $x = \pi/2$ and $3\pi/2$ and differentiable at all other values in $(0, 2\pi)$
(D) $f$ is discontinuous at $x = \pi/2, \pi$ and $3\pi/2$

**Solution:** We have

$$f(x) = \begin{cases} \sin x + \cos x & \text{if } 0 < x \le \dfrac{\pi}{2} \\ \sin x - \cos x & \text{if } \dfrac{\pi}{2} < x \le \pi \\ -\sin x - \cos x & \text{if } \pi < x \le \dfrac{3\pi}{2} \\ -\sin x + \cos x & \text{if } \dfrac{3\pi}{2} < x < 2\pi \end{cases}$$

Therefore

$$f'(x) = \begin{cases} \cos x - \sin x & \text{if } 0 < x \le \dfrac{\pi}{2} \\ \cos x + \sin x & \text{if } \dfrac{\pi}{2} < x \le \pi \\ -\cos x + \sin x & \text{if } \pi < x \le \dfrac{3\pi}{2} \\ -\cos x - \sin x & \text{if } \dfrac{3\pi}{2} < x < 2\pi \end{cases}$$

Now it is clear that

**(i)** $f'\left(\dfrac{\pi}{2} - 0\right) = -1, \quad f'\left(\dfrac{\pi}{2} + 0\right) = 1$

**(ii)** $f'(\pi - 0) = -1, \quad f'(\pi + 0) = 1$

**(iii)** $f'\left(\dfrac{3\pi}{2} - 0\right) = -1, \quad f'\left(\dfrac{3\pi}{2} + 0\right) = 1$

Therefore, $f$ is not differentiable at $x = \pi/2, \pi$ and $3\pi/2$.

**Answer: (B)**

**29.** Let $f(x) = |x^3|$ and $g(x) = x^3$ both being defined in the open interval $(-1, 1)$. Then

(A) $f'(x) = g'(x)$ for all $x \in (-1, 1)$
(B) $f'(x) = -g'(x)$ for all $x \in (-1, 1)$
(C) $f'(x) = |g'(x)|$ for all $x \in (-1, 1)$
(D) $g'(x) = |f'(x)|$ for all $x \in (-1, 1)$

**Solution:** We have

$$f(x) = \begin{cases} -x^3 & \text{if } -1 < x < 0 \\ x^3 & \text{if } 0 \le x < 1 \end{cases}$$

Therefore

$$f'(x) = \begin{cases} -3x^2 & \text{if } -1 < x < 0 \\ 3x^2 & \text{if } 0 \le x < 1 \end{cases}$$

Also

$$f'(0-0) = 0 = f'(0+0)$$

so that $f$ is differentiable at $x = 0$ and hence differentiable for all $x$ in $(-1, 1)$. But

$$g'(x) = 3x^2 \quad \forall x \in (-1, 1)$$

Therefore

$$g'(x) = |f'(x)| \quad \forall x \in (-1, 1)$$

**Answer: (D)**

**30.** Function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f(x - y) = \frac{f(x)}{f(y)}$$

If $f'(0) = p$ and $f'(a) = q$, then $f'(-a)$ is

(A) $\dfrac{p^2}{q}$ (B) $\dfrac{q}{p}$

(C) $\dfrac{p}{q}$ (D) $q$

**Solution:** We have

$$f(0) = f(0-0) = \frac{f(0)}{f(0)} \Rightarrow f(0) = 0 \text{ or } 1$$

If $f(0) = 0$ then

$$0 = f(0) = f(0) = f(x - x) = \frac{f(x)}{f(x)}$$

$$\Rightarrow f(x) = 0 \ \forall x$$

This is not possible, because

$$f(x - y) = \frac{f(x)}{f(y)}$$

Therefore

$$f(0) = 1 \tag{2.6}$$

Also

$$f(-a)=f(0-a)=\frac{f(0)}{f(a)}=\frac{1}{f(a)}$$

$$\Rightarrow\ f(a)f(-a)=1 \tag{2.7}$$

Now

$$\begin{aligned}
q&=f'(a)\\
&=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\
&=\lim_{h\to 0}\frac{f(h-(-a))-f(a)}{h}\\
&=\lim_{h\to 0}\frac{\dfrac{f(h)}{f(-a)}-f(a)}{h}\\
&=f(a)\lim_{h\to 0}\frac{f(h)-1}{h}\quad\text{[By Eq. (2.7)]}\\
&=f(a)\lim_{h\to 0}\frac{f(h)-f(0)}{h}\\
&\qquad[\because f(0)=1 \text{ according to Eq. (2.6)}]\\
&=f(a)f'(0)\\
&=f(a)(p)
\end{aligned}$$

Therefore

$$f(a)=\frac{q}{p} \tag{2.8}$$

Now

$$\begin{aligned}
f'(-a)&=\lim_{h\to 0}\frac{f(-a+h)-f(-a)}{h}\\
&=\lim_{h\to 0}\frac{\dfrac{f(h)}{f(a)}-\dfrac{1}{f(a)}}{h}\quad\text{[By Eq. (2.7)]}\\
&=\frac{1}{f(a)}\lim_{h\to 0}\frac{f(h)-f(0)}{h}\quad[\because f(0)=1]\\
&=\frac{f'(0)}{f(a)}\\
&=p\left(\frac{p}{q}\right)\quad\text{[By Eq. (2.8)]}\\
&=\frac{p^2}{q}
\end{aligned}$$

**Answer: (A)**

**31.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying the relation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

for all $x, y\in\mathbb{R}$. If $f(0)=1$ and $f'(0)$ exists and is equal to $-1$, then $f(2)$ is equal to

(A) 1 (B) $-1$
(C) $-2$ (D) 2

**Solution:** The given relation is

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}(f(x)+f(y)) \tag{2.9}$$

In Eq. (2.9), put $y = 0$ and replace $x$ with $2x$. Then we have

$$f(x)=\frac{1}{2}(f(2x)+1)\quad[\because f(0)=1]$$

Therefore

$$f(2x)=2f(x)-1 \tag{2.10}$$

Now,

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{f\left(2\left(\dfrac{x+h}{2}\right)\right)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{2f\left(\dfrac{x+h}{2}\right)-1-f(x)}{h}\quad\text{[By Eq. (2.10)]}\\
&=\lim_{h\to 0}\frac{2\left[\dfrac{f(x)+f(h)}{2}\right]-1-f(x)}{h}\quad\text{[By Eq. (2.9)]}\\
&=\lim_{h\to 0}\frac{f(h)-1}{h}\\
&=\lim_{h\to 0}\frac{f(h)-f(0)}{h}\quad(\because f(0)=1)\\
&=f'(0)\\
&=-1\quad\text{(By hypothesis)}
\end{aligned}$$

Therefore, $f'(x)=-1$ for all real $x$. So $f(x)=-x+k$ where $k$ is a constant. Again,

$$1=f(0)=0+k\Rightarrow k=1$$

Therefore

$$f(x)=-x+1$$

or

$$f(2)=-2+1=-1$$

**Answer: (B)**

**32.** Let $f$ be real-valued function defined on $\mathbb{R}$ as follows:

$$f(x)=\begin{cases}1-x & \text{for } x<1\\ (1-x)(2-x) & \text{for } 1\le x\le 2\\ 3-x & \text{for } x>2\end{cases}$$

Let $m$ be the number of values of $x$ at which $f$ is discontinuous and $n$ be the number of values of $x$ at which $f$ is not differentiable. Then $m+n$ is equal to

(A) 2 (B) 1
(C) 0 (D) 4

**Solution:** It is enough if we check the continuity and differentiability of $f$ at $x = 1$ and 2 only.

**(i)** $\lim_{x\to 1-0} f(x) = 1-1 = 0 = \lim_{x\to 1+0} f(x) \Rightarrow f$ is continuous at $x = 1$.

**(ii)** $\lim_{x\to 2-0} f(x) = (1-2)(2-2) = 0$ and $\lim_{x\to 2+0} f(x) = 3-2 = 1 \Rightarrow f$ is discontinuous at $x = 2$. Therefore $m = 1$. Now

$$f'(x) = \begin{cases} -1 & \text{for } x < 1 \\ 2x-3 & \text{for } 1 \le x \le 2 \\ -1 & \text{for } x > 2 \end{cases}$$

**(iii)** $f'(1-0) = -1$ and $f'(1+0) = 2(1) - 3 = -1$. That is $f'(1-0) = f'(1+0) = -1 \Rightarrow f$ is differentiable at $x = 1$ and $f'(1) = -1$.

**(iv)** Since $f$ is discontinuous at $x = 2$, it is not differentiable at $x = 2$. Therefore $n = 1$. Hence $m + n = 2$.

**Answer: (A)**

**33.** Let $f(x) = 2^{x(x-1)}$ for all $x \ge 1$. Then $(f^{-1})'(4)$ is $(1/k)\log_2 e$ where the value of $k$ is

(A) 4 (B) 8
(C) 9 (D) 12

**Solution:** Clearly $f(x) \ge 1$ for $x \ge 1$. Let $y \ge 1$ and $y = 2^{x(x-1)}$. Therefore

$$\log_2 y = x(x-1) = x^2 - x$$

This implies that $x^2 - x - \log_2 y = 0$ has real roots. Hence

$$x = \frac{1}{2}(1 \pm \sqrt{1 + 4\log_2 y})$$

Now

$$x \ge 1 \Rightarrow x = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 y})$$

Therefore

$$f^{-1}(x) = \frac{1}{2}(1 + \sqrt{1 + 4\log_2 x}) \quad \text{for } x \ge 1$$

$$(f^{-1})'(x) = \frac{1}{2}\left(0 + \frac{1}{2\sqrt{1+4\log_2 x}}\right)\left[0 + \frac{4}{x}\log_2 e\right]$$

$$(\because \log_2 x = \log_e x \cdot \log_2 e)$$

So

$$(f^{-1})'(4) = \frac{1}{4\sqrt{1+4\log_2 4}}\left(0 + \frac{4}{4}\log_2 e\right)$$

$$= \frac{1}{12}\log_2 e$$

**Answer: (D)**

**34.** The derivative of the $f(x) = \sin^4 x + \cos^4 x, 0 \le x \le 2\pi$ is positive if

(A) $0 < x < \frac{\pi}{8}$ (B) $\frac{\pi}{4} < x < \frac{\pi}{2}$
(C) $\frac{\pi}{2} < x < \frac{5\pi}{8}$ (D) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

**Solution:** We have

$$\begin{aligned} f(x) &= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x \\ &= 1 - 2\sin^2 x\cos^2 x \\ &= 1 - \frac{1}{2}\sin^2 2x \\ &= 1 - \frac{1}{2}\left(\frac{1-\cos 4x}{2}\right) \\ &= \frac{3}{4} + \frac{1}{4}\cos 4x \end{aligned}$$

Therefore

$$\begin{aligned} f'(x) > 0 &\Rightarrow -\sin 4x > 0 \\ &\Rightarrow \sin 4x < 0 \\ &\Rightarrow \pi < 4x < 2\pi \\ &\Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2} \end{aligned}$$

**Answer: (B)**

**35.** If the function

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

is continuous for all real $x$, then

(A) $a = 2, b = -1$ and $f$ is differentiable for all $x$
(B) $a = -2, b = 1$ and $f$ is not differentiable at $x = -1, 1$
(C) $a = 2, b = -1$ and $f$ is not differentiable at $x = -1, 1$
(D) $a = -2, b = -1$ and $f$ is not differentiable at $x = -1, 1$

**Solution:** We have

$$f(x) = \begin{cases} x^2 + ax + b & \text{for } x < -1 \\ 2x & \text{for } -1 \le x \le 1 \\ x^2 + ax + b & \text{for } x > 1 \end{cases}$$

$f$ is continuous at $x = -1 \Rightarrow f(-1-0) = f(-1+0)$

$$\Rightarrow +1 - a + b = -2$$

$$\Rightarrow a - b = 3 \qquad (2.11)$$

$f$ is continuous at $x = 1 \Rightarrow f(1-0) = f(1+0)$

$$\Rightarrow a + b = 1 \qquad (2.12)$$

From Eqs. (2.11) and (2.12), $a = 2$ and $b = -1$. Now

$$f(x) = \begin{cases} x^2 + 2x - 1 & \text{if } x < -1 \\ 2x & \text{if } -1 \le x \le 1 \\ x^2 + 2x - 1 & \text{if } x > 1 \end{cases}$$

Therefore

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x < -1 \\ 2 & \text{if } -1 \le x \le 1 \\ 2x + 2 & \text{if } x > 1 \end{cases}$$

Now,

$$f'(-1-0) = 2(-1) + 2 = 0$$

and $$f'(-1+0) = 2$$

Therefore $f$ is not differentiable at $x = -1$. Again

$$f'(1-0) = 2$$

and $$f'(1+0) = 2(1) + 2 = 4$$

Hence $f$ is not differentiable at $x = 1$.

**Answer: (C)**

**36.** Let

$$f(x) = \begin{cases} e^{\cos x} \sin x & \text{if } |x| \le 1 \\ 2 & \text{otherwise} \end{cases}$$

If $m$ and $p$ are, respectively, the number of points of discontinuity and the number of points at which $f$ is not differentiable, then

(A) $m = 0, p = 2$  (B) $m = 1, p = 1$
(C) $m = 2, p = 1$  (D) $m = 2, p = 2$

**Solution:** We have

$$f(x) = \begin{cases} 2 & \text{if } x < -1 \\ e^{\cos x} \sin x & \text{if } -1 \le x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$$

Clearly

$$f(-1-0) = 2$$

and $$f(-1+0) = e^{\cos(-1)} \cdot \sin(-1) = -e^{\cos 1} \sin 1$$

Therefore $f$ is discontinuous at $x = -1$ and hence $f$ is not differentiable at $x = -1$. Also $f(1-0) = e^{\cos 1} \sin 1$ and $f(1+0) = 2$ so that $f$ is discontinuous at $x = 1$ and therefore $f$ is not differentiable at $x = 1$. Hence $m = 2, p = 2$.

**Answer: (D)**

**37.** The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at

(A) $-1$  (B) 0
(C) 1  (D) 2

**Solution:** We have

$$|(x-1)(x-2)| = \begin{cases} (x-1)(x-2) & \text{for } x < 1 \\ -(x-1)(x-2) & \text{for } 1 \le x \le 2 \\ (x-1)(x-2) & \text{for } x > 2 \end{cases}$$

and $$\cos(|x|) = \cos(\pm x) = \cos x$$

Therefore

$$f(x) = \begin{cases} (x^2-1)(x-1)(x-2) + \cos x & \text{for } x < 1 \\ -(x^2-1)(x-1)(x-2) + \cos x & \text{for } 1 \le x \le 2 \\ (x^2-1)(x-1)(x-2) + \cos x & \text{for } x > 2 \end{cases}$$

Now

$$f'(x) = \begin{cases} 2x(x-1)(x-2) + (x^2-1)(x-2) + (x^2-1)(x-1) - \sin x & \text{for } x < 1 \\ -2x(x-1)(x-2) - (x^2-1)(x-2) - (x^2-1)(x-1) - \sin x & \text{for } 1 \le x \le 2 \\ 2x(x-1)(x-2) + (x^2-1)(x-2) + (x^2-1)(x-1) - \sin x & \text{for } x > 2 \end{cases}$$

Clearly

$$f'(1-0) = -\sin 1 = f'(1+0)$$

so that $f$ is differentiable at $x = 1$ and

$$f'(2-0) = -(4-1)(2-1) - \sin 2 = -3 - \sin 2$$

and $$f'(2+0) = (4-1)(2-1) - \sin 2 = 3 - \sin 2$$

Therefore $f'(2-0) \ne f'(2+0)$ and hence $f$ is not differentiable at $x = 2$.

**Answer: (D)**

**38.** Let $f(x) = xe^{x(1-x)}, x \in \mathbb{R}$ Then

(A) $f'(x) \ge 0$ in $\left[\frac{-1}{2}, 1\right]$  (B) $f'(x) < 0$ in $\left[\frac{-1}{2}, 1\right]$

(C) $f'(x) \ge 0 \; \forall x \in \mathbb{R}$  (D) $f'(x) < 0 \; \forall x \in \mathbb{R}$

**Solution:** We have

$$\begin{aligned} f'(x) &= e^{x(1-x)} + x e^{x(1-x)}(1-2x) \\ &= e^{x(1-x)}(1 + x - 2x^2) \\ &= -e^{x(1-x)}(2x+1)(x-1) \end{aligned}$$

Since $e^{x(1-x)} > 0$ for all real $x$,

$$f'(x) \geq 0 \Leftrightarrow (2x+1)(x-1) \leq 0$$
$$\Leftrightarrow x \in \left[-\frac{1}{2}, 1\right]$$

**Answer: (A)**

**39.** If $f(x) = \dfrac{\log_e(x+3)}{x^2+3x+2}$, then the domain of $f'(x)$ is

(A) $\mathbb{R} - \{-1, -2\}$ (B) $(-2, \infty)$

(C) $\mathbb{R} - \{-1, -2, -3\}$ (D) $(-3, \infty) - \{-1, -2, \}$.

**Solution:** Clearly $\log_e(x+3)$ is defined when $x > -3$. That is

$$\log_e(x+3) \text{ is defined in } (-3, \infty) \tag{2.13}$$

Also

$$x^2 + 3x + 2 = 0 \Leftrightarrow x = -1, -2 \tag{2.14}$$

From Eqs. (2.13) and (2.14), the domain of $f(x)$ is $(-3, \infty) - \{-1, -2\}$. Now

$$f'(x) = \frac{\dfrac{x^2+3x+2}{x+3} - (2x+3)\log_e(x+3)}{(x^2+3x+2)^2}$$
$$= \frac{(x^2+3x+2) - (x+3)(2x+3)\log_e(x+3)}{(x+3)(x^2+3x+2)^2}$$

Therefore, $f'(x)$ is defined for $x > -3$ and $x \neq -2, -1$. So domain of $f'(x)$ is $(-3, \infty) - \{-1, -2\}$.

**Answer: (D)**

**40.** $f : (0, \infty) \to \mathbb{R}$ is continuous. If $F(x)$ is a differentiable function such that $F'(x) = f(x)\ \forall\ x > 0$ and $F(x^2) = x^2 + x^3$, then $f(4)$ equals

(A) $\dfrac{5}{4}$ (B) 7

(C) 4 (D) 2

**Solution:** We have

$$F'(x) = f(x) \Rightarrow F'(x^2) = f(x^2) \tag{2.15}$$

Now, since

$$F(x^2) = x^2 + x^3$$

we get

$$F'(x^2)(2x) = 2x + 3x^2$$

Therefore

$$F'(x^2) = 1 + \frac{3}{2}x$$

From Eq. (2.15) we get

$$f(x^2) = F'(x^2) = 1 + \frac{3}{2}x$$

Put $x = 2$ so that $f(4) = 1 + 3 = 4$.

***Note:*** If we put $x = -2$, then $f(4) = -2$ which cannot be done, because $f$ is defined for $x > 0$ and $F'(x) = f(x)$ for $x > 0$. Also one can replace $x$ with $\sqrt{x}$ in $F(x^2) = x^2 + x^3$ so that $F(x) = x + x^{3/2}$ and hence

$$f(x) = F'(x) = 1 + \frac{3}{2}\sqrt{x}$$

**Answer: (C)**

**41.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \text{Max}\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is

(A) $\{-1, 1\}$ (B) $\{-1, 0\}$

(C) $\{0, 1\}$ (D) $\{-1, 0, 1\}$

**Solution:** Explicit form of $f$ is

$$f(x) = \begin{cases} x & \text{if } x < -1 \\ x^3 & \text{if } -1 \leq x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ x^3 & \text{if } x > 1 \end{cases}$$

See Fig. 2.1. The graphs $y = x^3$ and $y = x$ intersect in three points, namely $(0, 0)$, $(-1, -1)$ and $(1, 1)$. Therefore

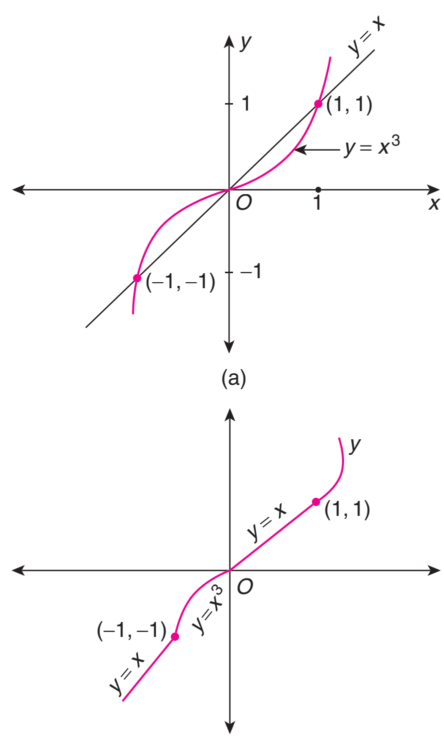



**FIGURE 2.1** Single correct choice type question 41: (a) Curve of $y = x$ and $y = x^3$. (b) Graph of $f(x) = \text{Max}\{x, x^3\}$.

$$f'(x)=\begin{cases}1 & \text{if } x<-1\\ 3x^2 & \text{if } -1<x<0\\ 1 & \text{if } 0<x<1\\ 3x^2 & \text{if } x<1\end{cases}$$

Clearly

**(i)** $f'(-1-0)=1$ and $f'(-1+0)=3$

**(ii)** $f'(0-0)=0$ and $f'(0+0)=1$

**(iii)** $f'(1-0)=1$ and $f'(1+0)=3$

Therefore $f$ is not differentiable at $x=-1, 0$ and 1.

**Answer: (D)**

**42.** Which of the following function(s) is differentiable at $x=0$?

(A) $\cos(|x|)+|x|$ (B) $\cos(|x|)-|x|$

(C) $\sin(|x|)+|x|$ (C) $\sin(|x|)-|x|$

**Solution:**

(A) We have

$$f_1(x)=\cos(|x|)+|x|$$
$$=\begin{cases}\cos x-x & \text{if } x<0\\ \cos x+x & \text{if } x\ge 0\end{cases}$$

Differentiating w.r.t. $x$ we get

$$f_1'(x)=\begin{cases}-\sin x-1 & \text{if } x<0\\ -\sin x+1 & \text{if } x>0\end{cases}$$

This implies

$$f_1'(0-0)=-1 \quad \text{and} \quad f_1'(0+0)=1$$

Therefore, $f_1(x)$ is not differentiable at $x=0$. Hence (A) is not correct.

(B) We have

$$f_2(x)=\cos(|x|)-|x|$$
$$=\begin{cases}\cos x+x & \text{if } x<0\\ \cos x-x & \text{if } x\ge 0\end{cases}$$

Differentiating w.r.t. $x$ we get

$$f_2'(x)=\begin{cases}-\sin x+1 & \text{if } x<0\\ -\sin x-1 & \text{if } x>0\end{cases}$$

This implies that

$$f_2'(0-0)=1 \text{ and } f_2'(0+0)=-1$$

Therefore, $f_2(x)$ is not differentiable at $x=0$. Hence (B) is not correct.

(C) We have

$$f_3(x)=\sin(|x|)+|x|$$
$$=\begin{cases}-\sin x-x & \text{if } x<0\\ \sin x+x & \text{if } x\ge 0\end{cases}$$

Differentiating w.r.t. $x$ we get

$$f_3'(x)=\begin{cases}-\cos x-1 & \text{if } x<0\\ \cos x+1 & \text{if } x>0\end{cases}$$

Now,

$$f_3'(0-0)=-1-1=-2$$

and $$f_3'(0+0)=1+1=2$$

Therefore $f_3$ is not differentiable at $x=0$. So (C) is not correct.

(D) We have

$$f_4(x)=\sin(|x|)-|x|$$
$$=\begin{cases}-\sin x+x & \text{if } x<0\\ \sin x-x & \text{if } x\ge 0\end{cases}$$

Differentiating w.r.t. $x$ we get

$$f_4'(x)=\begin{cases}-\cos x+1 & \text{if } x<0\\ \cos x-1 & \text{if } x\ge 0\end{cases}$$

This implies

$$f_4'(0-0)=-1+1=0$$

and $$f_4'(0+0)=1-1=0$$

Therefore $f_4(x)$ is differentiable at $x=0$. Hence (D) is correct.

**Answer: (D)**

**43.** Let $f(x)=3\sin x-4\sin^3 x$. The length of the longest interval in which $f'(x)\ge 0$ is

(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{2}$

(C) $\dfrac{3\pi}{2}$ (D) $\pi$

**Solution:** We have $f(x)=\sin 3x, 0\le x\le 2\pi$. Differentiating w.r.t. $x$ we get

$$f'(x)=3\cos 3x\ge 0$$
$$\Rightarrow 0\le 3x\le\frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2}\le 3x\le 2\pi$$

Therefore, length of the longest interval in which $f'(x)\ge 0$ is

$$\frac{\pi}{6}+\left(\frac{2\pi}{3}-\frac{\pi}{2}\right)=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$$

**Answer: (A)**

**44.** Suppose that $f(x)=(x+1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g'(x)$ is

(A) $\sqrt{x}-1, x \ge 0$ (B) $\dfrac{1}{2\sqrt{x}}, x>0$

(C) $\dfrac{1}{\sqrt{x}}-1, x>0$ (D) $2\sqrt{x}-1, x \ge 0$

**Solution:** By hypothesis $g(x)$ is the inverse of $f$ for $x \ge 0$ (Fig. 2.2). If $y \ge 0$ and $y=(x+1)^2$ we get

$$x=\sqrt{y}-1$$

Therefore

$$g(x)=f^{-1}(x)=\sqrt{x}-1 \qquad \text{for } x \ge 0$$

So, for $x > 0$

$$g'(x)=\frac{1}{2\sqrt{x}}$$

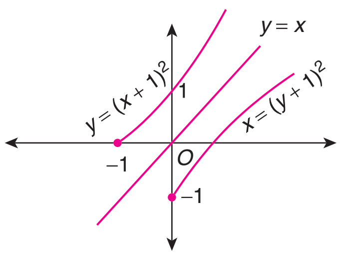



**FIGURE 2.2** Single correct choice type question 44.

**Answer: (B)**

**45.** The domain of the derivative of the function

$$f(x)=\begin{cases} \operatorname{Tan}^{-1}x & \text{if } |x| \le 1 \\ \dfrac{1}{2}(|x|-1) & \text{if } |x|>1 \end{cases}$$

is

(A) $\mathbb{R}-\{0\}$ (B) $\mathbb{R}-\{1\}$

(C) $\mathbb{R}-\{-1\}$ (D) $\mathbb{R}-\{-1, 1\}$

**Solution:** We have

$$f(x)=\begin{cases} \dfrac{1}{2}(-x-1) & \text{if } x<-1 \\ \operatorname{Tan}^{-1}x & \text{if } -1 \le x \le 1 \\ \dfrac{1}{2}(x-1) & \text{if } x>1 \end{cases}$$

Clearly $f$ is discontinuous at $x=-1, 1$ and hence at $-1, 1$, the function $f$ is not differentiable. Also

$$f'(x)=\begin{cases} \dfrac{-1}{2} & \text{if } x<-1 \\ \dfrac{1}{1+x^2} & \text{if } -1 \le x \le 1 \\ \dfrac{1}{2} & \text{if } x>1 \end{cases}$$

Therefore domain of $f'(x)$ is $\mathbb{R}-\{-1, 1\}$.

**Answer: (D)**

**46.** If $y$ is a function of $x$ and $\log(x+y)=2xy$, then $\left(\dfrac{dy}{dx}\right)_{x=0}$ is

(A) 1 (B) –1

(C) 2 (D) 0

**Solution:** Differentiating the given equation w.r.t. $x$ we get

$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=2y+2x\left(\frac{dy}{dx}\right) \tag{2.16}$$

From the given equation, $x=0 \Rightarrow y=1$. From Eq. (2.16),

$$\frac{dy}{dx}\left(\frac{1}{x+y}-2x\right)=2y-\frac{1}{x+y}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{x=0} \times \left[\frac{1}{0+1}-0\right]=2(1)-\frac{1}{0+1}=2-1=1$$

So

$$\left(\frac{dy}{dx}\right)_{x=0}=1$$

**Answer: (A)**

**47.** Let

$$f(x)=\begin{cases} xe^{ax}, & x \le 0 \\ x+ax^2-x^3, & x>0 \end{cases}$$

where $a$ is a positive constant. The interval in which $f''(x) \ge 0$, is

(A) $\left[\dfrac{-a}{3}, \dfrac{2a}{3}\right]$ (B) $\left[\dfrac{a}{2}, a\right]$

(C) $\left[\dfrac{-2}{a}, \dfrac{a}{3}\right]$ (D) $\left[\dfrac{a}{3}, \infty\right)$

**Solution:** We have

$$\begin{aligned} f(0-0) &= 0e^0 = 0 \\ &= 0+a(0)-0 \\ &= f(0+0) \end{aligned}$$

Therefore $f$ is continuous at $x = 0$. Now

$$f'(x)=\begin{cases} e^{ax}+axe^{ax} & \text{for } x<0 \\ 1+2ax-3x^2 & \text{for } x>0 \end{cases}$$

Clearly

$$f'(0-0)=e^0+0=1$$

and $\qquad f'(0+0) = 1 + 2a(0) - 3(0) = 1$

So $f$ is differentiable at $x = 0$ and $f'(0) = 1$. Hence $f'(x)$ exists for all real $x$. Now

$$f''(x) = \begin{cases} ae^{ax}(2+ax) & \text{for } x < 0 \\ 2a & \text{for } x = 0 \\ 2a - 6x & \text{for } x > 0 \end{cases}$$

Clearly $f''(0) = 2a > 0$ ($\because a > 0$). For $x < 0$,

$$f''(x) \ge 0 \Leftrightarrow 2 + ax \ge 0$$

$$\Leftrightarrow x \ge \frac{-2}{a} \tag{2.17}$$

Again for $x > 0$,

$$f''(x) \ge 0 \Leftrightarrow 2a - 6x \ge 0$$

$$\Leftrightarrow x \ge \frac{a}{3} \tag{2.18}$$

From Eqs. (2.17) and (2.18), $f''(x) \ge 0$ in the interval $\left[\frac{-2}{a}, \frac{a}{3}\right]$.

**Answer: (C)**

**48.** The function $f: \mathbb{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is given by

$$f(x) = 2\,\mathrm{Tan}^{-1}(e^x) - \frac{\pi}{2}$$

Then

(A) $f$ is even and $f'(x) > 0$ for $x > 0$
(B) $f$ is odd and $f'(x) > 0$ for all $x \in \mathbb{R}$
(C) $f$ is odd and $f'(x) < 0$ for all $x \in \mathbb{R}$
(D) $f$ is neither even nor odd, but $f'(x) > 0\ \forall x \in \mathbb{R}$

**Solution:** We have

$$f(x) = 2\,\mathrm{Tan}^{-1}(e^x) - \frac{\pi}{2}$$

Therefore

$$\begin{aligned} f(-x) &= 2\,\mathrm{Tan}^{-1}(e^{-x}) - \frac{\pi}{2} \\ &= 2\,\mathrm{Cot}^{-1}(e^x) - \frac{\pi}{2} \\ &= 2\left[\frac{\pi}{2} - \mathrm{Tan}^{-1}(e^x)\right] - \frac{\pi}{2} \\ &\qquad \left(\because \mathrm{Tan}^{-1}x + \mathrm{Cot}^{-1}x = \frac{\pi}{2}\ \forall x \in \mathbb{R}\right) \\ &= \frac{\pi}{2} - 2\,\mathrm{Tan}^{-1}(e^x) \\ &= -f(x) \end{aligned}$$

Therefore $f$ is an odd function. Also

$$f'(x) = \frac{2e^x}{1+e^{2x}} > 0\ \forall x \in \mathbb{R}\ \left(\because (\mathrm{Tan}^{-1}x)' = \frac{1}{1+x^2}\right)$$

Therefore $f$ is odd and $f'(x) > 0\ \forall x \in \mathbb{R}$.

**Answer: (B)**

**49.** Let $f(x) = \dfrac{\log_a(\pi + x)}{\log_a(e+x)}\ \forall x > -e$ where $a > 1$. Then

(A) $f'(x) > 0$ for $x \ge 0$
(B) $f'(x) < 0$ for $x \ge 0$
(C) $f'(x) > 0$ in $\left[0, \frac{\pi}{e}\right)$ and $f'(x) \le 0$ in $\left[\frac{\pi}{e}, \infty\right)$
(D) $f'(x) < 0$ in $\left[0, \frac{\pi}{e}\right]$ and $f'(x) \ge 0$ in $\left[\frac{\pi}{e}, \infty\right)$

**Solution:** We have

$$f'(x) = \frac{\dfrac{1}{\pi+x}\log(e+x) - \dfrac{1}{e+x}\log(\pi+x)}{(\log(e+x)^2)}$$

$\pi > e$ and $x \ge 0 \Rightarrow \pi + x \ge e + x$

$$\Rightarrow \log(\pi + x) > \log(e+x)$$

$$(\because \log_a t \text{ is increasing because } a > 1)$$

$$\Rightarrow \frac{\log(\pi+x)}{e+x} > \frac{\log(e+x)}{\pi+x}$$

Therefore $f'(x) < 0$ and $x \ge 0$.

**Answer: (B)**

**50.** Let $f(x) = \mathrm{Max}\{2\sin x, 1 - \cos x\}$ for $0 < x < \pi$. Then the value of $x$ at which $f$ is not differentiable is

(A) $\mathrm{Cos}^{-1}x\left(\frac{3}{5}\right)$ (B) $\mathrm{Sin}^{-1}\left(\frac{3}{5}\right)$

(C) $\pi - \mathrm{Sin}^{-1}\left(\frac{3}{5}\right)$ (D) $\pi - \mathrm{Cos}^{-1}\left(\frac{3}{5}\right)$

**Solution:** See Fig. 2.3. The two curves $y = 2\sin x$ and $y = 1 - \cos x$ intersect at the point

$$x = \pi - \mathrm{Cos}^{-1}\left(\frac{3}{5}\right)$$

which can be obtained by solving the equation $2\sin x = 1 - \cos x$. Also

$$f(x) = \begin{cases} 2\sin x & \text{for } 0 < x \le \mathrm{Cos}^{-1}\left(\frac{-3}{5}\right) \\ 1 - \cos x & \text{for } \mathrm{Cos}^{-1}\left(\frac{-3}{5}\right) < x < \pi \end{cases}$$

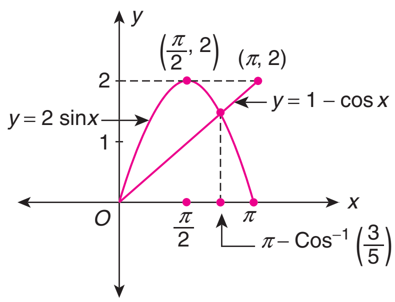



FIGURE 2.3 Single correct choice type question 50.

Differentiating w.r.t. $x$ we get

$$f'(x) = \begin{cases} 2\cos x & \text{for } 0 < x \le \pi - \text{Cos}^{-1}\left(\frac{3}{5}\right) \\ \sin x & \text{for } \pi - \text{Cos}^{-1}\frac{3}{5} < x < \pi \end{cases}$$

Therefore at $x = \pi - \text{Cos}^{-1}(3/5)$, we have

$$\text{Left derivative } = 2\cos\left(\pi - \text{Cos}^{-1}\left(\frac{3}{5}\right)\right)$$
$$= -2 \times \frac{3}{5} = -\frac{6}{5}$$

At $x = \pi - \text{Cos}^{-1}(3/5)$, we have

$$\text{Right derivative } = \sin\left(\pi - \text{Cos}^{-1}\left(\frac{3}{5}\right)\right)$$
$$= \sin\left(\text{Cos}^{-1}\left(\frac{3}{5}\right)\right)$$
$$= \sin\left(\text{Sin}^{-1}\left(\frac{4}{5}\right)\right)$$
$$= \frac{4}{5}$$

Therefore, $f'(\pi - \text{Cos}^{-1}(3/5))$ does not exist.

**Answer: (D)**

***Note:*** In general, if the graph of a function is union of branches of intersecting curves, then at the points of intersection the function is not differentiable.

**51.** Let $f(x) = \text{Max}\{\sin x, \cos x, 0\}$ for $0 \le x \le 2\pi$. Then, the number of points at which $f$ is not differentiable in $(0, 2\pi)$ is

(A) 0 (B) 1
(C) 2 (D) 3

**Solution:** See Fig. 2.4 and note that the thick portion is the graph of $y = f(x)$. Draw the graph of $y = \sin x$, $y = \cos x$ so that

$$f(x) = \begin{cases} \cos x & \text{for } 0 \le x \le \frac{\pi}{4} \\ \sin x & \text{for } \frac{\pi}{4} < x \le \pi \\ 0 & \text{for } \pi < x \le \frac{3\pi}{2} \\ \cos x & \text{for } \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

As said in the above note, there are three **sharp points** on the graph at $x = \pi/4, \pi, 3\pi/2$ and hence $f$ is not differentiable at these points.

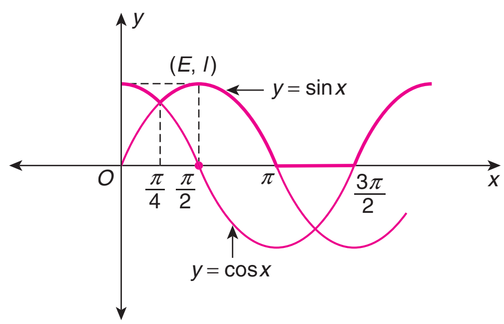



FIGURE 2.4 Single correct choice type question 51.

**Answer: (D)**

**52** $f:(0,\infty) \to (0,\infty)$ is a twice differentiable function and satisfies the functional relation $f(x+1) = xf(x)$ for all positive $x$. If $g(x) = \log(f(x))$, then for any positive integer $n$,

$$g''\left(n + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n-1)^2}\right)$

(B) $4\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n-1)^2}\right)$

(C) $-4\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n+1)^2}\right)$

(D) $4\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n+1)^2}\right)$

**Solution:** Since $g(x) = \log(f(x))$, we have

$$g(x+1) = \log(f(x+1))$$
$$= \log(xf(x))$$
$$= \log x + \log(f(x))$$
$$= \log x + g(x)$$

Therefore, for all $x > 0$ we have

$$g(x+1) - g(x) = \log x$$

Replacing $x$ with $x - (1/2)$ $(x > 1/2)$ we have

$$g\left(x+\frac{1}{2}\right)-g\left(x-\frac{1}{2}\right)=\log\left(x-\frac{1}{2}\right)$$

Differentiating w.r.t. $x$ we get

$$g'\left(x+\frac{1}{2}\right)-g'\left(x-\frac{1}{2}\right)=\frac{1}{x-\frac{1}{2}}=\frac{2}{2x-1}$$

Differentiating again, we get

$$g''\left(x+\frac{1}{2}\right)-g''\left(x-\frac{1}{2}\right)=\frac{-4}{(2x-1)^2} \qquad (2.19)$$

Now, substituting $x=1,2,3,\ldots,n$ in Eq. (2.19) and adding them, we get that

$$g''\left(n+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4\left(1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2n-1)^2}\right)$$

**Answer: (A)**

**53.** Let $f(x)$ be a polynomial function and $g(x)=e^x$. If $h(x)=(f\circ g)(x)$, then $h''(x)$ is

(A) $f''(e^x)\cdot e^x+f'(e^x)$

(B) $f''(e^x)\cdot e^{2x}+f'(x)\cdot e^x$

(C) $f''(e^x)$

(D) $f''(e^x)\cdot e^{2x}+f'(e^x)\cdot e^x$

**Solution:** We have

$$h(x)=(f\circ g)(x)=f(g(x))=f(e^x).$$

Differentiating w.r.t. $x$ (using chain rule) we get

$$h'(x)=f'(e^x)\cdot e^x$$

Again differentiating we get

$$\begin{aligned}h''(x)&=f''(e^x)\cdot e^x\cdot e^x+f'(e^x)\cdot e^x\\&=f''(e^x)\cdot e^{2x}+f'(e^x)\cdot e^x\end{aligned}$$

**Answer: (D)**

**54.** If

$$f(x)=2\,\mathrm{Sin}^{-1}\sqrt{1-x}+\mathrm{Sin}^{-1}(2\sqrt{x(1-x)})$$

for $0<x<1/2$ then $f'(x)$ is equal to

(A) $\dfrac{2}{\sqrt{x(1-x)}}$ (B) 0

(C) $\sqrt{\dfrac{1-x}{x}}$ (D) $x\dfrac{-1}{\sqrt{x(1-x)}}$

**Solution:** We have $0<x<1/2$. This implies that

$$\mathrm{Sin}^{-1}\sqrt{1-x} \text{ and } \mathrm{Sin}^{-1}(2\sqrt{x(1-x)})$$

are defined because

$$\frac{1}{\sqrt{2}}<\sqrt{1-x}<1 \quad\text{and}\quad 0<2\sqrt{x(1-x)}<1$$

$$\Leftrightarrow 4x(1-x)<1$$

$$\Leftrightarrow 4x^2-4x+1=(1-2x)^2>0$$

Therefore

$$\begin{aligned}f'(x)&=\frac{2}{\sqrt{1-(1-x)}}\times\frac{1}{2\sqrt{1-x}}(-1)\\&\quad+\frac{1}{\sqrt{1-4x(1-x)}}\times\frac{2}{2\sqrt{x(1-x)}}(1-2x)\\&=\frac{-1}{\sqrt{x(1-x)}}+\frac{1-2x}{(1-2x)\sqrt{x(1-x)}} \qquad (\because 0<x<1/2)\\&=\frac{-1}{\sqrt{x(1-x)}}+\frac{1}{\sqrt{x(1-x)}}=0\end{aligned}$$

**Answer: (B)**

**55.** If $y^2=P(x)$ is a polynomial of degree 3, then

$$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)=$$

(A) $P'''(x)+P'(x)$ (B) $P''(x)\cdot P'''(x)$

(C) $P(x)P'''(x)$ (D) a constant

**Solution:** It is given that $y^2=P(x)$. In this problem, for comfort sake, we denote $dy/dx$ by $y_1$ and $d^2y/dx^2$ by $y_2$. Differentiating the given equation $y^2=P(x)$ w.r.t. $x$, we have

$$2yy_1=P'(x) \qquad (2.20)$$

Again differentiating both sides w.r.t. $x$ we get

$$2y_1^2+2yy_2=P''(x) \qquad (2.21)$$

Multiplying both sides of Eq. (2.21) with $y^2$, we get

$$2y^2y_1^2+2y^3y_2=y^2P''(x)=P(x)P''(x)$$

Therefore

$$\begin{aligned}2y^3y_2&=P(x)P''(x)-2y^2y_1^2\\&=P(x)P''(x)-\frac{1}{2}(P'(x))^2 \qquad \text{[By Eq. (2.20)]}\end{aligned}$$

So

$$\begin{aligned}2\frac{d}{dx}(y^3y_2)&=P'(x)P''(x)+P(x)P'''(x)-\frac{1}{2}\times 2P'(x)\times P''(x)\\&=P(x)P'''(x)\end{aligned}$$

**Answer: (C)**

**56.** If $y=\sin(m\,\mathrm{Sin}^{-1}x)$, then

$$(1-x^2)\frac{d^2y}{dx^2}=$$

(A) $xy - m^2y^2$ (B) $x\dfrac{dy}{dx} - m^2y$

(C) $x\dfrac{dy}{dx} + m^2y$ (D) $x\dfrac{dy}{dx} - my^2$

**Solution:** We have $y = \sin(m\,\text{Sin}^{-1}x)$. Differentiating w.r.t. $x$ we get

$$\frac{dy}{dx} = \cos(m\,\text{Sin}^{-1}x) \times \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{dy}{dx} = m\cos(m\,\text{Sin}^{-1}x)$$

Again differentiating both sides w.r.t. $x$ we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{1}{2\sqrt{1-x^2}}(-2x)\frac{dy}{dx}$$
$$= m[-\sin(m\,\text{Sin}^{-1}x)] \times \frac{m}{\sqrt{1-x^2}}$$
$$= \frac{-m^2y}{\sqrt{1-x^2}}$$

Therefore

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$$
$$(1-x^2)\frac{d^2y}{dx} = x\frac{dy}{dx} - m^2y$$

**Answer: (B)**

**57.** If $xy = ax^2 + (b/x)$, then

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} =$$

(A) $\dfrac{y}{x}$ (B) $\dfrac{-y}{x}$

(C) $\dfrac{2y}{x}$ (D) $\dfrac{-2y}{x}$

**Solution:** It is given that

$$xy = ax^2 + \frac{b}{x}$$

Differentiating both sides w.r.t. $x$ we get

$$y + x\frac{dy}{dx} = 2ax - \frac{b}{x^2} \quad (2.22)$$

$$\Rightarrow x^2y + x^3\frac{dy}{dx} = 2ax^3 - b \quad (2.23)$$

Again differentiating both sides w.r.t. $x$ we get

$$2yx + x^2\frac{dy}{dx} + 3x^2\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} = 6ax^2$$

$$\Rightarrow 2xy + 4x^2\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} = 6ax^2$$

$$\Rightarrow 2ax^2 + \frac{2b}{x} + 4x^2\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} = 6ax^2 \quad \left(\because xy = ax^2 + \frac{b}{x}\right)$$

$$\Rightarrow 4x^2\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} = 4ax^2 - \frac{2b}{x}$$

$$\Rightarrow 4x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 4ax - \frac{2b}{x^2}$$
$$= \frac{4ax^3 - 2b}{x^2}$$
$$= \frac{2(2ax^3 - b)}{x^2}$$
$$= \frac{2}{x^2}\left(x^2y + x^3\frac{dy}{dx}\right) \quad \text{[By Eq. (2.23)]}$$
$$= 2y + 2x\frac{dy}{dx}$$

Therefore

$$2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 2y$$
$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{2y}{x}$$

**Answer: (C)**

**58.** If $0 < x < 1$, then

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots\infty =$$

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{x}{1-x}$

(C) $\dfrac{x}{1+x}$ (D) $\dfrac{1-x}{1+x}$

**Solution:** Let

$$u_n = \frac{2^{n-1}x^{2^{n-1}-1}}{1+x^{2^{n-1}}}$$

Therefore

$$\frac{u_{n+1}}{u_n} = \frac{2\cdot x^{2^{n-1}-1}(1+x^{2^{n-1}})}{1+x^{2^n}} \to 0 \quad \text{as } n \to \infty$$

Combining Theorems 1.57 and 1.58 (i.e., applying Cauchy's Root Test and D'Alemberts Test), we get that the given series is convergent. Now let

$$y = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^{n-1}})$$

so that

$$(1-x)y = (1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})$$
$$= (1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})$$

Therefore, finally

$$(1-x)y = 1-x^{2^n}$$

and hence

$$y = \frac{1-x^{2^n}}{1-x} \to \frac{1}{1-x} \quad \text{as } n \to \infty$$

Therefore the infinite product

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots\infty = \frac{1}{1-x}$$

Using logarithmic differentiation, we have

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots + \infty = \frac{1}{1-x}$$

**Answer: (A)**

***Note:*** For more information about convergence of series of positive terms, see pages 66 and 258 of *Introduction To Real Analysis*, Robert G. Bartle and Donald R. Sherbert, Wiley Student Edition, Wiley India.

**59.** Let $f(x) = [x]\sin(\pi x)$ where $[x]$ denotes the integer part of $x$. Then at $x = k$, $k$ being an integer, the left derivative of $f(x)$ is

(A) $(-1)^k(k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$

(C) $(-1)^k(k\pi)$ (D) $(-1)^{k-1}(k\pi)$

**Solution:** We have

$$f'(k-0) = \lim_{\substack{h\to 0\\ h>0}} \frac{f(k-h)-f(0)}{-h}$$
$$= \lim_{h\to 0} \frac{[k-h]\sin(\pi(k-h))}{-h}$$
$$= \lim_{h\to 0} \frac{(k-1)\sin(k\pi - h\pi)}{-h}$$
$$= \lim_{h\to 0} \frac{(k-1)(-1)^{k-1}\sin(h\pi)}{-h}$$
$$= \lim_{h\to 0}(k-1)(-1)^k\left(\frac{\sin h\pi}{\pi h}\right)\pi$$
$$= (-1)^k(k-1)\pi$$

**Answer: (A)**

**60.** Let $f(x) = \begin{cases} x\left(1+\dfrac{1}{3}\sin(\log x^2)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then

(A) $f$ is discontinuous at $x = 0$
(B) $f$ is differentiable at $x = 0$
(C) $f$ is continuous and differentiable at $x = 0$
(D) $f$ is continuous at $x = 0$, but not differentiable at $x = 0$

**Solution:** Observe that

$$f(x) = \begin{cases} x\left(1+\dfrac{1}{3}\sin(2\log|x|)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Since $\sin(2\log|x|)$ is a bounded function, we have

$$\lim_{x\to 0} x\sin(2\log|x|) = 0$$

Therefore

$$\lim_{x\to 0} f(x) = 0+0 = 0 = f(0)$$

Also

$$f'(x) = 1 + \frac{1}{3}\sin(2\log|x|) + \frac{x}{3}\cdot\frac{2}{x}\cos(2\log|x|) \quad \text{for} \quad x \neq 0$$

and $f'(0)$ does not exist.

**Answer: (D)**

**61.** Consider the following two statements:

**Statement I:** The function $f(x) = \dfrac{x}{1+|x|}$ is differentiable for real $x$.

**Statement II:** The function $g(x) = x^2|x|$ is thrice differentiable for all real $x$.

Then

(A) Both statements are true.
(B) Statement I is true but statement II is false.
(C) Statement I is false and statement II is true.
(D) Both statements are false.

**Solution:** We have

$$f(x) = \frac{x}{1+|x|} = \begin{cases} \dfrac{x}{1-x} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \dfrac{x}{1+x} & \text{if } x > 0 \end{cases}$$

$$g(x) = x^2|x| = \begin{cases} -x^3 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^3 & \text{if } x > 0 \end{cases}$$

Clearly, $f(x)$ is continuous for all $x \neq 0$ and at $x = 0$

$$f(0-0) = f(0+0) = 0 = f(0)$$

Hence $f$ is continuous for all real $x$. Also

$$f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} & \text{if } x < 0 \\ \dfrac{1}{(1+x)^2} & \text{if } x > 0 \end{cases}$$

and $\quad f'(0-0) = 1 = f'(0+0)$

Therefore $f$ is differentiable for all real $x$. Thus, Statement I is true. Now

$$g'(x) = \begin{cases} -3x^2 & \text{if } x < 0 \\ 3x^2 & \text{if } x > 0 \end{cases}$$

$$\Rightarrow g'(0-0) = g'(0+0) = 0$$

and

$$g''(x) = \begin{cases} -6x & \text{if } x < 0 \\ 6x & \text{if } x > 0 \end{cases}$$

$$\Rightarrow g''(0-0) = g''(0+0) = 0$$

But

$$g'''(x) = \begin{cases} -6 & \text{if } x < 0 \\ 6 & \text{if } x > 0 \end{cases}$$

$$\Rightarrow g'''(0-0) = -6;\ g'''(0+0) = 6$$

Therefore $g''$ is not differentiable at $x = 0$. Hence $g'''(x)$ exists for all real $x$ is false. So Statement II is false.

**Answer: (B)**

## *Multiple Correct Choice Type Questions*

**1.** Let $f(x) = [x] + |1-x|$ for $-1 \leq x \leq 3$ where $[x]$ denotes the integer part of $x$. Then

(A) In the open interval $(-1, 3)$, $f$ has three points of discontinuity

(B) $f$ is right continuous at $x = -1$ and has right derivative at $x = -1$

(C) $f$ is left continuous at $x = 3$ and has left derivative at $x = 3$

(D) $f$ has right derivative at $x = -1$ and is not differentiable at $x = 0, 1, 2, 3$.

**Solution:** We have

$$f(x) = \begin{cases} -1+1-x = -x & \text{for } -1 \leq x < 0 \\ 0+1-x = 1-x & \text{for } 0 \leq x < 1 \\ 1+x-1 = x & \text{for } 1 \leq x < 2 \\ 2+x-1 = x+1 & \text{for } 2 \leq x < 3 \\ 3+2 = 5 & \text{at } x = 3 \end{cases}$$

See Fig. 2.5 which shows the graph of $y = f(x)$. On the graph clearly there are breaks at $x = 0, 1, 2, 3$.

**(i)** At $x = -1$, the function is right continuous.

**(ii)** At $x = 3$, $f(x)$ is not left continuous.

**(iii)** At discontinuous points the function is not differentiable.

**Answers: (A), (B), (D)**

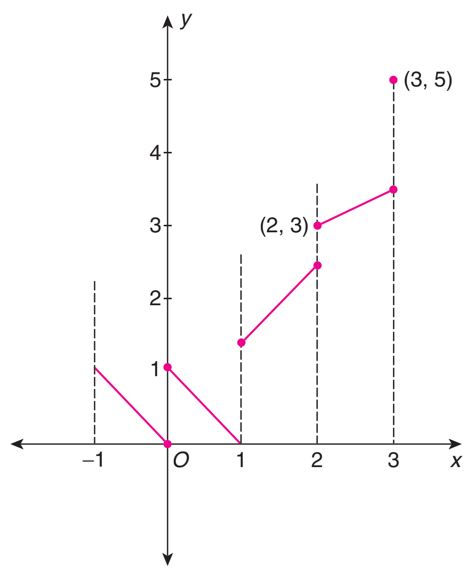



**FIGURE 2.5** Multiple correct choice type question 1.

**2.** Let

$$f(x) = \begin{cases} 3^x & \text{for } -1 \leq x \leq 1 \\ 4-x & \text{for } 1 < x < 4 \end{cases}$$

Then

(A) $f$ is discontinuous at $x = 1$

(B) $f$ is continuous at $x = 1$

(C) $f$ is differentiable at $x = 1$

(D) $f$ is not differentiable at $x = 1$

**Solution:** See Fig. 2.6. We have $f(0) = 1$ and $f(1) = 3$. The graph has no break in $[-1, 4)$. But at the point $(1, 3)$, there is a sharp point. Now

$$f(1-0) = 3^1 = 3 = 4 - 1 = f(1+0)$$

Therefore $f$ is continuous at $x = 1$. Again

$$f'(1-0) = 3\log 3 \quad \text{and} \quad f'(1+0) = -1$$

Therefore $f$ is not differentiable at $x = 1$.

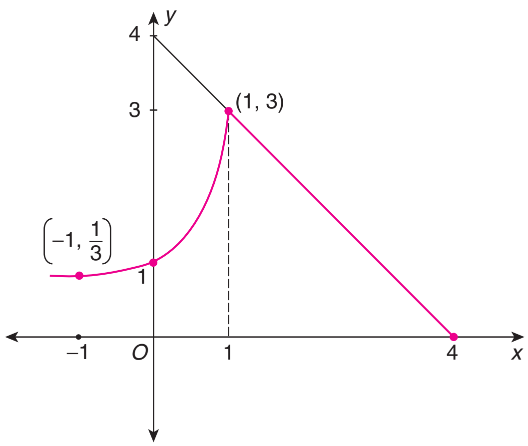



**FIGURE 2.6** Multiple correct choice type question 2.

**Answers: (B), (D)**

**3.** Let $f(x) = |\log x|$. Then
(A) $f$ is continuous at $x = 1$
(B) $f$ is discontinuous at $x = 1$
(C) $f$ is differentiable at $x = 1$
(D) $f$ is not differentiable at $x = 1$

**Solution:** We have

$$f(x) = \begin{cases} -\log x & \text{for } 0 < x < 1 \\ \log x & \text{for } 1 \le x \end{cases}$$

Now

$$f(1-0) = -\log 1 = 0$$

and $$f(1+0) = \log 1 = 0$$

Therefore $f$ is continuous at $x = 1$. Now

$$f'(x) = \begin{cases} -\dfrac{1}{x} & \text{for } 0 < x < 1 \\ \dfrac{1}{x} & \text{for } 1 < x \end{cases}$$

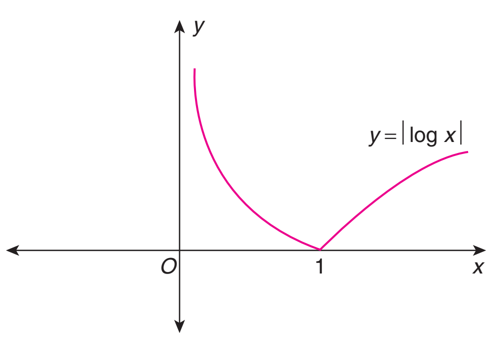



**FIGURE 2.7** Multiple correct choice type question 3.

Therefore

$$f'(1-0) = -\frac{1}{1} = -1$$

and $$f'(1+0) = \frac{1}{1} = 1$$

Hence $f$ is not differentiable at $x = 1$. Also you can notice that $(1, 0)$ is a sharp point on the graph in Fig. 2.7.

**Answers: (A), (D)**

**4.** The function $\mathrm{Sin}^{-1}(\cos x)$
(A) has infinite number of discontinuities
(B) has finite number of discontinuities
(C) has no discontinuities
(D) has infinite number of points at which the function is not differentiable

**Solution:** Let $f(x) = \mathrm{Sin}^{-1}(\cos x)$. Since cos $x$ and $\mathrm{Sin}^{-1} x$ are, respectively, continuous on their respective domains, it follows that $f(x)$ is continuous for all real $x$. Also

$$f'(x) = \frac{1}{\sqrt{1-\cos^2 x}}(-\sin x) = \frac{-\sin x}{|\sin x|}$$

Therefore, $f'(x)$ does not exist at $x = n\pi$, where $n \in \mathbb{Z}$.

**Answers: (C), (D)**

**5.** Let $f(x) = \begin{cases} x\,\mathrm{Tan}^{-1}\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Then $f$ is
(A) continuous at $x = 0$
(B) not continuous at $x = 0$
(C) continuous and differentiable at $x = 0$
(D) not differentiable at $x = 0$

**Solution:** Put $\mathrm{Tan}^{-1}(1/x) = \theta, x \ne 0$. Therefore

$$\frac{1}{x} = \tan\theta$$

so that $\theta \to \pi/2$ as $x \to 0$. In this case

$$f(x) = \begin{cases} \theta\cot\theta, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

which gives

$$\theta\cot\theta \to 0 \text{ as } \theta \to \frac{\pi}{2}$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0 = f(0)$$

Hence $f$ is continuous at $x = 0$.

For $x \neq 0$,

$$f'(x) = \mathrm{Tan}^{-1}\frac{1}{x} + \frac{x}{1+x^2}\left(-\frac{1}{x^2}\right)$$
$$= \mathrm{Tan}^{-1}\frac{1}{x} - \frac{1}{x(1+x^2)}$$

Therefore neither $f'(0-0)$ nor $f'(0+0)$ exist because of the presence of $x$ in the denominator of the second term, whereas

$$\lim_{x\to 0-0} \mathrm{Tan}^{-1}\frac{1}{x} = -\infty$$

and
$$\lim_{x\to 0+0} \mathrm{Tan}^{-1}\frac{1}{x} = +\infty$$

Hence $f'(0)$ does not exist.

**Answers: (A), (D)**

**6.** Let $g(x) = \begin{cases} \dfrac{\sqrt{x+1}-1}{\sqrt{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

Then $g$ is

(A) continuous and differentiable at $x = 0$
(B) continuous and differentiable for all $x > 0$
(C) continuous for all $x > 0$
(D) not right differentiable at $x = 0$

**Solution:** For $x > 0$,

$$g(x) = \frac{x}{\sqrt{x}(\sqrt{x+1}+1)}$$
$$= \frac{\sqrt{x}}{\sqrt{x+1}+1}$$

Therefore, $g$ is continuous and differentiable for all $x > 0$ and

$$\lim_{x\to 0} g(x) = \frac{0}{1+1} = 0 = g(0)$$

So $g$ is continuous at $x = 0$. Now

$$g'(0+0) = \lim_{\substack{h\to 0\\ h>0}} \frac{\sqrt{h}}{(\sqrt{h+1}+1)h} = +\infty$$

Neither $g'(0-0)$ nor $g'(0+0)$ exist finitely. Hence $g$ is not differentiable at $x = 0$.

**Answers: (B), (C), (D)**

**7.** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then

(A) $f$ is continuous at $x = 0$
(B) $f$ is differentiable at $x = 0$
(C) $f''(0)$ exists
(D) $f'(x)$ is not continuous at $x = 0$

**Solution:** Since $\sin(1/x)$ is bounded and $x^2 \to 0$ as $x \to 0$, we have

$$\lim_{x\to 0} x^2 \sin(1/x) = 0 = f(0)$$

Thus $f$ is continuous at $x = 0$. Also,

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2 \sin\frac{1}{h}}{h}$$
$$= \lim_{h\to 0}\left(h \sin\frac{1}{h}\right)$$
$$= 0$$

Therefore $f'(0)$ exists and $f'(0) = 0$. Now

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Hence $\lim_{x\to 0} f'(x)$ does not exist because $\lim_{x\to 0} \cos(1/x)$ does not exist. Thus $f'(x)$ is not continuous at $x = 0$.

**Answers: (A), (B), (D)**

**8.** Suppose $f: \mathbb{R} \to \mathbb{R}$ is a function satisfying

(i) $f(x+y) = f(x)f(y)$ for all real $x, y$.
(ii) $f(x) = 1 + xg(x)$ where $\lim_{x\to 0} g(x) = 1$.

Then

(A) $f$ is differentiable for all $x$
(B) $f$ is twice differentiable for all $x$ and $f''(0) = 1$
(C) $f$ is differentiable for all $x$ and $f'(x) = f(x)$
(D) $f(0) = 1$

**Solution:** We have

$$f(0) = 1 + 0g(0) = 1$$

So (D) is true. Now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$
$$= \lim_{h\to 0} f(x)\frac{(f(h)-1)}{h}$$

$$= \lim_{h\to 0} f(x)\frac{(hg(h))}{h}$$
$$= f(x)\lim_{h\to 0} g(h)$$
$$= f(x)\times 1$$
$$= f(x)$$

Therefore $f'(x)$ exists for all real $x$ and $f'(x) = f(x)$. Hence (C) is true. Now

$$f''(x) = f'(x) = f(x)$$
$$\Rightarrow f''(0) = f(0) = 1$$

So (B) is true.

**Answers: (A), (B), (C), (D)**

**9.** $f:\mathbb{R}\to\mathbb{R}$ is a function satisfying the relation

$$f(x+y) = f(x)f(y)\ \forall\, x, y\in\mathbb{R}$$

and $\quad f(x)\neq 0\ \forall$ real $x$

Suppose the function is differentiable at $x = 0$ and $f'(0) = 2$. Then

(A) $f$ is differentiable for all $x\in\mathbb{R}$
(B) $f'(x) = 2f(x)\ \forall\, x\in\mathbb{R}$
(C) $2f'(x) = f(x)\ \forall\, x\in\mathbb{R}$
(D) $f(0) = 1$

**Solution:** We have

$$x = 0 = y \Rightarrow f(0) = (f(0))^2$$
$$\Rightarrow f(0) = 0 \quad\text{or}\quad f(0) = 1$$

Now $f(0) = 0$ contradicts the hypothesis that $f(x)\neq 0\ \forall\, x$. Therefore $f(0)\neq 0$. So $f(0) = 1$. Hence (D) is correct. Now

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$
$$= \lim_{h\to 0} f(x)\frac{(f(h)-1)}{h}$$
$$= \lim_{h\to 0} f(x)\frac{(f(h)-f(0))}{h}$$
$$= f(x)f'(0)$$
$$= 2f(x)$$

So (B) is true. Therefore $f$ is differentiable for all $x$ and $f'(x) = 2f(x)$.

**Answers: (A), (B), (D)**

**QUICK LOOK**

$$f(x) = e^{2x}$$

**10.** Let $f(x) = \begin{cases}\dfrac{x}{1+e^{1/x}} & \text{if } x\neq 0\\ 0 & \text{if } x = 0\end{cases}$

Then

(A) f is continuous at x = 0
(B) $f'(0-0) = 1$
(C) $f'(0+0) = 0$
(D) $f'(0) = 1$

**Solution:** We have $e^{1/x}\to +\infty$ as $x\to 0+$ and $e^{1/x}\to 0$ as $x\to 0-$. In any case, $\lim_{x\to 0} f(x) = 0 = f(0)$. Thus $f$ is continuous at $x = 0$. So (A) is true. Now

$$f'(0-0) = \lim_{\substack{h\to 0\\ h>0}}\frac{f(0-h)-f(0)}{-h}$$
$$= \lim_{h\to 0}\left(\frac{\dfrac{0-h}{1+e^{-1/h}}-0}{-h}\right)$$
$$= \lim_{h\to 0}\frac{1}{1+e^{-1/h}}$$
$$= \frac{1}{1+0} = 1$$

Thus $f'(0-0) = 1$ and so (B) is true. Again

$$f'(0+0) = \lim_{\substack{h\to 0\\ h>0}}\frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0}\left(\frac{\dfrac{0+h}{1+e^{1/h}}-0}{h}\right)$$
$$= \lim_{h\to 0}\frac{1}{1+e^{1/h}}$$
$$= 0 \quad \left(\because \lim_{\substack{h\to 0\\ h>0}} e^{1/h} = +\infty\right)$$

Therefore (C) is true.

**Answers: (A), (B), (C)**

**11.** Let $f:\mathbb{R}\to\mathbb{R}$ be any function and $g(x) = |f(x)|$. Then which of following is/are *not* true?

(A) If $f$ is onto, then $g$ is onto
(B) If $f$ is one-one, then $g$ is one-one
(C) If $f$ is continuous, then $g$ is continuous
(D) If $f$ is differentiable, then $g$ is differentiable

**Solution:** $f$ is onto $\Rightarrow f(\mathbb{R}) = \mathbb{R}$. But $g(\mathbb{R}) = \mathbb{R}^+\cup\{0\}$. Therefore (A) is not true. Now

$$|\pm f(x)| = |f(x)| \Rightarrow g \text{ is not one-one}$$

Thus (B) is not true.

Suppose $f$ is continuous at "$a$". Then to each $\varepsilon > 0,$ there corresponds $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for} \quad |x - a| < \delta \tag{2.24}$$

Now

$$\begin{aligned} |g(x) - g(a)| &= \big||f(x)| - |f(a)|\big| \\ &\le |f(x) - f(a)| < \varepsilon \text{ for } |x - a| < \delta \end{aligned}$$

[By Eq. (2.24)]

Therefore $g(x)$ is continuous at $a$. Thus (C) is true. Now, $f(x) = x$ is differentiable at $x = 0$, but $g(x) = |x|$ is not differentiable at $x = 0$. Hence (D) is not true.

**Answers: (A), (B), (D)**

**12.** Let $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$

Then

(A) $f$ is continuous at $x = 0$

(B) $f$ is differentiable at $x = 0$

(C) $f'(x)$ is continuous on $\mathbb{R}$

(D) $f''(x)$ exists for all $x \in \mathbb{R}$

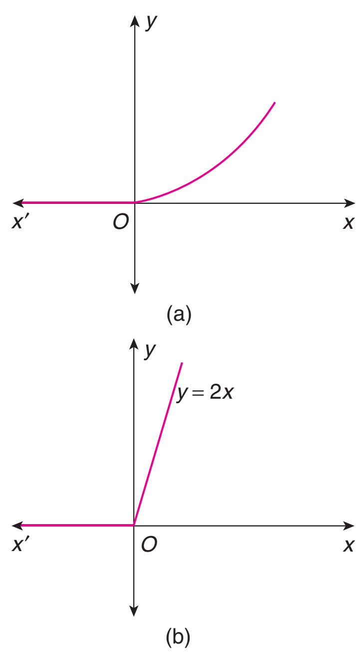



**FIGURE 2.8** Multiple correct choice type question 12.

**Solution:** See Fig. 2.8(a). The graph of $y = f(x)$ is the union of negative $x$-axis and the branch of the parabola $y = x^2$ in the first quadrant. Clearly

$$f(0 - 0) = 0 = f(0 + 0)$$

so that $f$ is continuous at $x = 0$. Hence (A) is true. Now

$$\lim_{\substack{h\to 0 \\ h>0}} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0}\left(\frac{0}{-h}\right) = 0$$

$$\Rightarrow f'(0 - 0) = 0$$

and

$$\lim_{\substack{h\to 0 \\ h>0}} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$\Rightarrow f'(0 + 0) = 0$$

Therefore

$$f'(0 - 0) = f'(0 + 0) = 0$$

So $f$ is differentiable at $x = 0$ and $f'(0) = 0$. This implies that (B) is true. Now [see Fig. 2.8(b)]

$$f'(x) = \begin{cases} 0 & \text{for } x < 0 \\ 2x & \text{for } x \ge 0 \end{cases}$$

Clearly $f'(x)$ is continuous at $x = 0$ and hence $f'$ is continuous on $\mathbb{R}$. Therefore (C) is true. Again

$$f''(x) = \begin{cases} 0 & \text{for } x < 0 \\ 2 & \text{for } x \ge 0 \end{cases}$$

So $f''$ is not continuous at $x = 0$ and hence $f''(0)$ does not exist. So (D) is false.

**Answers: (A), (B), (C)**

**QUICK LOOK**

1. In Fig. 2.8(a), even though at origin the point looks like a sharp point, but actually, at origin $x$-axis is a tangent to the curve $y = x^2$.
2. The only sharp point on the graph of $y = f'(x)$ is at the origin so that $f'$ is not differentiable at $x = 0$.

**13.** Let $f$ be a function defined for all real $x$ and let it satisfy the relation

$$f(x + y) = f(x) + f(y) + xy(x + y)$$

If $f'(0) = -1$, then

(A) $f$ is differentiable for all real $x$

(B) $f'$ is differentiable for all real $x$

(C) $f'(3) = 8$

(D) $f'$ satisfies the relation

$$f'(x) + f'\left(\frac{1}{x}\right) = -f'(x)f'\left(\frac{1}{x}\right) \; \forall\, x \ne 0$$

**Solution:** In the given relation, substituting $x = y = 0$, we have $f(0) = 0$. Also substituting $y = -x$, we have $f(-x) = -f(x)$ which reveals that $f$ is odd. Now,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\left(\frac{f(x)+f(h)+hx(x+h)-f(x)}{h}\right)$$
$$= \lim_{h\to 0}\left[\frac{f(h)}{h}+x(x+h)\right]$$
$$= \lim_{h\to 0}\left[\frac{f(h)-f(0)}{h}+x(x+h)\right] \quad [\because f(0)=0]$$
$$= f'(0)+x^2$$
$$= -1+x^2$$

Therefore $f$ is differentiable, $f'$ is differentiable, $f'(3)=8$. Also

$$f'(x)+f'\left(\frac{1}{x}\right) = (x^2-1)+\left(\frac{1}{x^2}-1\right) = x^2+\frac{1}{x^2}-2$$

and
$$f'(x)f'\left(\frac{1}{x}\right) = (x^2-1)\left(\frac{1}{x^2}-1\right)$$
$$= 1-x^2-\frac{1}{x^2}+1$$
$$= 2-x^2-\frac{1}{x^2}$$
$$= -\left(f(x)+f\left(\frac{1}{x}\right)\right)$$

Therefore (A), (B), (C) and (D) are all true.

**Answers: (A), (B), (C), (D)**

**14.** Let $y=f(x)$ be a function defined parametrically by

$$x = 2t-|t-1|$$
and
$$y = 2t^2+t|t|$$

Then $f$ is

(A) continuous at $x=-1$
(B) continuous at $x=2$
(C) differentiable at $x=1$
(D) not differentiable at $x=2$

**Solution:**

**Case I:** $t \le 0$. Then

$$x = 2t-(1-t) = 3t-1$$
and
$$y = 2t^2-t^2 = t^2 = \frac{1}{9}(x+1)^2$$

Also $t\le 0$ implies

$$x+1 = 3t \le 0 \Rightarrow x \le -1$$

**Case II:** $0 < t \le 1$. Therefore

$$x = 2t-(1-t) = 3t-1$$
and
$$y = 2t^2+t^2 = 3t^2 = \frac{1}{3}(x+1)^2$$

So
$$x = 3t-1 < 3-1 = 2 \quad (\because 0<t<1)$$

**Case III:** $t>1$. Therefore

$$x = 2t-(t-1) = t+1 > 2$$
and
$$y = 2t+t^2 = 3t^2 = 3(x-1)^2$$

Therefore

$$f(x) = \begin{cases} \frac{1}{9}(x+1)^2 & \text{for } x \le -1 \\ \frac{1}{3}(x+1)^2 & \text{for } -1<x\le 2 \\ 3(x-1)^2 & \text{for } x>2 \end{cases}$$

**(i)** Clearly

$$f(-1-0) = \frac{1}{3}(-1+1)^2 = 0$$
and
$$f(-1+0) = \frac{1}{3}(-1+1)^2 = 0$$

So $f$ is continuous at $x=-1$. Hence (A) is true.

**(ii)** We have

$$f(2-0) = \frac{1}{3}\times 9 = 3$$
and
$$f(2+0) = 3(2-1)^2 = 3$$

Therefore $f$ is continuous at $x=2$.

**(iii)** We have

$$f'(-1-0) = \frac{2}{9}(-1+1) = 0$$
and
$$f'(-1+0) = \frac{2}{3}(-1+1) = 0$$

Therefore $f$ is differentiable at $x=-1$. So (C) is true.

**(iv)** We have

$$f'(2-0) = \frac{2}{3}(2+1) = 2$$
and
$$f'(2+0) = 3(2-1)^2 = 3$$

Therefore $f$ is not differentiable at $x = 2$. So (D) is true.

**Answers: (A), (B), (C), (D)**

**15.** Let $f(x)$ be a polynomial with positive degree satisfying the relation

$$f(x)f(y) = f(x) + f(y) + f(xy) - 2$$

for all real $x$ and $y$. Suppose $f(4) = 65$. Then

(A) $f'(x)$ is a polynomial of degree two

(B) roots of the equation $f'(x) = 2x + 1$ are real

(C) $xf'(x) = 3[f(x) - 1]$

(D) $f'(-1) = 3$

**Solution:** We have

$$f(x)f(y) = f(x) + f(y) + f(xy) - 2$$

for all real $x$ and $y$. Substituting $x = y = 1$, we have

$$f(1)^2 - 3f(1) + 2 = 0$$

Therefore

$$f(1) = 1 \quad \text{or} \quad f(1) = 2$$

If $f(1) = 1$, then

$$f(x)f(1) = f(x) + f(1) + f(x) - 2$$
$$\Rightarrow f(x) = 1 \ \forall \text{ real } x$$

so that $f(x)$ will be a constant polynomial which is a contradiction. Therefore $f(1) \neq 1$ and hence $f(1) = 2$. Now replacing $y$ with $1/x$ in the given relation we get

$$f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) - 2$$
$$= f(x) + f\left(\frac{1}{x}\right) \quad [\because f(1) = 2]$$

We know that any polynomial satisfying the relation

$$f(x) + f(1/x) = f(x)f(1/x) \ \forall x \neq 0$$

must be a polynomial of the form $x^n + 1$ or $1 - x^n$ where $n$ is its degree (Problem 19, page 61, Vol. 1). Therefore

$$f(x) = x^n + 1 \quad \text{or} \quad 1 - x^n$$

But

$$f(4) = 65 \Rightarrow f(x) = x^3 + 1$$

Therefore

$$f'(x) = 3x^2$$

**Answers: (A), (B), (C), (D)**

**16.** The function

$$f(x) = \begin{cases} |x - 3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$$

is

(A) continuous at $x = 1$

(B) differentiable at $x = 1$

(C) continuous at $x = 3$

(D) differentiable at $x = 3$

**Solution:** We can write the given function as

$$f(x) = \begin{cases} \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & \text{for } x < 1 \\ 3 - x & \text{for } 1 \leq x \leq 3 \\ x - 3 & \text{for } x > 3 \end{cases}$$

Now

$$f(1-0) = \frac{1}{4} - \frac{3}{2} + \frac{13}{4}$$
$$= \frac{1 - 6 + 13}{4} = \frac{8}{4} = 2$$
$$f(1+0) = 3 - 1 = 2$$

Therefore $f$ is continuous at $x = 1$. Now

$$f'(x) = \begin{cases} \dfrac{x}{2} - \dfrac{3}{2} & \text{for } x < 1 \\ -1 & \text{for } 1 \leq x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

Therefore

$$f'(1-0) = \frac{1}{2} - \frac{3}{2} = -1$$

and $$f'(1+0) = -1$$

So $f$ is differentiable at $x = 1$. Actually, the line $y = 3 - x$ is a tangent to the parabola

$$y = \frac{x^2}{4} - \frac{3x}{2} + 13$$

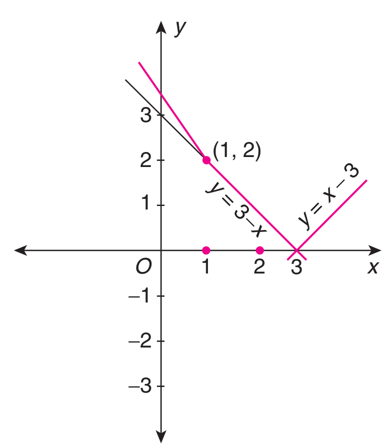



**FIGURE 2.9** Multiple correct choice type question 16.

at the point (1, 2) (this fact will be realized in the next chapter). See Fig. 2.9. Therefore $f$ is continuous and differentiable at $x = 1$ and so (A) and (B) are true. Now,

$$f(3-0) = f(3+0) = 0$$

This implies $f$ is continuous at $x = 3$ and so (C) is true. Again

$$f'(3-0) = -1 \quad \text{and} \quad f'(3+0) = 1$$

imply $f$ is *not* differentiable at $x = 3$.

**Answers: (A), (B), (C)**

**17.** Which of the following is/are correct?

(A) The function

$$f(x) = \frac{x}{1+x} + \frac{x}{(1+x)(1+2x)} + \frac{x}{(1+2x)(1+3x)} + \cdots \infty$$

is discontinuous at $x = 0$

(B) $f(x) = \operatorname{Sin}^{-1}\left(\dfrac{2x}{1+x^2}\right)$ is not differentiable at two points which are $\pm 1$

(C) $f(x) = x \log|x|$ is not differentiable for $0 < x < 1$

(D) $f(x) = \begin{cases} \sin\left(\dfrac{\pi x}{2}\right) & \text{for } x < 1 \\ |2x-3|[x] & \text{for } x \ge 1 \end{cases}$

where $[x]$ denotes part of $x$ continuous at $x = 1$

**Solution:**

(A) Let

$$s_n(x) = \frac{x}{1+x} + \frac{x}{(1+x)(1+2x)} + \cdots + \frac{x}{(1+nx)(1+(n+1)x)}$$

$$= \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{1+2x}\right) + \left(\frac{1}{1+2x} - \frac{1}{1+3x}\right) + \cdots$$

$$+ \left(\frac{1}{1+nx} - \frac{1}{1+(n+1)x}\right)$$

$$= 1 - \frac{1}{1+(n+1)x}$$

Now $n \to \infty$ implies

$$\frac{1}{1+(n+1)x} \to 0$$

Therefore

$$\lim_{n\to\infty} s_n(x) = 1$$

So

$$f(x) = \begin{cases} 1 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Therefore $f$ is discontinuous at $x = 0$. This implies (A) is true.

(B) We have

$$f(x) = \operatorname{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$= \begin{cases} 2\operatorname{Tan}^{-1} x & \text{for } -1 \le x \le 1 \\ \pi - 2\operatorname{Tan}^{-1} x & \text{for } x > 1 \\ -\pi - 2\operatorname{Tan}^{-1} x & \text{for } x < -1 \end{cases}$$

Therefore

$$f(-1-0) = -\pi - 2\left(-\frac{\pi}{4}\right)$$

$$= -\pi + \frac{\pi}{2}$$

$$= -\frac{\pi}{2}$$

$$f(-1+0) = 2\operatorname{Tan}^{-1}(-1)$$

$$= 2 \times \left(\frac{-\pi}{4}\right)$$

$$= -\frac{\pi}{2}$$

Therefore $f$ is continuous at $x = -1$. Similarly

$$f(1-0) = 2\operatorname{Tan}^{-1}(1) = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

and $\quad f(1+0) = \pi - 2\operatorname{Tan}^{-1}(1) = \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2}$

So $f$ is continuous at $x = 1$. Now

$$f'(x) = \begin{cases} \dfrac{-2}{1+x^2} & \text{for } x < -1 \\ \dfrac{2}{1+x^2} & \text{for } -1 \le x \le 1 \\ \dfrac{-2}{1+x^2} & \text{for } x > 1 \end{cases}$$

Therefore

$$f'(-1-0) = -1, \quad f'(-1+0) = 1$$

and $\quad f'(1-0) = 1, \quad f'(1+0) = -1$

So $f$ is not differentiable at $x = \pm 1$. Hence (B) is true.

(C) We have

$$f(x) = x\log x \quad \text{for} \quad x > 0$$

and $$f'(x) = \log x + 1$$

Therefore (C) is false.

(D) We have

$$f(x) = \begin{cases} \sin\dfrac{\pi x}{2} & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ 3-2x & \text{for } 1 < x \le \dfrac{3}{2} \end{cases}$$

Now

$$f(1-0) = \sin \pi/2 = 1$$

$$f(1+0) = 3-2 = 1$$

and $$f(1) = 1$$

Therefore $f$ is continuous at $x = 1$ and so (D) is true.

**Answers: (A), (B), (D)**

**18.** Let $f(x) = \begin{cases} 1-\sqrt{1-x^2} & \text{for } -1 \le x \le 1 \\ 1+\log\dfrac{1}{x} & \text{for } x > 1 \end{cases}$

Then

(A) $f$ is continuous at $x = 1$

(B) $f$ is not differentiable at $x = 1$

(C) $f$ is continuous and differentiable at $x = 1$

(D) $f'(x)$ exists for all $x \in (0, 1)$

**Solution:** We have

$$f(1-0) = 1-\sqrt{1-1} = 1$$

and $$f(1+0) = 1+\log 1 = 1$$

Therefore $f$ is continuous at $x = 1$. Hence (A) is true. Now

$$f'(x) = \frac{x}{\sqrt{1-x^2}}$$

for $0 \le x < 1$ and at $x = 1$, $f'(1-0)$ does not exist. Hence $f$ is not differentiable at $x = 1$. Therefore (B) is true. Also $f'(x)$ exists for all $x \in (0, 1)$ and so (D) is true.

**Answers: (A), (B), (D)**

**19.** Which of the following are true?

(A) $f(x) = \dfrac{\sin x}{x}$ is not defined at $x = 0$, but $\lim\limits_{x\to 0} f(x)$ exists

(B) $g(x) = \dfrac{|x|}{x}$ is not defined at $x = 0$ and $\lim\limits_{x\to 0} f(x)$ does not exist

(C) $h(x) = \begin{cases} \dfrac{1}{2}-x & \text{for } 0 < x < \dfrac{1}{2} \\ \left(\dfrac{1}{2}-x\right)^2 & \text{for } \dfrac{1}{2} \le x < 1 \end{cases}$

is continuous at $x = 1/2$, but not differentiable at $x = 1/2$

(D) $Q(x) = |x-1| + |x-2|$ is continuous for all $x \in \mathbb{R}$, but not differentiable at $x = 1, 2$

**Solution:** Clearly (A) is true because

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$$

Let

$$g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$g$ is not defined at $x = 0$ and also $g(0+0) = 1$ and $g(0-0) = -1$ which shows that $\lim\limits_{x\to 0} g(x)$ does not exist. So (B) is true. For the function given in (C)

$$h\left(\frac{1}{2}-0\right) = \frac{1}{2}-\frac{1}{2} = 0$$

and $$h\left(\frac{1}{2}+0\right) = \left(\frac{1}{2}-\frac{1}{2}\right)^2 = 0$$

so that $h$ is continuous at $x = 1/2$. Further

$$h'(x) = \begin{cases} -1 & \text{for } 0 < x < \dfrac{1}{2} \\ -2\left(\dfrac{1}{2}-x\right) & \text{for } \dfrac{1}{2} \le x < 1 \end{cases}$$

so that

$$h'\left(\frac{1}{2}-0\right) = 1$$

and $$h'\left(\frac{1}{2}+0\right) = 0$$

Thus (C) is true. For the function given in (D), we have

$$Q(x) = \begin{cases} 3-2x & \text{for } x < 1 \\ 1 & \text{for } 1 \le x \le 2 \\ 2x-3 & \text{for } x > 2 \end{cases}$$

Therefore $Q$ is continuous at $x = 1$ and 2, but $Q'(1-0) = -2$, $Q'(1+0) = 0$. Also $Q'(2-0) = 0$ and $Q'(2+0) = 2$. Hence $Q$ is not differentiable at $x = 1, 2$. This implies that (D) is true.

**Answers: (A), (B), (C), (D)**

**20.** Consider the function $f(x) = |(x+1)[x]|$ for $-1 \le x \le 2$ where $[x]$ is the integral part of $x$. Then $f$ is

(A) right continuous at $x = -1$

(B) not continuous at $x = 0$

(C) continuous at $x = 1$

(D) not left continuous at $x = 2$

**Solution:** We have

$$f(x) = \begin{cases} 0 & \text{at } x = -1 \\ x+1 & \text{for } -1 < x < 0 \\ 0 & \text{for } 0 \le x < 1 \\ 2 & \text{at } x = 1 \\ x+1 & \text{for } 1 < x < 2 \\ 6 & \text{at } x = 2 \end{cases}$$

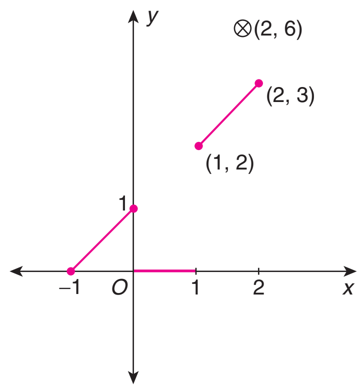



**FIGURE 2.10** Multiple correct choice type question 20.

See Fig. 2.10

**(i)** $f(-1+0) = -1+1 = 0 = f(-1)$. Therefore $f$ is right continuous at $x = -1$. So (A) is true.

**(ii)** $f(0-0) = 1$ and $f(0+0) = 0$. Therefore $f$ is not continuous at $x = 0$ and so (B) is true.

**(iii)** $f(1-0) = 0$, $f(1+0) = 2$. Therefore $f$ is not continuous at $x = 1$. So (C) is false.

**(iv)** $f(2-0) = 3 \ne f(2)$. Hence $f$ is not left continuous at $x = 2$ and so (D) is true.

**Answers: (A), (B), (D)**

## *Matrix-Match Type Questions*

**1.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $f(x) = \begin{cases} 3^x & \text{for } -1 \le x \le 1 \\ 4-x & \text{for } 1 < x < 4 \end{cases}$ is | (p) continuous at $x = 1$ |
| | (q) differentiable at $x = 1$ |
| (B) If $g(x) = \text{Min}\{x, x^3\}$ then $g'(x) = 1$ for all $x$ greater than | (r) not differentiable at $x = 1$ |
| (C) The function $h(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & x > 1 \end{cases}$ is | (s) 1 |
| (D) The function $Q(x) = \text{Max}\{x, x^3\}$ is | (t) continuous for all $x$ in their respective domains |

**Solution:**

(A) We have

$$f(1-0) = 3^1 = 3$$

and

$$f(1+0) = 4 - 1 = 3$$

Therefore $f$ is continuous at $x = 1$. Also

$$f'(x) = \begin{cases} 3^x \log 3, & -1 \le x \le 1 \\ -1, & 1 < x < 4 \end{cases}$$

Hence

$$f'(1-0) = 3 \log 3$$

and

$$f'(1+0) = -1$$

Thus $f$ is not differentiable at $x = 1$.

**Answer: (A) → (p), (r), (t)**

(B) See Fig. 2.11(a). We have

$$g(x) = \text{Min}\{x, x^3\}$$

$$= \begin{cases} x^3 & \text{for } x \le -1 \\ x & \text{for } -1 < x \le 0 \\ x^3 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$$

Clearly $g'(x) = 1$ for $x > 1$.

**Answer: (B) → (s)**

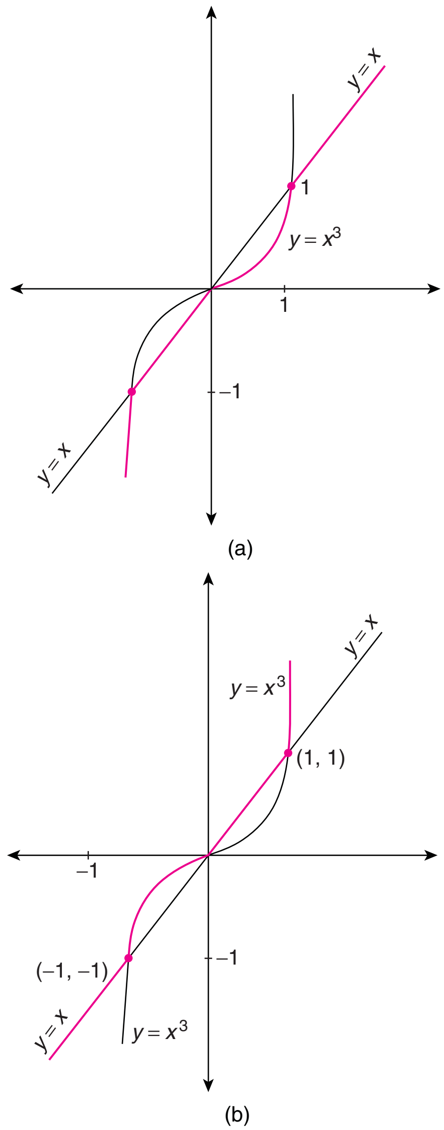



**FIGURE 2.11** Matrix-match type question 1: (a) Graph of $g(x) = \text{Min}\{x, x^3\}$; (b) graph of $Q(x) = \text{Max}\{x, x^3\}$.

(C) We have

$$h(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 2 - x & \text{for } x > 1 \end{cases}$$

$$h(1-0) = 1 \quad \text{and} \quad h(1+0) = 2 - 1 = 1$$

Therefore $h$ is continuous at $x = 1$. Further $h'(1-0) = 1$ whereas $h'(1+0) = -1$. So $h$ is not differentiable at $x = 1$.

**Answer: (C) → (p), (r), (t)**

(D) We have [see Fig. 2.11(b)]

$$Q(x) = \begin{cases} x & \text{for } \quad x \le -1 \\ x^3 & \text{for } -1 < x \le 0 \\ x & \text{for } \quad 0 < x \le 1 \\ x^3 & \text{for } \quad x > 1 \end{cases}$$

Now

$$\begin{aligned} Q(-1-0) &= -1 \quad \text{and} \quad Q(-1+0) = -1 \\ Q(0-0) &= 0 \quad \text{and} \quad Q(0+0) = 0 \\ Q(1-0) &= 1 \quad \text{and} \quad Q(1+0) = 1 \\ Q'(1-0) &= 1 \quad \text{and} \quad Q'(1+0) = 3 \end{aligned}$$

**Answer: (D) → (p), (r), (t)**

**2.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) If $f(x)$ is a polynomial satisfying the relation $$f(x) + f(2x) = 5x^2 - 18$$ then $f'(1)$ equals | (p) 0 |
| (B) Let $y = x^3 - 2$ and $x = 3z^2 + 5$. Then the value of $dy/dz$ at $z = 0$ is | (q) 1 |
| (C) If $f(x) = x^2 - 4x$, then $\lvert f(x)\rvert$ is not differentiable at $x$ equals | (r) 2 |
| (D) Suppose that $f$ and $g$ are differentiable functions such that $f'(x) = -g(x)$ and $g'(x) = -f(x)$. Let $h(x) = (f(x))^2 - (g(x))^2$. Then $h'(2)$ equals | (s) 4 |
| | (t) 3 |

**Solution:**

(A) Let $f(x) = ax^2 + bx + c$ (Why?). By hypothesis,

$$(ax^2 + bx + c) + (4ax^2 + 2bx + c) = 5x^2 - 18$$

Solving we get

$$\begin{aligned} 5a &= 5 \Rightarrow a = 1 \\ 3b &= 0 \Rightarrow b = 0 \\ 2c &= -18 \Rightarrow c = -9 \end{aligned}$$

Therefore

$$f(x) = x^2 - 9$$

So

$$f'(1) = 2$$

**Answer: (A) → (r)**

(B) We have

$$y = x^3 - 2, \quad x = 3z^2 + 5$$

Differentiating we get

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{dx}{dz} = 6z$$

So

$$\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = 3x^2(6z) = 3(3z^2+5)^2 6z$$

Therefore

$$\left(\frac{dy}{dz}\right)_{z=0} = 0$$

**Answer: (B) → (p)**

(C) Figure 2.12(a) is the graph of $y = f(x)$ and Fig. 2.12(b) is the graph of $y = |f(x)|$. Now

$$|f(x)| = |x(x-4)| = \begin{cases} x^2 - 4x & \text{for } x \le 0 \\ 4x - x^2 & \text{for } 0 < x \le 4 \\ x^2 - 4x & \text{for } x > 4 \end{cases}$$

Therefore

$$f'(0-0) = -4 \quad \text{and} \quad f'(0+0) = 4$$

Also

$$f'(4-0) = 4 - 2(4) = -8$$

and

$$f'(4+0) = 2(4) - 4 = 8$$

Therefore $f$ is not differentiable at $x = 0, 4$. Observe that the portion of the graph of $y = f(x)$ below the $x$-axis is reflected through $x$-axis for $y = |f(x)|$.

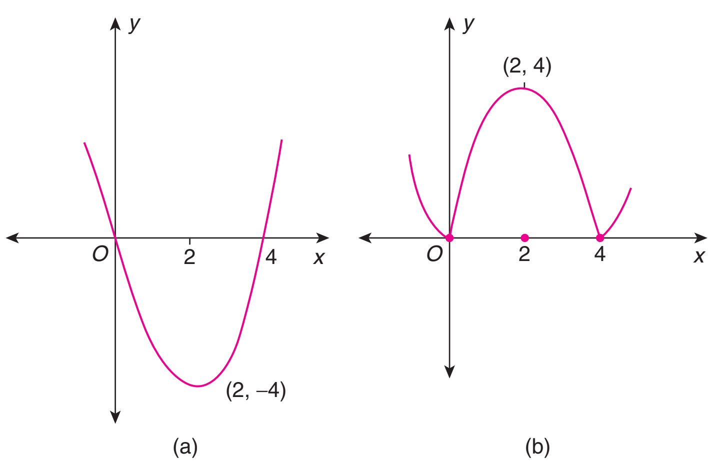



**FIGURE 2.12** Matrix-match type question 2: (a) Graph of $y = f(x)$; (b) graph of $y = |f(x)|$.

**Answer: (C) → (p), (s)**

(D) We have

$$h'(x) = 2f(x)f'(x) - 2g(x)g'(x)$$
$$= 2(-g'(x))(-g(x)) - 2g(x)g'(x) = 0$$

Therefore

$$h'(2) = 0$$

**Answer: (D) → (p)**

**3.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) The function $\sin(\pi[x])$ where $[x]$ is the integer part of $x$ is | (p) continuous everywhere |
| (B) $x\|x\|$ is | (q) differentiable everywhere |
| (C) The function $\|x-2\|+\|x+2\|$ is | (r) not differentiable at exactly one point |
| (D) The function $f(x) = \text{Min}\{1, x^2, x^3\}$ | (s) not differentiable exactly at two points |

**Solution:**

(A) Since $[x]$ is an integer, $\sin(\pi[x]) = 0$ for all $x \in \mathbb{R}$. Hence the function is continuous and differentiable for all real $x$.

**Answer: (A) → (p), (q)**

(B) We have

$$x|x| = \begin{cases} -x^2 & \text{for } x \le 0 \\ x^2 & \text{for } x > 0 \end{cases}$$

Hence it is continuous and differentiable for all real $x$ [see Fig. 2.13(a)].

**Answer: (B) → (p), (q)**

(C) We have

$$f(x) = \begin{cases} -2x & \text{for } x \le -2 \\ 4 & \text{for } -2 < x < 2 \\ 2x & \text{for } x \ge 2 \end{cases}$$

**Answer: (C) → (p), (s)**

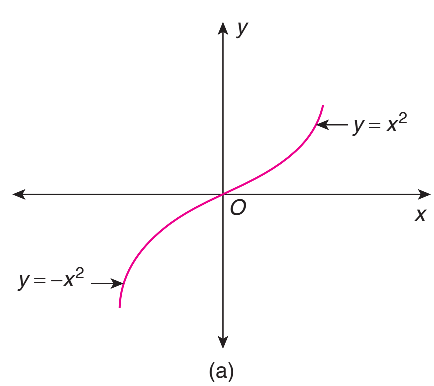
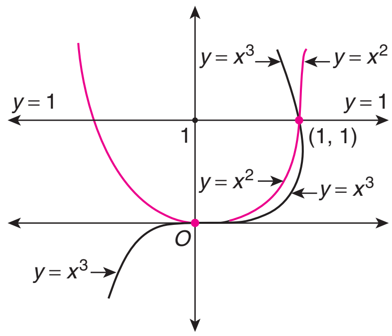



**FIGURE 2.13** Matrix-match type question 3.




(b)

**FIGURE 2.13** (*Continued*)

(D) We have

$$f(x) = \text{Min}\{1, x^2, x^3\}$$

See Fig. 2.13(b). Now

$$f(x) = \begin{cases} x^3 & \text{for } x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

So $f$ is continuous for all $x$ and is differentiable for all $x$ except at $x = 1$.

**Answer: (D) → (p), (r)**

**4.** Functions are given in Column I and their derived functions are given in Column II. Match them.

| *Column I* | *Column II* |
|---|---|
| (A) $y = x\sqrt{a^2 - x^2} + a^2 \text{Sin}^{-1}\dfrac{x}{a}$ | (p) $\text{Sin}^{-1} x + \dfrac{x}{\sqrt{1-x^2}}$ |
| (B) $y = \text{Tan}^{-1}\left(\dfrac{4\sin x}{3+5\cos x}\right)$ | (q) $\dfrac{x^2}{1-x^4}$ |
| (C) $y = \log\left(\dfrac{1+x}{1-x}\right)^{1/4} - \dfrac{1}{2}\text{Tan}^{-1} x$ | (r) $\dfrac{4}{5+3\cos x}$ |
| (D) $y = x\,\text{Sin}^{-1} x$ | (s) $2\sqrt{a^2 - x^2}$ |

**Solution:**

(A) We have

$$y = x\sqrt{a^2 - x^2} + a^2 \text{Sin}^{-1}\frac{x}{a}$$

Put $x = a\sin\theta$ so that

$$\frac{dx}{d\theta} = a\cos\theta = a\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{a^2 - x^2}$$

Now

$$y = a^2 \sin\theta\cos\theta + a^2\theta$$

Therefore

$$\begin{aligned}\frac{dy}{d\theta} &= \left(\frac{a^2}{2}\cos 2\theta\right)2 + a^2 \\ &= a^2(1 + \cos 2\theta) \\ &= 2a^2\cos^2\theta \\ &= 2a^2\left(1 - \frac{x^2}{a^2}\right) \\ &= 2(a^2 - x^2)\end{aligned}$$

Now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2(a^2 - x^2)}{\sqrt{a^2 - x^2}} = 2\sqrt{a^2 - x^2}$$

**Answer: (A) → (s)**

(B) We have

$$y = \text{Tan}^{-1}\left(\frac{4\sin x}{3 + 5\cos x}\right)$$

Put

$$u = \frac{4\sin x}{3 + 5\cos x}$$

Therefore

$$\begin{aligned}\frac{du}{dx} &= \frac{4[\cos x(3 + 5\cos x) - \sin x(-5\sin x)]}{(3 + 5\cos x)^2} \\ &= \frac{4(3\cos x + 5)}{(3 + 5\cos x)^2} \qquad (2.25)\end{aligned}$$

Also

$$y = \text{Tan}^{-1} u$$

Differentiating w.r.t. $u$, we get

$$\begin{aligned}\frac{dy}{du} &= \frac{1}{1 + u^2} \\ &= \frac{1}{1 + \dfrac{16\sin^2 x}{(3 + 5\cos x)^2}} \\ &= \frac{(3 + 5\cos x)^2}{9 + 30\cos x + 25\cos^2 x + 16\sin^2 x} \\ &= \frac{(3 + 5\cos x)^2}{9 + 30\cos x + 16 + 9\cos^2 x} \\ &= \frac{(3 + 5\cos x)^2}{(5 + 3\cos x)^2}\end{aligned}$$

Therefore from Eqs. (2.24) and (2.25), we get

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{4}{5+3\cos x}$$

**Answer: (B) → (r)**

(C) We have

$$y=\log\left(\frac{1+x}{1-x}\right)^{1/4}-\frac{1}{2}\mathrm{Tan}^{-1}x$$
$$=\frac{1}{4}\log\left(\frac{1+x}{1-x}\right)-\frac{1}{2}\mathrm{Tan}^{-1}x \quad \text{for } x>-1$$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx}=\frac{1}{4}\left(\frac{1-x}{1+x}\right)\left[\frac{(1-x)+(1+x)}{(1-x)^2}\right]-\frac{1}{2}\left(\frac{1}{1+x^2}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1-x^2}-\frac{1}{1+x^2}\right)$$
$$=\frac{x^2}{1-x^4}$$

**Answer: (C) → (q)**

(D) We have

$$y=x\,\mathrm{Sin}^{-1}x$$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx}=\mathrm{Sin}^{-1}x+\frac{x}{\sqrt{1-x^2}}$$

**Answer: (D) → (p)**

**5.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) Let $y=f(u)=\dfrac{1}{u^2+u-2}$ where $u=\dfrac{1}{x-1}$. Then $y$ has a removable discontinuity at $x$ equals | (p) 3 |
| | (q) 0 |
| (B) If $f(x)=\dfrac{1}{1-x}$, then the number of removable discontinuities of $y=f(f(f(x)))$ is | (r) 1 |
| | (s) 4 |
| (C) If $x+y=e^{x-y}$, then at (1/2, 1/2) the value of $dy/dx$ is | |
| (D) If $f(x+y+z)=f(x)f(y)f(z)$ for all $x, y$, and $z$, $f(2)=4$, and $f'(0)=1$, and $f(0)>0$. Then $f'(2)$ is | (t) 2 |

**Solution:**

(A) We have

$$y=f(u)=\frac{1}{u^2+u-2}=\frac{1}{(u+2)(u-1)}$$

Now $u=1$ and $u=-2$ are points of discontinuity:

$$u=1\Rightarrow\frac{1}{x-1}=1\Rightarrow x=2$$
$$u=-2\Rightarrow\frac{1}{x-1}=-2\Rightarrow -2x+2=1\Rightarrow x=\frac{1}{2}$$

Originally $x=1$ is a point of discontinuity for $u=1/(x-1)$. Therefore $x=-2, 1/2$ and 1 are points of discontinuity for the composite function $y=f(u)$ where $u=1/(x-1)$. Now $x\to 1\Rightarrow u\to\infty$ and hence $\lim_{u\to\infty}f(u)=0$. By defining

$$y=f(u)=\begin{cases}\dfrac{1}{u^2+u-2} & \text{where } u=\dfrac{1}{x-1}, x\neq 1\\ 0 & \text{when } x=1\end{cases}$$

we get that $y$ is continuous at $x=1$.

**Answer: (A) → (r)**

(B) We have $y=f(f(f(x)))$ where

$$f(x)=\frac{1}{1-x}$$
$$=f\left(f\left(\frac{1}{1-x}\right)\right)$$
$$=f\left(\frac{1}{1-\frac{1}{1-x}}\right)$$
$$=f\left(\frac{1-x}{-x}\right)\quad \text{when } x\neq 0$$
$$=\frac{1}{1-\frac{1-x}{-x}}$$
$$=x$$

$x=1$ is a discontinuity of $f(x)$ and $x=0$ is a point of discontinuity of $f(f(x))$. Hence $x=0$ and 1 are discontinuities of $f(f(f(x)))$. Now by defining

$$Q(x)=\begin{cases}f(f(f(x))) & \text{if } x\neq 0 \text{ and } 1\\ 0 & \text{if } x=0\\ 1 & \text{if } x=1\end{cases}$$

we get that $Q(x)$ is continuous at $x=0, 1$.

**Answer: (B) → (t)**

(C) We have

$$x+y=e^{x-y}$$

Therefore

$$\log(x+y)=x-y$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1+\frac{dy}{dx}}{x+y}=1-\frac{dy}{dx}$$

$$\Rightarrow 1+\frac{dy}{dx}=x+y-(x+y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(x+y+1)=x+y-1$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y-1}{x+y+1}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(1/2,\,1/2)}=0$$

**Answer: (C) → (q)**

(D) We have

$$f(0)=(f(0))^3$$

$$\Rightarrow f(0)=1 \quad [\because f(0)>0]$$

Therefore

$$\begin{aligned} f(x+2) &= f(x+2+0) \\ &= f(x)f(2)f(0) \\ &= 4f(x) \end{aligned}$$

Now

$$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{f(2)f(h)f(0)-f(2)}{h} \\ &= \lim_{h\to 0}4\left(\frac{f(h)-f(0)}{h}\right) \qquad [\because f(0)=0,\ f(2)=4] \\ &= 4f'(0) \\ &= 4\ [\because\ f'(0)=1] \end{aligned}$$

**Answer: (D) → (s)**

## *Comprehension-Type Questions*

**1. Passage:** Let $f$ be a real-valued function defined on a closed interval $[a, b]$ and $c \in (a, b)$. Then

(a) $f$ is continuous at $c$ if and only if $\lim\limits_{\substack{h\to 0\\ h>0}} f(c-h)$ and $\lim\limits_{\substack{h\to 0\\ h>0}} f(c+h)$ exist, are equal, and in turn are equal to $f(c)$.

(b) $f$ is differentiable at $c$ if and only if

$$\lim_{\substack{h\to 0\\ h>0}}\frac{f(c-h)-f(c)}{-h} \quad \text{and} \quad \lim_{\substack{h\to 0\\ h>0}}\frac{f(c+h)-f(c)}{h} \text{ exist}$$

and are equal or

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

($h$ may be positive or negative) exists finitely.
Answer the following questions.

**(i)** If $f(x)=\begin{cases} ax^2+b, b\neq 0 & \text{for } x\le 1 \\ bx^2+ax+c & \text{for } x>1 \end{cases}$

is continuous and differentiable at $x=1$, then

(A) $c=0, a=2b$
(B) $a=b$, and $c$ is real number
(C) $a=b, c=0$
(D) $a=b$ and $c\neq 0$

**(ii)** Suppose $f(x+y)=f(x)f(y)$ for all real numbers $x, y$ and $f(3)=3,\ f'(0)=11,$ then $f'(3)$ is equal to

(A) 22 (B) 28
(C) 44 (D) 33

**(iii)** Suppose $f(x+y)=f(x)+f(y)$ and $f(x)=x^2 g(x)$ where $g(x)$ is continuous, then $f'(x)$ equals

(A) 0 (B) $g(x)$
(C) $g(0)$ (D) $g(x)+g(0)$

**Solution:**

**(i)** By hypothesis

$$\lim_{\substack{h\to 0\\ h>0}} f(1-h)=\lim_{\substack{h\to 0\\ h>0}} f(1+h)$$

Therefore

$$a+b=b+a+c$$

$$\Rightarrow c=0$$

Again $f$ is differentiable at $x=1$. Therefore

$$\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{a(1-h)^2+b-(a+b)}{-h}$$

$$=\lim_{h\to 0}\frac{b(1+h)^2+a(1+h)-(a+b)}{h} \quad (\because c=0)$$

$$\Rightarrow \lim_{h\to 0}\frac{-2ah+ah^2}{-h}=\lim_{h\to 0}\frac{2bh+bh^2+ah}{h}$$

$$\Rightarrow 2a=2b+a$$

$$\Rightarrow a=2b$$

Therefore $a=2b$ and $c=0$.

**Answer: (A)**

**(ii)** We have

$$x=y=0$$

$$\Rightarrow f(0)=(f(0))^2$$

$$\Rightarrow f(0)=0 \quad \text{or} \quad 1$$

If $f(0)=0$, then

$$\begin{aligned} f(x) &= f(x+0)\\ &= f(x)f(0)\\ &= f(x)\times 0\\ &= 0\ \forall\ x \end{aligned}$$

which cannot be, because $f(3)=3$. Hence $f(0)=1$. Now

$$\begin{aligned} f'(3) &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\ &= \lim_{h\to 0}\frac{f(3)f(h)-f(3)}{h}\\ &= 3\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right) \quad [\because f(3)=3, f(0)=1]\\ &= 3f'(0)\\ &= 3\times 11\\ &= 33 \end{aligned}$$

**Answer: (D)**

**(iii)** We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{f(h)}{h}\\ &= \lim_{h\to 0}\frac{h^2g(h)}{h}\\ &= \lim_{h\to 0}hg(h)\\ &= 0\times g(0) \quad (\because g \text{ is continous})\\ &= 0 \end{aligned}$$

**Answer: (A)**

**2. Passage:** Consider the function

$$g(x)=\begin{cases} x+2, & x<0\\ -2-x^2, & 0\le x<1\\ x, & x\ge 1 \end{cases}$$

Answer the following questions.

**(i)** The number of points at which $g$ is discontinuous is

(A) 0 (B) 1 (C) 2 (D) 3

**(ii)** The number of points where $|g|$ is discontinuous is

(A) 0 (B) 1 (C) 2 (D) infinite

**(iii)** The number of points at which $|g|$ is not differentiable is

(A) 2 (B) (C) 1 (D) 3

**Solution:**

We have

$$g(0-0)=2 \quad \text{and} \quad g(0+0)=-2$$

This implies that $g$ is discontinuous at $x=0$.

Also,

$$g(1-0)=-3 \quad \text{and} \quad g(1+0)=1$$

This implies $g$ is discontinuous at $x=1$. Thus $g$ is discontinuous at two points. Now

$$|g(x)|=\begin{cases} |x+2| & \text{for } x<0\\ x^2+2 & \text{for } 0\le x<1\\ x & \text{for } x\ge 1 \end{cases}$$

$$=\begin{cases} -x-2 & \text{for } x<-2\\ x+2 & \text{for } -2\le x<0\\ x^2+2 & \text{for } 0\le x<1\\ x & \text{for } x\ge 1 \end{cases}$$

From the graph of $y=|g(x)|$ (Fig. 2.14), one can see that the answers for (ii) and (iii) are (B) and (D), respectively.

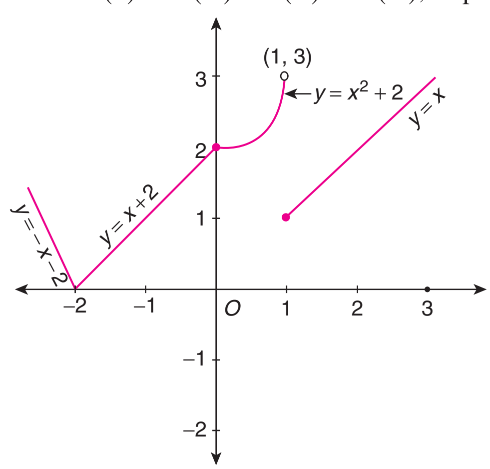



**FIGURE 2.14** Comprehension-type question 2.

**Answer: (i) → (C), (ii) → (B), (iii) → (D)**

**3. Passage:** For $x>0$, let

$$f(x)=\lim_{n\to\infty}\frac{\log(2+x)+x^{2^n}\sin x}{1+x^{2^n}}$$

Answer the following questions:

**(i)** $\lim_{x\to 0+0} f(x)$ is equal to

(A) 0 (B) log1

(C) log2 (D) does not exist

**(ii)** At $x=1, f$

(A) is continuous

(B) not continuous

(C) both continuous and differentiable

(D) continuous but not differentiable

**(iii)** In $[0, \pi/2]$, the number of points at which $f$ vanishes is

(A) 0 (B) 1

(C) 2 (D) 3

**Solution:**

**Case I:** $0<x<1$. In this case

$$f(x)=\log(2+x) \quad (\because \lim_{n\to\infty} x^{2^n} = 0)$$

**Case II:** At $x=1$, $f(x)=\frac{1}{2}(\log 3+\sin 1)$

**Case III:** $x>1$. Then

$$\lim_{n\to\infty}\frac{\log(2+x)+x^{2^n}\sin x}{1+x^{2^n}} = \lim_{n\to\infty}\frac{\left[\dfrac{\log(2+x)}{x^{2^n}}+\sin x\right]}{\dfrac{1}{x^{2^n}}+1}$$

$$=\sin x \left(\because \lim_{n\to\infty}\frac{1}{x^{2^n}}=0\right)$$

Therefore

$$f(x)=\begin{cases}\log(2+x) & \text{for } 0<x<1\\ \dfrac{1}{2}(\log 3+\sin 1) & \text{at } x=1\\ \sin x & \text{for } x>1\end{cases}$$

Hence

**(i)** $\lim_{x\to 0+0} = \log 2$

**(ii)** $\lim_{x\to 1-0} f(x)=\log 3$ and $\lim_{x\to 1+0} f(x)=\sin 1$

$\Rightarrow f$ is not continuous at $x=1$

**(iii)** Clearly $f(x)\neq 0$ for $x\in[0, \pi/2]$.

**Answers: (i) → (C), (ii) → (B), (iii) → (A)**

## Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation for Statement I.

(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.

(C) Statement I is true and Statement II is false.

(D) Statement I is false and Statement II is true.

**1. Statement I:** For

$$0<x<1, \frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\cdots\infty=\frac{x}{1-x}$$

**Statement II:** $\dfrac{x^{2^{n-1}}}{1-x^{2^n}}=\dfrac{1}{1-x^{2n-1}}-\dfrac{1}{1-x^{2^n}}$ for $n=1, 2, 3, \ldots$.

**Solution:** Let

$$u_1=\frac{x}{1-x^2}=\frac{1}{1-x}-\frac{1}{1-x^2}$$

$$u_2=\frac{x^2}{1-x^4}=\frac{1}{1-x^2}-\frac{1}{1-x^4}$$

$$u_3=\frac{x^4}{1-x^8}=\frac{1}{1-x^4}-\frac{1}{1-x^8}$$

and so on. Therefore

$$s_n=u_1+u_2+u_3+\cdots+u_n = \frac{1}{1-x}-\frac{1}{1-x^{2^n}}$$

Hence

$$\lim_{n\to\infty} s_n=\frac{1}{1-x}-\frac{1}{1-0}=\frac{x}{1-x}$$

**Answer: (A)**

**2. Statement I:** If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}$, $x\neq 0$ is to be continuous at $x=0$, then $f(0)$ is to be defined as $-1/8$.

**Statement II:** $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)=1$

**Solution:** We have

$$f(x) = \frac{2-\sqrt{x+4}}{\sin 2x}$$
$$= \frac{4-(x+4)}{\sin 2x(2+\sqrt{x+4})}$$
$$= \frac{-x}{\sin 2x(2+\sqrt{x+4})}$$
$$= \frac{-1}{\left(\dfrac{\sin 2x}{2x}\right)\cdot 2(2+\sqrt{x+4})}$$

Therefore

$$\lim_{x\to 0} f(x) = \frac{-1}{1\cdot 2(2+\sqrt{0+4})} = \frac{-1}{8}$$

**Answer: (A)**

3. **Statement I:** The function $\sin(|x|)$ is not differentiable at $x=0$.

**Statement II:** If $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$, then $g \circ f$ is differentiable at $c$.

**Solution:** Statement II is a theorem (see Theorem 2.3)

$$\sin|x| = \begin{cases} -\sin x & \text{if } x<0 \\ \sin x & \text{if } x \ge 0 \end{cases}$$

Therefore

$$f'(0-0) = -\cos(0) = -1$$

and $$f'(0+0) = \cos 0 = 1$$

So at $x=0$, $\sin|x|$ is not differentiable.

**Answer: (D)**

4. **Statement I:** The function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x=0$, but the derivative is not continuous at $x=0$.

**Statement II:** If a function $f(x)$ is differentiable at $x=a$, its derivative is continuous at $x=a$.

**Solution:** We have

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Now

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at $x=0$, because $\cos(1/x)$ has no limit as $x\to 0$. Statement I is true. This example shows that Statement II is false.

**Answer: (C)**

**QUICK LOOK**

$f'(0)=0$ because

$$\lim_{h\to 0}\frac{f(0-h)-f(0)}{h} = \lim_{h\to 0}\left(h\sin\frac{1}{h}\right) = 0$$

for

$$-h \le h\sin\frac{1}{h} \le h \quad \text{where } h>0$$

and $$h \le h\sin\frac{1}{h} \le -h \quad \text{where } h<0$$

so that by squeezing theorem

$$\lim_{h\to 0} h\sin\frac{1}{h} = 0$$

5. **Statement I:** Let $f(x)=\cos x$ and $g(x)=\sin x$ for $0 \le x \le \pi/2$. Then $f(x)=g(x)$ for at least one point in $(0, \pi/2)$.

**Statement II:** If $f$ and $g$ are continuous on $[a, b]$ and $f(a) \ge g(a)$ and $f(b) \le g(b)$, then $f(x_0)=g(x_0)$ for at least one $x_0$ in $[a, b]$.

**Solution:** Statement I is true because

$$f\left(\frac{\pi}{4}\right) = g\left(\frac{\pi}{4}\right)$$

We prove that Statement II is also true and it is a correct explanation of Statement I. If either $f(a)=g(a)$ or $f(b)=g(b)$ we are through. Suppose $f(a)>g(a)$ and $f(b)<g(b)$. Define

$$Q(x) = f(x)-g(x) \quad \text{for } x \in [a, b]$$

Since $f$ and $g$ are continuous, $Q(x)$ is also continuous on $[a, b]$. Also

$$Q(a)Q(b) = [f(a)-g(a)][f(b)-g(b)] < 0$$

Therefore $Q(x)$ must vanish at some point $x_0 \in (a, b)$ (see Corollary 1.10). So

$$f(x_0) = g(x_0) \text{ of some } x_0 \in (a,b)$$

Now, Statement I is true if we take $f(x)=\cos x$, $g(x)=\sin x$ and $[a, b]=[0, \pi/2]$.

**Answer: (A)**

## *Integer Answer Type Questions*

**1.** If $f(x) = \text{Sin}^{-1}(3x - 4x^3)$, then $f'(0)$ equals ________.

**Solution:** Put $x = \sin\theta$ so that

$$\frac{dx}{d\theta} = \cos\theta = \sqrt{1-x^2}$$

Therefore

$$f(x) = \text{Sin}^{-1}(\sin 3\theta) = 3\theta$$

Differentiating we get

$$f'(x) = 3 \cdot \frac{d\theta}{dx} = \frac{3}{\sqrt{1-x^2}}$$

So

$$f'(0) = 3$$

**Answer: 3**

**2.** Let $f:[0,1] \to [0,1]$ be continuous function. Then the number of fixed points of $f$ is at least ________.

**Solution:** If $f(0) = 0$ or $f(1) = 1$, the purpose is served. Assume that $f(0) \neq 0$ and $f(1) \neq 1$. Let

$$h(x) = f(x) - x \quad \text{for } x \in [0,1]$$

Therefore

$$h(0) = f(0) > 0$$

and

$$h(1) = f(1) - 1 < 0$$

So

$$h(0)h(1) = f(0)(f(1) - 1) < 0$$

Therefore $h(x) = 0$ for at least one $x \in (0,1)$. Thus $f(x) = x$ for at least one $x \in (0,1)$.

**Answer: 1**

**3.** $f(x)$ is a real-valued function defined on $\mathbb{R}$ such that

$$|f(x) - f(y)| < \frac{1}{3}|x - y|$$

for all $x$, $y$. Then the number of fixed points of $f$ is ________.

**Solution:** See Integer Answer Type Question 1 (Chapter 1, Worked-Out Problems) and take $\alpha = 1/3$.

**Answer: 1**

**4.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. If $f(a) = 2, f'(a) = 1, g(a) = -1$ and $g'(a) = 2$, then

$$\lim_{x \to a}\left(\frac{g(x)f(a) - g(a)f(x)}{x - a}\right) =$$

**Solution:** We have

$$\frac{g(x)f(a) - g(a)f(x)}{x-a} = \frac{(g(x) - g(a))f(a) - g(a)(f(x) - f(a))}{x-a}$$

Therefore

$$\lim_{x \to a}\left(\frac{g(x)f(a) - g(a)f(x)}{x-a}\right) = g'(a)f(a) - g(a)f'(a)$$

$$= 2(2) - (-1)(1) = 5$$

**Answer: 5**

**5.** Suppose $f$ is twice differentiable function satisfying $f''(x) = -f(x)$. Define two functions $g(x)$ and $h(x)$ by $g(x) = f'(x)$ and $h(x) = (f(x))^2 + (g(x))^2$. If $h(5) = 5$, then $h(10)$ is equal to ________.

**Solution:** We have

$$h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$$
$$= -2f''(x)f'(x) + 2f'(x)f''(x) = 0 \ \forall\, x$$

Therefore $h(x)$ is a constant function. Hence

$$h(5) = 5 \Rightarrow h(10) = 5$$

**Answer: 5**

**6.** If $f:\mathbb{R} \to \mathbb{R}$ is a function satisfying $f(-x) = f(x)$ for all real $x$ and is differentiable at $x = 0$, then $f'(0)$ equals ________.

**Solution:** We have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{h} \quad (\because f(h) = f(-h))$$

$$= -\lim_{h \to 0}\left(\frac{f(-h) - f(0)}{-h}\right)$$

$$= -f'(0-0) = -f'(0)$$

Therefore $f'(0) = 0$.

**Answer: 0**

**7.** The only integer at which $f(x) = [x^2] - [x]^2$ where $[\cdot]$ denotes the integer part is continuous is ________.

**Solution:** Let $n$ be a positive integer. For small positive values of $h$,

$$[n-h]^2 = (n-1)^2;\ n^2 - 1 < (n-h)^2 < n^2$$

$$\Rightarrow [(n-h)^2] = n^2 - 1$$

Therefore

$$f(n-0) = \lim_{\substack{h\to 0\\ h>0}} ([(n-h)^2] - [n-h]^2)$$

$$= n^2 - 1 - (n-1)^2$$

$$= 2n - 2$$

Again for small positive values of $h$,

$$n^2 < (n+h)^2 \Rightarrow [(n+h)^2] = n^2$$

Now

$$f(n+0) = \lim_{\substack{h\to 0\\ h>0}} ([(n+h)^2] - [n+h]^2)$$

$$= n^2 - n^2 = 0$$

$$f(n-0) = f(n+0)$$

$$\Leftrightarrow 2n - 2 = 0$$

$$\Leftrightarrow n = 1$$

Therefore the only integer at which $f$ is continuous is 1.

**Answer: 1**

**8.** Let $f(x) = \begin{cases} \dfrac{(4^x-1)^3}{\sin\dfrac{x}{4}\log\left(1+\dfrac{x^2}{3}\right)}, & x \neq 0 \\ \lambda, & x = 0 \end{cases}$

If $f$ is to be continuous at $x = 0$, then $\lambda = 12(\log a)^b$ where $a + b$ is equal to ________.

**Solution:** We have

$$\frac{(4^x-1)^3}{\sin\dfrac{x}{4}\log\left(1+\dfrac{x^2}{3}\right)} = \frac{\left(\dfrac{4^x-1}{x}\right)^3}{\left(\dfrac{\sin\dfrac{x}{4}}{\dfrac{x}{4}}\right)\dfrac{\log\left(1+\dfrac{x^2}{3}\right)}{\dfrac{x^2}{3}}} \times 12$$

Therefore

$$\lim_{x\to 0} f(x) = \frac{(\log 4)^3 \times 12}{1\times 1}$$

This gives $a = 4$ and $b = 3$, so that $a + b = 7$.

**Answer: 7**

**9.** Let $f(x) = \begin{cases} x^p\cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then, the minimum value of $p$ for which $f$ is differentiable at $x = 0$, where $p$ is a positive integer, is ________.

**Solution:** If $p = 1$, then $f(x) = x\cos(1/x)$ is continuous at $x = 0$, because

$$-h \le h\cos\frac{1}{h} \le h \quad \text{when } h > 0$$

and

$$h \le h\cos\frac{1}{h} \le -h \quad \text{when } h < 0$$

so that

$$|h| \le \left|h\cosh\frac{1}{h}\right| \le |h|$$

and hence

$$\lim_{h\to 0}\left(h\cos\frac{1}{h}\right) = 0 = f(0)$$

Therefore $f$ is continuous at $x = 0$. Now

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\left(\cos\frac{1}{h}\right) \quad \text{does not exist}$$

This implies that $f$ is not differentiable at $x = 0$ when $p = 1$. Now, let $p = 2$. Therefore

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\cos\dfrac{1}{h}}{h}$$

$$= \lim_{h\to 0}\left(h\cos\frac{1}{h}\right) = 0$$

So $f'(0)$ exists when $p = 2$. Hence minimum value of $p$ is 2.

**Answer: 2**

**10.** Let

$$f(x) = \begin{cases} \cos\dfrac{\pi}{4}(1-x+[x]) & \text{if } x \ge 0 \text{ and } [x] \text{ is even} \\ \sin\dfrac{\pi}{4}(x-[x]) & \text{if } x \ge 0 \text{ and } [x] \text{ is odd} \end{cases}$$

where $[\cdot]$ denotes the integral part of $x$. In the interval $(0, 5)$, if $m$ is the number of points of discontinuity of $f$ and $n$ is the number of points where $f$ is differentiable, then $m + n$ equals ________.

**Solution:** We have

$$f(x)=\begin{cases}\cos\frac{\pi}{4}(1-x) & \text{if } 0\le x<1\\ \sin\frac{\pi}{4}(x-1) & \text{if } 1\le x<2\\ \cos\frac{\pi}{4}(3-x) & \text{if } 2\le x<3\\ \sin\frac{\pi}{4}(x-3) & \text{if } 3\le x<4\\ \cos\frac{\pi}{4}(5-x) & 4\le x<5\end{cases}$$

Clearly $f$ is discontinuous at $x = 1$ and 3 and $f$ is continuous at $x = 2, 4$. Also

$$f'(2-0)=\frac{\pi}{4}\cos\frac{\pi}{4}(2-1)=\frac{\pi}{4\sqrt{2}}$$

and $$f'(2+0)=-\left(-\frac{\pi}{4}\right)\sin\frac{\pi}{4}(3-2)=\frac{\pi}{4\sqrt{2}}$$

Therefore $f$ is differentiable at $x = 2$. Similarly $f$ is differentiable at $x = 4$. Hence $m = 2$ and $n = 2$.

**Answer: 4**

**QUICK LOOK**

In $(0, \infty)$, $f$ is continuous and differentiable at even integers and not continuous and not differentiable at odd numbers.

**11.** Let $f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x^2 & \text{if } x \text{ is irrational}\end{cases}$

Then, the number of points at which $f$ is differentiable is ________.

**Solution:** Let $a \ne 0$. Now

$$\frac{f(x)-f(a)}{x-a}=\begin{cases}\dfrac{0-a^2}{x-a} & \text{if } x \text{ is rational and } a \text{ is irrational}\\ \dfrac{x^2-0}{x-a} & \text{if } x \text{ irrational and } a \text{ is rational}\end{cases}$$

$$=\begin{cases}\dfrac{-a^2}{x-a}\\ \dfrac{x^2}{x-a}\end{cases}$$

Therefore

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\pm\infty$$

When $a = 0$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to 0}\frac{f(x)-0}{x-0}$$
$$=\lim_{x\to 0} 0 \quad\text{or}\quad \lim_{x\to 0} x$$

In any case

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$$

Therefore $f'(0)$ exists and is equal to 0.

**Answer: 1**

***Note:*** The result is same if

$$f(x)=\begin{cases}x^2 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

**12.** In the open interval $(-2, 2)$, the number of points at which $f(x)=[x^2-1]$ ($[\cdot]$ denotes integral part) is ________.

**Solution:** First observe that $-1 \le x^2-1<3$ for $-2<x<2$ and also $x^2-1$ assumes integer values at $x=-\sqrt{3}, -\sqrt{2}, -1, 0, 1, \sqrt{2}$ and $\sqrt{3}$. Now, we write explicit form of $f(x)$.

$$f(x)=\begin{cases}2 & \text{if } -2<x\le-\sqrt{3} & \text{because } 2<x^2-1<3\\ 1 & \text{if } -\sqrt{3}<x\le-\sqrt{2} & \text{because } 1<x^2-1\le 2\\ 0 & \text{if } -\sqrt{2}<x\le-1 & \text{because } 0<x^2-1\le 1\\ -1 & \text{if } -1<x\le 0 & \text{because } -1<x^2-1\le 0\\ -1 & \text{if } 0<x<1 & \text{because } -1<x^2-1<0\\ 0 & \text{if } 1\le x<\sqrt{2} & \text{because } 0<x^2-1<1\\ 1 & \text{if } \sqrt{2}\le x<\sqrt{3} & \text{because } 1\le x^2-1<2\\ 2 & \text{if } \sqrt{3}\le x<2 & \text{because } 2\le x^2-1<3\end{cases}$$

From the above, we can see that $f$ is continuous at $x = 0$ and at other 6 points $f$ is discontinuous.

***Advice:*** The reader can draw the graph and check it.

**Answer: 6**

## SUMMARY

**2.1** ***Derivative:*** Suppose $f:[a,b]\to IR$ is a function and $a<c<b$. If $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists, then we say that *f* is differentiable at "*c*" and this limit is denoted by $f'(c)$ derivative or differential coefficient at *c*. If we write $y=f(x)$, then $f'(c)$ is also denoted by $(dy/dx)_{x=c}$. If *f* is differentiable at each point of (*a*, *b*) then we say that *f* is differentiable in (*a*, *b*) and $f'(x)$ or $(dy/dx)$ is called derivative or derived function of $f(x)$.

**QUICK LOOK**

$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists and is equivalent to

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

**2.2** ***Left and Right Derivatives:*** *f* is defined on [*a*, *b*] and $a<c<b$.

**(i)** If $\lim_{\substack{h\to 0\\ h>0}}\frac{f(c-h)-f(c)}{-h}$ exists, then this limit is called the *left derivative* of *f* at *c* and is denoted by $f'(c-0)$.

**(ii)** If $\lim_{\substack{h\to 0\\ h>0}}\frac{f(c+h)-f(c)}{h}$ exists, then this limit is called the *right derivative* of *f* at *c* and is denoted by $f'(c+0)$.

**QUICK LOOK**

1. $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ (without mentioning $h>0$, i.e., $h>0$ or $h<0$) then this limit is $f'(c)$.
2. $f'(c)$ exists $\Leftrightarrow f'(c-0)$ and $f'(c+0)$ exist and are equal.

**2.3** If $f:[a,b]\to\mathbb{R}$ is differentiable at $c\in(a,b)$, then *f* is continuous at *c*.

***Note:***

1. *f* is differentiable at the end points *a* and *b* means $f'(a+0)$ and $f'(b-0)$ exist.
2. The converse of Theorem 2.3 is not true; for example, take $f(x)=|x|$ at $c=0$.
3. The function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}\cos(3^n x)$$

is continuous for all real *x*, but not differentiable at any $x\in\mathbb{R}$.

**2.4** Suppose *f* and *g* are differentiable at *c*, and $\lambda$ and $\mu$ be any two real numbers. Then

**(i)** $\lambda f+\mu g$ is differentiable at *c* and

$$(\lambda f+\mu g)'(c)=\lambda f'(c)+\mu g'(c)$$

**(ii)** *fg* is differentiable at *c* and

$$(fg)'(c)=f'(c)g(c)+g'(c)f(c)$$

**(iii)** If $g(c)\neq 0$, then *f*/*g* is differentiable at *c* and

$$\left(\frac{f}{g}\right)'(c)=\frac{f'(c)g(c)-g'(c)f(c)}{(g(c))^2}$$

**2.5** ***Differentiation of a Function or Chain Rule:*** If *f* is differentiable at *c* and *g* is differentiable at $f(c)$, then $g\circ f$ is differentiable at *c* and

$$(g\circ f)'(c)=g'(f(c))f'(c)$$

***Note:*** If $y=g(u)$ where $u=f(x)$, then

$$\left(\frac{dy}{dx}\right)_{x=c}=\left(\frac{dg}{du}\right)_{u=f(c)}\left(\frac{du}{dx}\right)_{x=c}$$

**2.6** ***Carathéodory Theorem:*** Suppose $f:[a,b]\to\mathbb{R}$ is a function and $c\in(a,b)$. Then *f* is differentiable at *c* if and only if

**(i)** *f* is continuous at *c*.

**(ii)** There exists a function $g:[a,b]\to\mathbb{R}$ such that *g* is continuous at *c* and $f(x)-f(c)=(x-c)g(x)$ for all $x\in[a,b]$. In this case $f'(c)=g(c)$.

### Example

If $f(x)=x^3, x\in\mathbb{R}$ and *a* is a real number then the function $g(x)$ in the above theorem is $g(x)=x^2+ax+a^2$.

**2.7** ***Differentiability of 1/f:*** Suppose $f:[a,b]\to\mathbb{R}$ and $f(x)\neq 0\ \forall\, x\in[a,b]$. Let $c\in(a,b)$ and suppose

$f$ is differentiable at $c$. Then the function $g = 1/f$ is differentiable at $c$ and

$$g'(c) = \frac{-f'(c)}{(f(c))^2}$$

**2.8** Suppose $f:[a,b] \to [c,d]$ is a bijection and $g = f^{-1}$. If $f$ is differentiable at $x_0 \in (a,b)$ and $f'(x_0) \neq 0$, and $g$ is continuous at $f(x_0)$, then $g$ is differentiable at $f(x_0)$ and

$$g'(f(x_0)) = \frac{1}{f'(x_0)}$$

***Note:*** If $y = f(x)$ is a bijection and is differentiable, then $x = f^{-1}(y)$ is differentiable and

$$(f^{-1})'(y) = \frac{1}{dy/dx}$$

which we denote by $dx/dy$ so that

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = f'(x)(f^{-1})'(y) = 1$$

It is in this context that we write $dx/dy$ for $1/(dy/dx)$.

**2.9** ***Increasing and Decreasing Functions:*** Let $A$ be a subset of $\mathbb{R}$ and $f: A \to \mathbb{R}$ be a function. Then

**(i)** $f$ is called increasing or monotonically increasing if $f(x) \leq f(y)$ whenever $x < y$ and $x, y \in A$. If $f(x) < f(y)$ whenever $x < y$, then $f$ is called strictly increasing

**(ii)** $f$ is called decreasing or monotonically decreasing if $f(x) \geq f(y)$ whenever $x < y$. If $f(x) > f(y)$ whenever $x < y$, then $f$ is called strictly decreasing.

**2.10** Suppose $f:[a,b] \to \mathbb{R}$ is continues and strictly increasing. Write $\alpha = f(a)$ and $\beta = f(b)$ so that $f^{-1}$ exists, is continuous and strictly increasing on $[\alpha, \beta]$. If $a < c < b$, $f$ is differentiable at $c$ and $f'(c) \neq 0$, then $f^{-1}$ is differentiable at $f(c)$ and

$$(f^{-1})'(c) = \frac{1}{f'(c)}$$

***Note:***

**(i)** The above result is valid if $f$ is continuous and strictly decreasing.

**(ii)** The above result is *not valid* if $f'(c) = 0$. For example, take $f(x) = x^3$ and $c = 0$, so that $f^{-1}(x) = \sqrt[3]{x}$ , In this case $f$ is not differentiable at $c = 0$.

**2.11** ***List of derivatives of some standard functions:***

| *S. No.* | *Function* | *Derivative* |
|---|---|---|
| 1. | $x^\alpha$ ($x > 0$ and $\alpha$ is real) | $\alpha x^{\alpha-1}$ |
| 2. | $\log\lvert x\rvert \ (x \neq 0)$ | $\frac{1}{x}$ |
| 3. | $a^x (a > 0, x \text{ is real})$ | $a^x \log_e a$ |
| 4. | $e^x$ | $e^x$ |
| 5. | $\sin x$ | $\cos x$ |
| 6. | $\cos x$ | $-\sin x$ |
| 7. | $\tan x \left(x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right)$ | $\sec^2 x$ |
| 8. | $\cot x \ (x \neq n\pi, n \in \mathbb{Z})$ | $-\mathrm{cosec}^2 x$ |
| 9. | $\sec x \left(x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right)$ | $\sec x \tan x$ |
| 10. | $\mathrm{cosec}\, x \ (x \neq n\pi, n \in \mathbb{Z})$ | $-\mathrm{cosec}\, x \cot x$ |
| 11. | $\mathrm{Sin}^{-1} x \ (-1 < x < 1)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| 12. | $\mathrm{Cos}^{-1} x \ (-1 < x < 1)$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| 13. | $\mathrm{Tan}^{-1} x \ (x \in \mathbb{R})$ | $\frac{1}{1+x^2}$ |

*(Continued)*

| S. No. | Function | Derivative |
|---|---|---|
| 14. | $\text{Cot}^{-1} x \ (x \in \mathbb{R})$ | $\dfrac{-1}{1+x^2}$ |
| 15. | $\text{Sec}^{-1} x (x < -1 \quad \text{or} \quad x > 1)$ | $\begin{cases} \dfrac{-1}{x\sqrt{x^2-1}}, & x < -1 \\ \dfrac{1}{x\sqrt{x^2-1}}, & x > 1 \\ \dfrac{1}{\lvert x \rvert \sqrt{x^2-1}}, & \lvert x \rvert > 1 \end{cases}$ |
| 16. | $\text{Cosec}^{-1} x$ | $\dfrac{-1}{\lvert x \rvert \sqrt{x^2-1}}, \ \lvert x \rvert > 1$ |
| 17. | $\sinh x \left( = \dfrac{e^x - e^{-x}}{2} \right)$ | $\cosh x \left( = \dfrac{e^x + e^{-x}}{2} \right)$ |
| 18. | $\cosh x$ | $\sinh x$ |
| 19. | $\tanh x$ | $\text{sech}^2 x$ |
| 20. | $\coth x$ | $-\text{cosech}^2 x$ |
| 21. | $\text{sech}\, x$ | $-\text{sech}\, x \tanh x$ |
| 22. | $\text{cosech}\, x$ | $-\text{cosech}\, x \coth x$ |
| 23. | $\text{Sinh}^{-1} x$ | $\dfrac{1}{\sqrt{1+x^2}}$ |
| 24. | $\text{Cosh}^{-1} x$ | $\dfrac{1}{\sqrt{x^2-1}}$ |
| 25. | $\text{Tanh}^{-1} x$ | $\dfrac{1}{1-x^2}$ |
| 26. | $\text{Coth}^{-1} x$ | $\dfrac{1}{1-x^2}$ |
| 27. | $\text{Sech}^{-1} x \ (0 < x < 1)$ | $\dfrac{-1}{x\sqrt{1-x^2}}$ |
| 28. | $\text{Cosech}^{-1} x$ | $\dfrac{1}{\lvert x \rvert \sqrt{1+x^2}}$ |

## Special Methods of Differentiation

**2.12** ***Substitution Method or Chain Rule or Composite Function:*** If $y$ is a function of $u$ and $u$ is function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

That is, $y = g(f(x)) = g(u)$ where $u = f(x)$. This implies

$$\frac{dy}{dx} = g'(u)u'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$$

**2.13** ***Logarithmic Differentiation***

**(i)** If $h(x) = (f(x))^{g(x)}$ where $f(x)$ is a positive function then

$$\log h(x) = g(x)\log f(x)$$

Hence differentiating w.r.t. $x$, we have

$$\frac{1}{h(x)}h'(x) = g'(x)\log f(x) + \frac{g(x)}{f(x)}\cdot f'(x)$$

so that we can obtain $h'(x)$.

**(ii)** If $h(x) = f_1(x)f_2(x)\cdots f_n(x)$ where $f_j(x) > 0$, then take logarithms on both sides and differentiate both sides w.r.t. $x$.

**2.14** ***Parametric Differentiation:*** Suppose $x$ and $y$ are functions of a parameter $t$, say $x = f(t)$ and $y = g(t)$ in any interval. Further assume that $f$ is invertible and $f, g, f^{-1}$ are differentiable in their relevant intervals. Then to find $dy/dx$, we proceed as follows:

$$y = g(t) = g(f^{-1}(x)) = (g \circ f^{-1})(x)$$

so that

$$\frac{dy}{dx} = g'(f^{-1}(x))(f^{-1})'(x) = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{g'(t)}{f'(t)}$$

**2.15** ***Implicit Function:*** Suppose $y$ is a function of $x$ which is not explicitly in terms of $x$, but $x$ and $y$ are connected through a relation $F(x, y) = 0$. Then by differentiating the equation $F(x, y) = 0$ w.r.t. $x$, we can find $dy/dx$.

**2.16** ***Differentiation of a Function w.r.t. Another Function:*** Suppose $u = f(x)$ and $v = g(x)$. Then

$$\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx} = \frac{f'(x)}{g'(x)}$$

## EXERCISES

### Single Correct Choice Type Questions

**1.** In the interval $[0, 3]$, the number of points at which the function $[x^2] \sin \pi x$ ($[\cdot]$ is the usual integral part) is discontinuous are

(A) 4 (B) 5
(C) 6 (D) 8

**2.** If $f(x) = \dfrac{e^x - 1}{e^x + 1}$, then $f'(0)$ equals

(A) 1 (B) 2
(C) 1/4 (D) 1/2

**3.** If $f(x) = \mathrm{Tan}^{-1}\left(\dfrac{4\sin x}{3 + 5\cos x}\right)$, then $f'(\pi/2)$ is

(A) 4/5 (B) 2/5
(C) 3/5 (D) 1

**4.** If $y = \cos(x + y)$, then $dy/dx =$

(A) $\dfrac{\sin(x+y)}{1+\sin(x+y)}$ (B) $\dfrac{-\sin(x+y)}{1+\sin(x+y)}$

(C) $\dfrac{\cos(x+y)}{1+\cos(x+y)}$ (D) $\dfrac{-\cos(x+y)}{1+\cos(x+y)}$

**5.** If $y = \log\left(\dfrac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2}\right) + 2\mathrm{Tan}^{-1}\left(\dfrac{x\sqrt{2}}{1 - x^2}\right)$, then at $x = 0$, $dy/dx =$

(A) 4 (B) $4\sqrt{2}$
(C) $2\sqrt{2}$ (D) 2

**6.** If $\cos(xy) = x$, then $dy/dx =$

(A) $-\left(\dfrac{1 + y\sin(xy)}{x\sin(xy)}\right)$ (B) $\dfrac{1 + y\sin(xy)}{x\sin xy}$

(C) $-\left(\dfrac{1 + y\sin(xy)}{\sin(xy)}\right)$ (D) $-\left(\dfrac{1 + \sin(xy)}{x\sin(xy)}\right)$

**7.** If $x = a\cos^3 t$, $y = b\sin^3 t$, then at the point $\left(\dfrac{a}{2\sqrt{2}}, \dfrac{a}{2\sqrt{2}}\right)$, $dy/dx =$

(A) $b/a$ (B) $-b/a$
(C) $a/b$ (D) $-a/b$

**8.** If a function $f(x)$ is continuous, $f(1) > 0$ and satisfies the relation $f(x) < f(y)$ whenever $x < y$ for all positive $x$ and $y$, then $f(x)$ has

(A) exactly one root (B) exactly two roots
(C) more than two roots (D) no roots

**9.** Let $f(x) = \text{Max}\{x^3, x^4\}$ for $-1 < x < 1$. Then $f(x)$

(A) is differentiable at $x = 0$
(B) is not differentiable at $x = 0$
(C) has exactly one point of discontinuity
(D) has exactly two points of non-differentiability

**10.** Consider the following two statements:
**P:** If $f(x)$ is differential at $x = a$ and $g(x)$ is not differentiable at $x = a$, then $f(x) + g(x)$ is differentiable at $x = a$.
**Q:** If $f(x)$ and $g(x)$ are not differentiable at $x = a$, then $f(x) + g(x)$ is not differentiable at $x = a$.

(A) both $P$ at $Q$ are true
(B) both $P$ at $Q$ are false
(C) $P$ is true and $Q$ is false
(D) $P$ is false and $Q$ is true

**11.** If $y^2 = 2xy$, then $y'''$ is

(A) 0 (B) $\dfrac{x+y}{x-y}$
(C) $x/y$ (D) $y/x$

**12.** If $x = a\cos 2t$, $y = b\sin^2 t$, then at $(-a, b)$, $d^2y/dx^2$ is equal to

(A) 1 (B) 1/2
(C) 0 (D) $1/\sqrt{2}$

**13.** If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ $(x, y > -1)$, then $\left(\dfrac{dy}{dx}\right)_{x=1}$ is

(A) 1/4 (B) −1/4
(C) 1/2 (D) −1/2

**14.** If $x^2 + y^2 = 1$, then

(A) $yy'' - 2(y')^2 + 1 = 0$ (B) $yy'' + (y')^2 + 1 = 0$
(C) $yy'' - (y')^2 - 1 = 0$ (D) $yy'' + 2(y')^2 + 1 = 0$

**15.** Let $f(x)$ and $g(x)$ be twice differentiable functions such that $f'(x)g'(x) = c$. If $h(x) = f(x)g(x)$, then $h''/h$ is equal to

(A) $\dfrac{f''}{f} + \dfrac{g''}{g} + \dfrac{2c}{fg}$ (B) $\dfrac{f''}{f} + \dfrac{g''}{g} - 2c$
(C) $\dfrac{f''}{f} + \dfrac{g''}{g} + \dfrac{c}{fg}$ (D) $\dfrac{f''}{f} + \dfrac{g''}{g} - c$

**16.** If $f(x) = \log_{x^2}(\log x)$, then $f'(e)$ is equal to

(A) 0 (B) 1
(C) $1/e$ (D) $1/2e$

**17.** Let $y = e^x(a\cos x + b\sin x)$, then $y'' + my' + ny = 0$ where $m + n$ value is

(A) 1 (B) 4
(C) −2 (D) 0

**18.** If $y = \text{Tan}^{-1}\left(\dfrac{2^x}{1+2^{x+1}}\right)$, then $\left(\dfrac{dy}{dx}\right)_{x=0}$ is

(A) 1 (B) 2
(C) log 2 (D) $\dfrac{-1}{10}\log 2$

**19.** If $x = 4t$ and $y = 2t^2$, then $\dfrac{d^2y}{dx^2}$ at $x = 1/2$ is

(A) 1/2 (B) 1/4
(C) 2 (D) 4

**20.** If $x = \text{Cos}^{-1}\dfrac{1}{\sqrt{1+t^2}}$ and $y = \text{Sin}^{-1}\dfrac{t}{\sqrt{1+t^2}}$, then $\dfrac{dy}{dx}$ is equal to

(A) 1 (B) $\text{Tan}^{-1}t$
(C) 0 (D) $\pi/2$

**21.** If $y = \text{Tan}^{-1}\left(\dfrac{x-\sqrt{x}}{1+x^{3/2}}\right)$, then $\left(\dfrac{dy}{dx}\right)_{x=1}$ is

(A) −1/4 (B) −1/2
(C) 1/4 (D) 1/2

**22.** If

$$f(x) = \text{Tan}^{-1}\frac{1}{x^2+x+1} + \text{Tan}^{-1}\frac{1}{x^2+3x+3} + \text{Tan}^{-1}\frac{1}{x^2+5x+7} + \cdots$$

upto 10 terms, then $f'(1)$ is equal to

(A) 30/61 (B) −30/61
(C) 40/61 (D) −40/61

**23.** If $y = \dfrac{1-x^4}{1+x^4}$, then $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}$ is equal to

(A) 1 (B) $xy$
(C) does not exist (D) $\dfrac{x+y}{xy}$

**24.** The number of points at which the function $f(x) = \text{Min}\{|x|, |x-2|, 2-|x-1|\}$ is not differential is

(A) 2 (B) 3
(C) 4 (D) 5

**25.** If $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x)+2f(y)}{3}$ for all $x, y \in \mathbb{R}$ and $f'(0) = 1$, then $f'(1)$ is equal to

(A) 1 (B) 0
(C) 2 (D) −1

## Multiple Correct Choice Type Questions

**1.** Which of the following are correct?

(A) If $x^m y^n = (x+y)^{m+n}$, then $x\frac{dy}{dx} = y$

(B) If $y = x\tan(x/2)$, then $\frac{dy}{dx} = \frac{x+\sin x}{1+\cos x}$

(C) If $f(x)\begin{cases} \frac{5e^{1/x}+2}{3-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

then $f(x)$ is differential at $x = 0$

(D) If

$$f(x) = \begin{cases} -2, & -3 \le x \le 0 \\ x-2, & 0 < x \le 3 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$

then the number of points at which $g$ is not differential for $-3 < x < 3$ is two.

**2.** Which of the following functions are continuous at the specified points?

(A) $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x \neq 0 \end{cases}$ at $x = 0$

(B) $f(x) = \lim\limits_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$ at $x = 1$

(C) $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & x \neq 0 \\ 1, & x \neq 0 \end{cases}$ at $x = 0$

(D) If $f(x) = \lim\limits_{n\to\infty}\left(\frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right)$ then $f$ is continuous at $x = 1$

**3.** Which of the following are true?

(A) If $u = \tan(\text{Sin}^{-1}x)$, then $\frac{du}{dx}$ is $\frac{\sec^2(\text{Sin}^{-1}x)}{\sqrt{1-x^2}}$

(B) The derivative of $x^{\text{Cos}^{-1}x}$ w.r.t. $x$ is

$x^{\text{Cos}^{-1}x}\left[\frac{\text{Cos}^{-1}x}{x} - \frac{\log x}{\sqrt{1-x^2}}\right]$ where $0 < x \le 1$

(C) If $y = \text{Tan}^{-1}\sqrt{1+x}$, then $\frac{dy}{dx}$ at $x = 0$ is $\frac{1}{4}$

(D) The function $[x] + \sqrt{x-[x]}$ where $[x]$ is the integral part of $x$ is continuous at all integers which are positive.

**4.** Let $f(x) = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \frac{1}{2^3}\tan\frac{x}{2^3} + \cdots\infty$.

Then

(A) $f(x) = \frac{1}{x} - \cot x$ (B) $f(x) = x - \tan x$

(C) $f(1) = 1 + \tan\left(\frac{\pi}{2}+1\right)$ (D) $f'(1) = -1 + \text{cosec}^2 1$

**5.** Which of the following are not true?

(A) Every continuous function defined on an open interval is bounded.

(B) If the function is differentiable at a point then its derivative is continuous at that point.

(C) The function $f(x) = \sqrt[3]{x^2}$ is continuous for all real $x$ and is not differentiable at $x = 0$.

(D) If a function is differentiable at infinite number of points, then it should not have infinite number of points of discontinuity.

**6.** Let $y = \text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$. Then $\frac{dy}{dx}$ is equal to

(A) $\frac{2}{1+x^2}$ for all $x$

(B) $\frac{-2}{1+x^2}$ for $|x| > 1$

(C) does not exist at $x = \pm 1$

(D) $\frac{2}{1+x^2}$ for $-1 < x < 1$

**7.** If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all real $x$, then

(A) $f'(1) = -5$ (B) $f''(2) = 2$

(C) $f'''(3) = 6$ (D) $f(0) = 6$

**8.** Let $f(x) = \begin{cases} \frac{[\tan^2 x]-1}{\tan^2 x - 1} & \text{if } x \neq n\pi \pm \frac{\pi}{4} \\ 0 & \text{if } x = n\pi \pm \frac{\pi}{4} \end{cases}$

Then

(A) $f(x)$ is discontinuous at $x = \frac{\pi}{4}$

(B) $f(x)$ is continuous at $x = \frac{\pi}{4}$

(C) $f$ is not differentiable at $x = \frac{\pi}{4}$

(D) $f$ is discontinuous at $x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

**9.** Let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying the following relations:

**(i)** $f(x+y) = f(x)f(y)\ \forall\ x, y \in \mathbb{R}$

**(ii)** $f(x) = 1 + xg(x)$ where $\lim\limits_{x\to 0} g(x) = 1$

Then

(A) $f(x)$ is continuous for all real $x$

(B) $f(x)$ is differentiable for all real $x$

(C) $\lim_{x\to 0} f(x) = 1$

(D) $\lim_{x\to -\infty} f(x) = 0$

[**Hint:** If $Q'(x) = Q(x)$, then $Q(x) = Ke^x$.]

**10.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

(i) $f\left(\dfrac{x+y+z}{3}\right) = \dfrac{f(x)+f(y)+f(z)}{3}$ for all $x, y, z \in \mathbb{R}$

(ii) $f(0) = 3$ and $f'(0) = 3$

Then

(A) the graph of $y = f(x)$ is a straight line

(B) the graph of $y = f(x)$ is represented by the equation $x^2 + y^2 = 9$

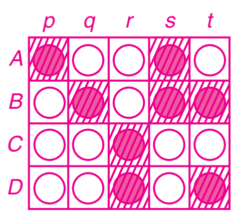
(C) $f(x)$ is unbounded

(D) $f(x)$ is differentiable for all $x$ in $\mathbb{R}$

## Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

**Example:** If the correct matches are (A) $\to$ (p), (s), (B) $\to$ (q), (s), (t), (C) $\to$ (r), (D) $\to$ (r), (t), that is if the matches are (A) $\to$ (p) and (s); (B) $\to$ (q), (s) and (t); (C) $\to$ (r); and (D) $\to$ (r), then the correct darkening of bubbles will look as follows:




**1.** In Column I, functions are given and in Column II their corresponding derivatives are given. Match them.

| *Column I* | *Column II* |
|---|---|
| (A) $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ | (p) $\dfrac{2}{x}\sin\log(x^4)$ |
| (B) $\sin^2(\log x^2)$ | (q) $\dfrac{2y}{\sqrt{x^2+1}}$ |
| (C) $\log\left[\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right]^{1/2}$ | (r) $\sec x$ |
| (D) $(x+\sqrt{x^2+1})^2$ | (s) $\dfrac{1}{\sqrt{1+x^2}}$ |

**2.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $f(x) = [x]$ where $[\cdot]$ is the integral part and $g(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ x^2 & \text{if } x \text{ is not an integer}\end{cases}$ then $(f \circ g)(x)$ is continuous at $x =$ | (p) 1 |
| (B) $f(x)\begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\end{cases}$ Then $f$ is continuous and differentiable at $x =$ | (q) 1/2 |
| (C) The function $f(x) = \lim_{n\to\infty} \dfrac{[1x]+[2x]+[3x]+\cdots+[nx]}{n^2}$ (where $[\cdot]$ has the usual meaning) is differentiable at $x =$ | (r) 3/2 |
| (D) $f(x) = \begin{cases} 2^x & \text{if } -1 \le x \le 1 \\ 3-x & \text{if } 1 < x \le 2\end{cases}$. | (s) 3 |
| Then $f$ is continuous and differentiable at $x =$ | (t) 0 |

**3.** Let

$$f(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & 1 \le x \le 2 \\ 3 - x, & 2 < x \le 3 \end{cases}$$

In the interval (0, 3), match the items of Column I with those in Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $f$ is discontinuous at $x =$ | (p) 1 |
| (B) $f$ is differentiable at $x =$ | (q) 1/2 |
| (C) $f'$ is continuous at $x =$ | (r) 3/2 |
| (D) $f'$ does not exist at $x =$ | (s) 2 |

**4.** Match the functions in Column I with the properties given in Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $x\|x\|$ | (p) Continuous in $(-1, 1)$ |
| (B) $\sqrt{\|x\|}$ | (q) Differentiable in $(-1, 1)$ |
| (C) $x+[x]$ | (r) Derivative is positive in $(-1, 1)$ |
| (D) $\|x-1\|+\|x+1\|$ | (s) Not differentiable at least at one point in $(-1, 1)$ |

## Comprehension-Type Questions

**1. Passage:** Let $f$ be a function defined in a neighbourhood of $a \in \mathbb{R}$. Then

(a) $f$ is continuous at $a$ if and only if left limit at $a$ [denoted by $f(a-0)$] = Right limit at $a$ [denoted by $f(a+0) = f(a)$]

(b) $f$ is differentiable at $a$ if and only if

Left derivative $f'(a-0) =$ Right derivative $f'(a+0)$

Answer the following questions

**(i)** If $f(x) = \cos\left(\dfrac{\pi}{2}[x] - x^3\right)$ for $1 < x < 2$ where $[x]$ is the integral part of $x$, then

(A) $f'\left(\left(\dfrac{\pi}{2}\right)^{1/3}\right)$ exists and is equal to 0

(B) $f'\left(\left(\dfrac{\pi}{2}\right)^{1/3}\right)$ exists and is equal to $-1$

(C) $f'\left(\left(\dfrac{\pi}{2}\right)^{1/3}\right)$ does not exist

(D) at $\left(\dfrac{\pi}{2}\right)^{1/3}$, the left derivative exists, but the right derivative does not exist

**(ii)** Which of the following is not true?

(A) If $f(x)$ and $g(x)$ are differentiable at $x = a$, then their sum $f(x) + g(x)$ is differentiable at $x = a$.

(B) If $f(x)$ and $g(x)$ are not differential at $x = a$, then their sum is not differentiable at $x = a$.

(C) If $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$

and $g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

then $f(x)g(x)$ is continuous at $x = 0$.

(D) If a function is differentiable for all real $x$ and $f(0) = f'(0) = 0$, then $f(x)$ need not be zero for all real $x$.

**(iii)** Let $f(x) = [x] + \sqrt{x-[x]}$ where $[x]$ denotes the integer part of $x$. Then

(A) $f$ is continuous at all integers

(B) $f$ is discontinues at all integers

(C) $f$ is a discontinues at $x = 0$

(D) $f$ is a bounded function

**2. Passage:** Let $f(x)$ be a function defined in a neighbourhood of $a \in \mathbb{R}$. Then

(a) $f$ is continuous at $a$ if and only if

$$\lim_{\substack{h\to 0\\ h>0}} f(a-h) = f(a) = \lim_{\substack{h\to 0\\ h>0}} f(a+h)$$

(b) $f$ is differentiable at $a$ if and only if

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(a-h)-f(a)}{-h} = \lim_{\substack{h\to 0\\ h>0}} \frac{f(a+h)-f(a)}{h}$$

or $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists finitely.

Answer the following questions.

**(i)** If a function is differentiable at a point $x = a$, then its derivative is continuous at $x = a$. An example of a function contradicting this statement is

(A) $f(x) = x^3 \ \forall\, x \in \mathbb{R}$

(B) $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(C) $f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(D) $f(x) = \begin{cases} 1+x^2 & \text{if } 0 \leq x \leq 1 \\ 2x & \text{if } 1 < x \leq 2 \end{cases}$

**(ii)** Let $f(x) > 0$ be a function defined for all $x > 0$ and differentiable for $x > 0$. Suppose

$$g(x) = \lim_{n\to\infty}\left(\frac{f(x+(1/n))}{f(x)}\right)^n$$

Then $g(x)$

(A) has infinitely many points of discontinuities

(B) is discontinuous at $x = 0$

(C) is not differentiable at $x = 1$

(D) is continuous and differentiable for all $x > 0$

[**Hint:** Show that $g(x) = e^{f'(x)/f(x)}$.]

**(iii)** Let

$$f(x) = \begin{cases} \dfrac{2(x-x^3)+|x-x^3|}{2(x-x^3)-|x-x^3|} & \text{if } x \neq 0, 1 \\ \dfrac{1}{3} & \text{if } x = 0 \\ 3 & \text{if } x = 1 \end{cases}$$

Then the number of points at which $f$ is discontinuous is

(A) 0 (B) 1

(C) 2 (D) infinitely many

**3.** Let $f(x) = e^{\sin x} + (\tan x)^x$. Answer the following questions.

**(i)** $f'\left(\dfrac{\pi}{4}\right)$ is equal to

(A) $\dfrac{e^{1/\sqrt{2}}}{\sqrt{2}} + \dfrac{\pi}{2}$ (B) $e^{1/\sqrt{2}}\sqrt{2} - \dfrac{\pi}{2}$

(C) $\dfrac{e^{\sqrt{2}}}{\sqrt{2}} + \dfrac{\pi}{4}$ (D) $e^{\sqrt{2}}\sqrt{2} + \dfrac{\pi}{4}$

**(ii)** $f'\left(\dfrac{\pi}{3}\right)$ is

(A) $2e^{\sqrt{3}/2} + (\sqrt{3})^{\sqrt{3}}\left(\log\tan\dfrac{\pi}{3} + \dfrac{2\pi}{3}\right)$

(B) $\dfrac{1}{2}e^{\sqrt{3}/2} + (\sqrt{3})^{\sqrt{3}}\left[\log\sqrt{3} + \dfrac{4\pi}{3\sqrt{3}}\right]$

(C) $\dfrac{1}{2}e^{\sqrt{3}/2} + (\sqrt{3})^{\sqrt{3}}\left(\log\sqrt{3} + \dfrac{2\pi}{3\sqrt{3}}\right)$

(D) $\dfrac{1}{2}e^{\sqrt{3}/2} + (\sqrt{3})^{\sqrt{3}}\left[\log\sqrt{3} - \dfrac{4\pi}{3\sqrt{3}}\right]$

**(iii)** $f'\left(\dfrac{\pi}{6}\right)$ equals

(A) $\dfrac{\sqrt{3}}{2}\sqrt{e} + \left(\dfrac{1}{\sqrt{3}}\right)^{1/\sqrt{3}}\left[-\log\sqrt{3} + \dfrac{\pi}{2\sqrt{3}}\right]$

(B) $\dfrac{\sqrt{3}}{2}\sqrt{e}^{1/2} + \left(\dfrac{1}{\sqrt{3}}\right)^{\sqrt{3}}\left[-\log\sqrt{3} + \dfrac{\pi}{\sqrt{3}}\right]$

(C) $\dfrac{2}{\sqrt{3}}\sqrt{e}^{1/2} + \left(\dfrac{1}{\sqrt{3}}\right)^{1/3}\left[\log\sqrt{3} + \dfrac{\pi}{3}\right]$

(D) $\dfrac{\sqrt{3}}{2}\sqrt{e}^{\sqrt{3}/2} + \left(\dfrac{1}{\sqrt{3}}\right)^{1/\sqrt{3}}\left[\dfrac{1}{2}\log 3 + \dfrac{\pi}{3}\right]$

## *Assertion–Reasoning Type Questions*

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation for Statement I.

(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.

(C) Statement I is true and Statement II is false.

(D) Statement I is false and Statement II is true.

**1. Statement I:** The function

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$.

**Statement II:** $\cos x$ is differentiable for all $x$.

**2. Statement I:** $f(x) = x^2$ is a differentiable function for all $x \in \mathbb{R}$.

**Statement II:** Every continuous function defined on a closed interval is differentiable.

**3. Statement I:** $f(x) = \begin{cases} 1+x, & 1 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$ is not differentiable at $x = 2$.

**Statement II:** If a function is discontinuous at point, then the function is not differentiable at that point.

**4. Statement I:** $f(x) = \begin{cases} (1+|\sin x|)^{a/|\sin x|}, & -\dfrac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \dfrac{\pi}{6} \end{cases}$

If $f(x)$ is to be continuous at $x = 0$, then $a = 2/3$ and $b = e^{2/3}$.

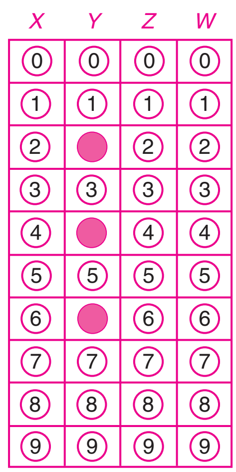
**Statement II:** A function is continuous at a point if and only if the left and right limits at that point exist and are equal as well as are equal to functional value at that point.

5. **Statement I:** If both functions $f(x)$ and $g(x)$ are not differentiable at $x = a$, then the function $f(x) + g(x)$ may be differentiable at $x = a$.

**Statement II:** $f(x) = [x]$ (integral part of $x$) and $g(x) = x - [x]$ are not differentiable at any integer, but $f(x) + g(x)$ is differentiable at all real $x$.

## *Integer Answer Type Questions*

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number $Y$ is 246, then the bubbles under $Y$ labeled as 2, 4, 6 are to be darkened.




1. Consider the function $y = f(x)$ defined parametrically by $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in \mathbb{R}$. Then in the interval $-1 \le x \le 1$, the number of points at which $f(x)$ is not differentiable is _______.

2. The function $\sqrt[3]{x}$ is not differentiable at $x =$ _______.

3. The function

$$f(x) = \begin{cases} x^p \cos\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is twice differentiable, but second derivative is not continuous if $p$ equals _______.

4. Let $f(x) = \begin{cases} ax^2 - ax + b, & x < 1 \\ x - 1, & 1 \le x \le 3 \\ cx^2 + dx + 2, & x > 3 \end{cases}$

be continuous for all $x$, then $b - (d/c)$ is equal to _______.

5. Let $f(x) = \begin{cases} x^2 & \text{for } x \le 0 \\ 1 & \text{for } 0 < x \le 1 \\ \dfrac{1}{x} & \text{for } x > 1 \end{cases}$

The number of points at which $f$ is not differentiable is _______.

6. Let $f(x) = \begin{cases} -x, & x \le 0 \\ x^2, & 0 \le x \le 1 \\ x^3 - x + 1, & x > 1 \end{cases}$

The number of points at which $f$ is continuous but not differentiable is _______.

7. In the interval $(0, \pi)$, the number of points at which the function integral part of $\sin x + \cos x$ is *not* continuous is _______.

8. Let $m$ be the value of the left derivative at $x = 2$ of the function $f(x) = [x]\sin(\pi x)$ ([] is the usual symbol), then $[m]$ is equal to _______.

9. Let $f(x) = \lim\limits_{n \to \infty} \dfrac{[1^2 x] + [2^2 x] + \cdots + [n^2 x]}{n^3}$ where [] denotes integer part, then $3(f(1) + f'(1))$ is _______.

10. If $f(x) = (e^x - e^{-x})\cos x - 2x$, then $f(0) + f'(0)$ is _______.

## ANSWERS

1. (C)
2. (D)
3. (A)
4. (B)

5. (B)
6. (A)
7. (B)
8. (D)
9. (A)
10. (B)
11. (A)
12. (C)
13. (B)
14. (B)
15. (A)
16. (D)
17. (D)
18. (D)
19. (B)
20. (A)
21. (C)
22. (B)
23. (A)
24. (D)
25. (A)

## Multiple Correct Choice Type Questions

1. (A), (B), (D)
2. (A), (D)
3. (A), (B), (C), (D)
4. (A), (B), (D)
5. (A), (b), (D)
6. (B), (C), (D)
7. (A), (B), (C), (D)
8. (A), (C), (D)
9. (A), (B), (D)
10. (A), (C), (D)

## Matrix-Match Type Questions

1. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (q)
2. (A) $\rightarrow$ (t); (B) $\rightarrow$ (t); (C) $\rightarrow$ (p),(q), (r), (s), (t); (D) $\rightarrow$ (p), (q), (r), (t)
3. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p), (q), (r); (C) $\rightarrow$ (p), (q), (r); (D) $\rightarrow$ (s)
4. (A) $\rightarrow$ (p), (q); (B) $\rightarrow$ (p), (s); (C) $\rightarrow$ (r), (s); (D) $\rightarrow$ (p), (q)

## Comprehension Type Questions

1. (i) (A); (ii) (B); (iii) (A)
2. (i) (B); (ii) (D); (iii) (C)
3. (i) (A); (ii) (B); (iii) (A)

## Assertion–Reasoning Type Questions

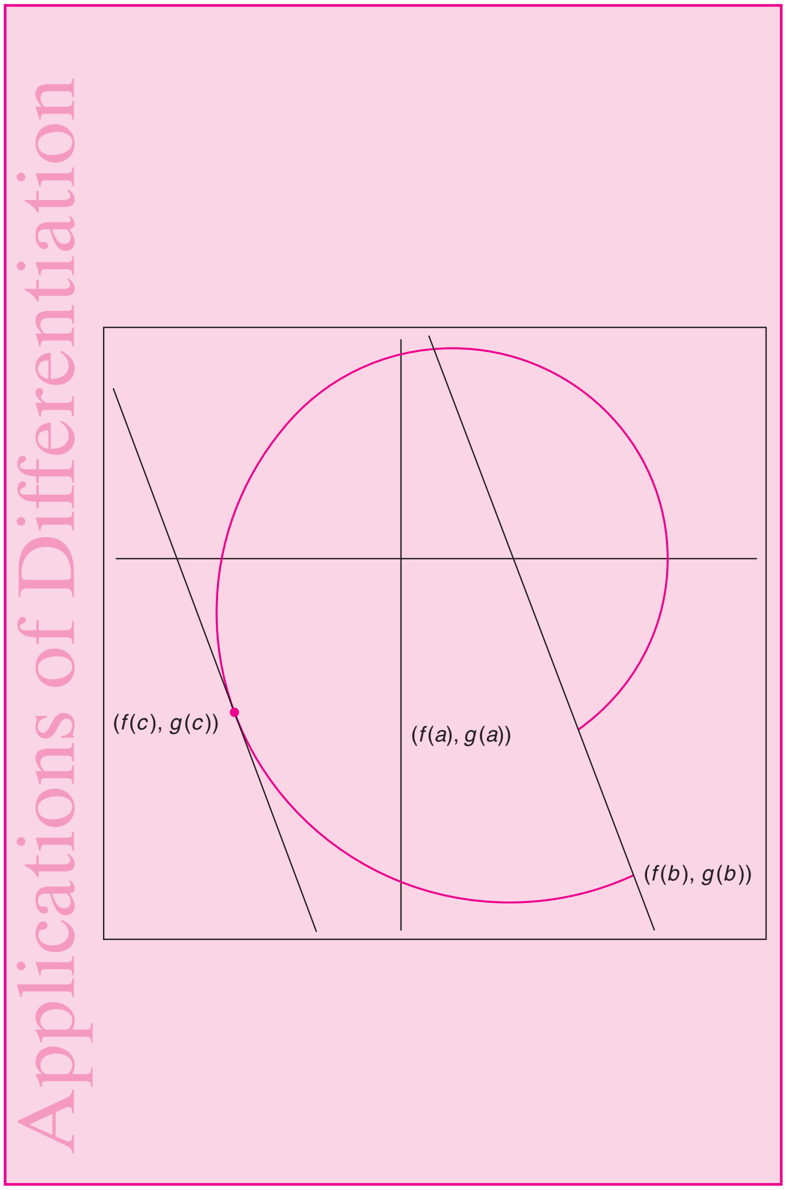
1. (B)
2. (C)
3. (A)
4. (A)
5. (A)

## Integer Answer Type Questions

1. 0
2. 0
3. 3
4. 3
5. 2
6. 1
7. 3
8. 3
9. 2
10. 0

# Applications of Differentiation

# 3




## Contents



Differentiation is all about finding **rates of change** of one quantity compared to another. We need differentiation when the rate of change is not constant.

The concept of a line tangent to a circle was introduced by ancient Greeks who used the same to some special curves. Greeks' concept of a tangent to a circle is as follows: *A line having just one point lying on a circle and all its other points outside the circle is considered to be a tangent.* This concept worked well in deriving many properties of tangent lines to a circle. However, it faced rough weather when extending it for more general curves where a tangent line to a curve at a point may intersect the curve at some more points. At this juncture people used the derivative to describe tangent and felt that this is far more satisfactory.

In this chapter we will discuss the geometric meaning of the derivative and study tangents and normals to curves. Further we will study the Mean Value Theorems, Maxima–Minima, Increasing and Decreasing functions, and L'Hospital's Rule which determine the limits of indeterminate forms.

## 3.1 | Tangents and Normals

In this section, we describe a tangent to a curve represented by a differentiable function $y = f(x)$ and obtain the equations of tangents and normal. First, we begin with the following definition.

**DEFINITION 3.1** **Tangent to a Curve** Let $P$ be a fixed point on a curve and $Q$ be a neighbouring point to $P$ on the curve. As $Q$ approaches point $P$ along the curve, suppose the chord $QP$ approaches a fixed line and finally coincides with the fixed line when $Q$ coincides with $P$. Then this fixed line is called the tangent to the curve at the point $P$. (See Fig. 3.1.)

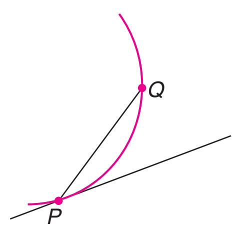



**FIGURE 3.1** Definition 3.1.

### 3.1.1 Geometric Meaning of the Derivative

Suppose $y = f(x)$ is a differentiable function. Consider Fig. 3.2. On the curve $y = f(x)$, let $P(a, f(a))$ and $Q(a + h, f(a + h))$ be two points. Draw $PM$ and $QN$ perpendicular to $x$-axis. Also draw $PR$ perpendicular to $QN$. Suppose the tangent at $P$ and the line $QP$ make angles $\theta$ and $\phi$, respectively, with the positive direction of the $x$-axis. Since PR is parallel to $x$-axis, we have that $\angle RPQ$ is equal to $\phi$. Hence $Q \to P$ along the curve, the chord $QP$ approaches the tangent $PT$ at $P$ so that $\phi \to \theta$ as $Q \to P$.

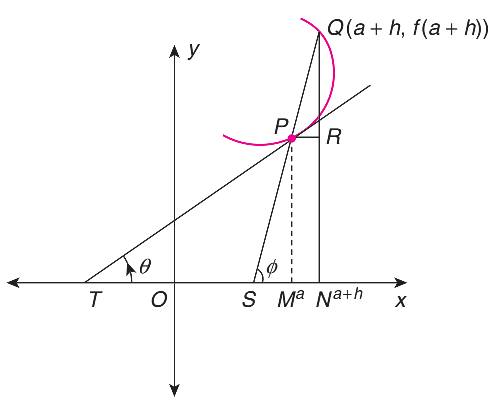



**FIGURE 3.2** Geometric meaning of the derivative.

Assume that chord $QP$ is not vertical. Now from the right-angled triangle $PQR$,

$$\tan\phi = \frac{QR}{PR} = \frac{f(a+h)-f(a)}{h}$$

$Q \to P$ along the curve implies that $h \to 0$. Therefore

$$\begin{aligned}\tan\theta &= \lim_{Q\to P} \tan\phi \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= f'(a)\end{aligned}$$

Thus $f'(a)$ is the slope of the tangent at $P(a, f(a))$.

***Note:***

**1.** $f'(a) = 0 \Leftrightarrow$ The tangent at $(a, f(a))$ is horizontal.

**2.** $|f'(a)| = \infty \Leftrightarrow$ The tangent at $(a, f(a))$ is vertical.

**3.** The equation of the tangent at $P(a, f(a))$ is

$$y - f(a) = f'(a)(x-a)$$

**4.** The equation of the tangent to the curve $y = f(x)$ at a point $(x_1, y_1)$ is

$$y - y_1 = f'(x_1)(x - x_1)$$

or

$$y - y_1 = \left(\frac{dy}{dx}\right)_{x=x_1} (x - x_1)$$

**DEFINITION 3.2** **Normal** Let $P$ be a point on a curve $y = f(x)$, where $f(x)$ is differentiable. The line perpendicular to the tangent at $P$ and passing through $P$ is called normal to the curve at $P$.

***Note:***

**1.** The slope of the normal at $P(x_1, y_1)$ is

$$\frac{-1}{f'(x_1)} = \frac{-1}{\left(\dfrac{dy}{dx}\right)_{x=x_1}} = -\left(\frac{dx}{dy}\right)_{x=x_1}$$

**2.** Equation to the normal at $P(x_1, y_1)$ is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

**DEFINITION 3.3** **Angle of Intersection of Curves** Let $C_1$ and $C_2$ be two curves and $P$ be their point of intersection. Then the angle between the tangents drawn to the curves at $P$ is called *angle of intersection* of $C_1$ and $C_2$. If the angle of intersection is a right angle, then the two curves are said to cut each other orthogonally at $P$.

***Note:*** Let $y = f(x)$ and $y = g(x)$ be two differentiable functions. Let $C_1$ and $C_2$ be the corresponding curves. Let $P(x_1, y_1)$ be a common point of $C_1$ and $C_2$ and $m_1 = f'(x_1), m_2 = g'(x_1)$. If $\theta$ is the angle of intersection of $C_1$ and $C_2$ at $P$, then

**1.** $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

**2.** $C_1$ and $C_2$ cut orthogonally at $(x_1, y_1) \Leftrightarrow m_1 m_2 = -1$

**3.** $C_1$ and $C_2$ touch each other at $P \Leftrightarrow m_1 = m_2$

**DEFINITION 3.4** Let $P$ be a point on $C$ represented by a differentiable function $y = f(x)$ (Fig. 3.3). Let the tangent and normal to the curve at $P$ meet the $x$-axis in $T$ and $N$, respectively. Draw $PG$ perpendicular to the $x$-axis. Then $PT$, $PN$, $TG$, and $GN$ are, respectively, called the lengths of tangent, the normal, the sub-tangent and the sub-normal at $P$.

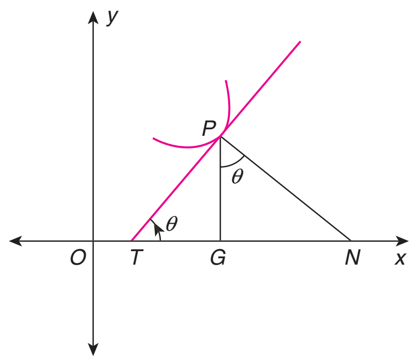



**FIGURE 3.3** Definition 3.4.

**THEOREM 3.1** Suppose $y = f(x)$ is a differentiable curve and $P(x_1, y_1)$ be a point on the curve in the $xy$-plane. Suppose the tangent and normal at $P$ meet the $x$-axis in $T$ and $N$, respectively. Draw $PG$ perpendicular to $x$-axis. Let $\theta$ be the angle made by $PT$ with the positive direction of the $x$-axis (see Fig. 3.3). Then

**(i)** Length of the tangent $PT$ is given by

$$|f(x_1)|\sqrt{1+\frac{1}{(f'(x_1))^2}} = |y_1|\sqrt{1+\left(\frac{dx}{dy}\right)^2_{(x_1,\,y_1)}}$$

**(ii)** Length of the normal $PN$ is given by

$$|f(x_1)|\sqrt{1+(f'(x_1))^2} = |y_1|\sqrt{1+\left(\frac{dy}{dx}\right)^2_{(x_1,\,y_1)}}$$

**(iii)** Length of the sub-tangent $TG$ is given by

$$\left|\frac{f(x_1)}{f'(x_1)}\right| = \left|y_1\left(\frac{dx}{dy}\right)_{(x_1,\,y_1)}\right|$$

**(iv)** Length of the sub-normal $GN$ is given by

$$|f(x_1)f'(x_1)| = \left|y_1\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}\right|$$

**PROOF** See Fig. 3.3.

**(i)** From $\Delta PGT$,

$$\begin{aligned}
PT &= \left|\frac{PG}{\sin\theta}\right| \\
&= |y_1|\sqrt{1+\cot^2\theta} \\
&= |y_1|\sqrt{1+\frac{1}{\tan^2\theta}} \\
&= |y_1|\sqrt{1+\frac{1}{\left(\dfrac{dy}{dx}\right)^2_{(x_1,\,y_1)}}} \\
&= |y_1|\sqrt{1+\left(\frac{dx}{dy}\right)^2_{(x_1,\,y_1)}}
\end{aligned}$$

**(ii)** From $\Delta PGN$,

$$PN = \left|\frac{PG}{\cos\theta}\right|$$
$$= |y_1|(\sec\theta)$$
$$= |y_1|\sqrt{1+\tan^2\theta}$$
$$= |y_1|\sqrt{1+\left(\frac{dy}{dx}\right)^2_{(x_1,\,y_1)}}$$

**(iii)** From $\Delta PGT$,

$$TG = |PG\cot\theta| = \left|y_1\left(\frac{dx}{dy}\right)_{(x_1,\,y_1)}\right|$$

**(iv)** Again from $\Delta PGN$,

$$GN = |PG\tan\theta| = \left|y_1\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}\right|$$

***Note:*** From (iii) and (iv) we have (sub-tangent) (sub-normal) is equal to $y_1^2$. That is, sub-tangent, ordinate and sub-normal are in GP.

## Example 3.1

Find the equations of tangent and normal to the curve $y^2 = 4ax$ at the point $(x_1, y_1)$ and at $(at^2, 2at)$.

***Solution:*** The given curve is $y^2 = 4ax$.

**Equation of the tangent**

Differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{2a}{y}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = \frac{2a}{y_1}, \text{ provided } y_1 \neq 0$$

So the equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

Now $(x_1, y_1)$ lies on the curve. This implies that $y_1^2 = 4ax_1$.

Therefore, the equation of the tangent at $(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1)$$

In this equation, put $x_1 = at^2, y_1 = 2at$ so that the equation of the tangent at $(at^2, 2at)$ is

$$ty = x + at^2$$

This equation is called *parametric form* of the tangent. If $y_1 = 0$, then the tangent at $(0, 0)$ is the $y$-axis.

**Equation of the normal**

Now, equation of the normal at $(x_1, y_1)$ is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

Substituting $x_1 = at^2$, $y_1 = 2at$ in the above equation, we get that the equation of the normal at $(at^2, 2at)$ is

$$tx + y = 2at + at^3$$

This form is called the *parametric form* of the normal to $y^2 = 4ax$ at $(at^2, 2at)$.

***Note:*** The parametric forms of the tangent and normal to $y^2 = 4ax$ at $(at^2, 2at)$ will be more useful in the chapter on "Parabola" (Vol. 4, Geometry).

## Example 3.2

Find the equations of the tangent and normal to the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the point $(x_1, y_1)$ and at the point $(a\cos\theta, b\sin\theta)$.

**Solution:**

**Equation of the tangent**

Differentiating both sides of the given equation w.r.t. $x$ we get

$$\frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0$$

and hence

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-\frac{b^2x_1}{a^2y_1}$$

So the equation of the tangent at $(x_1, y_1)$ is

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\Rightarrow b^2x_1x+a^2y_1y=b^2x_1^2+a^2y_1^2$$

Dividing by $a^2b^2$, we have

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

[since $(x_1, y_1)$ lies on the curve]. Therefore, the equation of the tangent at $(x_1, y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

In the above equation, put $x_1=a\cos\theta, y_1=b\sin\theta$. Then the equation of the tangent at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

This equation is called *parametric equation* of the tangent to the given curve at $(a\cos\theta, b\sin\theta)$.

**Equation of the normal**

Now, equation of the normal at $(x_1, y_1)$ is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

That is,

$$\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$$

In this equation if we put $x_1=a\cos\theta$ and $y_1=b\sin\theta$, the equation of the normal is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

This equation is called *parametric form* of the normal.

***Note:*** The equations of the tangent and normal at $(x_1, y_1)$ and at $(a\cos\theta, b\sin\theta)$ will be utilized in the chapter on "Ellipse" (Vol. 4, Geometry).

## Example 3.3

Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ cut orthogonally.

**Solution:** Suppose the curves

$$C_1: x^3-3xy^2=-2$$

and

$$C_2: 3x^2y-y^3=0$$

cut at $(x_1, y_1)$. Differentiating $C_1$ w.r.t. $x$, we get

$$3x^2-3y^2-6xy\frac{dy}{dx}=0$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-y_1^2+x_1^2}{2x_1y_1}=m_1 \quad \text{(say)}$$

Similarly, differentiating $C_2$ w.r.t. $x$, we have

$$\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{-2x_1y_1}{x_1^2 - y_1^2} = m_2 \quad \text{(say)}$$

Now

$$m_1m_2 = \left(\frac{x_1^2 - y_1^2}{2x_1y_1}\right)\left(\frac{-2x_1y_1}{x_1^2 - y_1^2}\right) = -1$$

By part (2) of the note under Definition 3.3, $C_1$ and $C_2$ cut orthogonally at $(x_1, y_1)$.

### Example 3.4

Find the length of the tangent, normal, sub-tangent and sub-normal to the curve $x = a\,(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = \pi/2$ [i.e., at the point $(a(\pi/2 - 1), a)$] where $a > 0$.

***Solution:*** Differentiating the two equations (given) we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$
$$\frac{dy}{d\theta} = a\sin\theta$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\frac{\theta}{2}$$

So

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = \cot\frac{\pi}{4} = 1$$

Now

**1.** Length of the tangent is

$$|\,y_1\,|\sqrt{1 + \left(\frac{dx}{dy}\right)^2_{x_1,\, y_1}} = a\sqrt{1 + 1^2} = a\sqrt{2}$$

**2.** Length of the normal is

$$|\,y_1\,|\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1,\, y_1)}} = a\sqrt{1+1} = a\sqrt{2}$$

**3.** Length of the sub-tangent is

$$\left|\, y_1 \cdot \left(\frac{dx}{dy}\right)_{x_1,\, y_1} \right| = |\,a \cdot 1\,| = a$$

**4.** Length of the sub-normal is

$$\left|\, y_1 \cdot \left(\frac{dy}{dx}\right)_{x_1,\, y_1} \right| = |\,a \cdot 1\,| = a$$

## 3.2 | Rate Measure

In Section 3.1, we have seen how the derivative of a function is useful in studying the properties of tangents. In this section we will discuss how the derivative of a function is also useful in the velocity problems or, more generally, the rate of change of a function. We begin with the following definition.

**DEFINITION 3.5** Let $f$ be a function defined on an open interval $(a, b)$ and let $c \in (a, b)$. Then the difference

$$\frac{f(c+h) - f(c)}{h}$$

where $h$ may take positive values or negative values is called **average change** or **change of *f*** in the interval $(c, c+h)$. If

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and is a finite number, which we denote by $f'(c)$, then $f'(c)$ is called the **rate of change of *f* at *c***.

**THEOREM 3.2** Suppose a participle moving on a straight line covers a distance $s(t)$ in time $t$. Then the velocity of the particle at time $t$ is $ds/dt$.

**PROOF** Since the velocity of a particle is the rate of change in distance, we have, the velocity $v$ at time $t$ is

$$\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}$$

which is nothing but $s'(t)$ or $ds/dt$. ■

***Note:*** The velocity $v$ is also a function of $t$.

**DEFINITION 3.6** **Acceleration** The rate of change of velocity is called *acceleration*. If $v = ds/dt$, then the acceleration is given by

$$\frac{dv}{dt}=\frac{d^2s}{dt^2} \quad \text{or} \quad s''(t)$$

**DEFINITION 3.7** **Angular Velocity and Angular Acceleration** If a particle $P$ is a moving on a plane curve, the angle made by $OP$ ($O$ being the origin) with the $x$-axis at time $t$ is denoted by $\theta(t)$. The rate at which the angle $\theta(t)$ is changing at time $t$ is called the *angular velocity* of the particle at time $t$ and is given by

$$\frac{d\theta}{dt} \quad \text{or} \quad \theta'(t)$$

The rate at which the angular velocity is changing at time $t$ is called *angular acceleration* and is given by

$$\frac{d^2\theta}{dt^2} \quad \text{or} \quad \theta''(t)$$

## Example 3.5

Consider a particle moving on a straight line. Let $s(t)$ be the distance traveled by it in time $t$ from a fixed point. Then $s(t)=\alpha\sin(\beta t)$, where $\alpha$ and $\beta$ are constants. If $v(t)$ is the velocity and $a(t)$ is the acceleration of the particle at time $t$, find

1. $v^2 - a\cdot s$
2. $da/ds$
3. $s\dfrac{da}{dt}$

***Solution:*** We first find the velocity and the acceleration. Now velocity is given by

$$v(t)=\frac{ds}{dt}=\alpha\beta\cos(\beta t)$$

The acceleration is given by

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d^2s}{dt^2} \\ &= -\alpha\beta^2\sin(\beta t) \\ &= -\beta^2 s(t) \end{aligned}$$

and hence

$$\begin{aligned} \frac{da}{dt} &= \frac{d}{dt}[-\beta^2 s(t)] \\ &= -\beta^2\frac{ds}{dt} \\ &= -\beta^2 v(t) \end{aligned}$$

1. We have

$$\begin{aligned} v^2-a\cdot s &= [v(t)]^2 - a(t)s(t) \\ &= \alpha^2\beta^2\cos^2(\beta t) - [-\beta^2 s(t)\cdot s(t)] \\ &= \alpha^2\beta^2\cos^2(\beta t) + \alpha^2\beta^2\sin^2(\beta t) \\ &= \alpha^2\beta^2 \end{aligned}$$

2. We have

$$\frac{da}{ds}=\frac{da}{dt}\div\frac{ds}{dt}=\frac{-\beta^2 v(t)}{v(t)}=-\beta^2$$

3. We have

$$\begin{aligned} s\frac{da}{dt} &= s(t)[-\beta^2 v(t)] \\ &= v(t)[-\beta^2 s(t)] \\ &= v(t)a(t) \end{aligned}$$

**Example 3.6**

Consider Fig. 3.4. $OAB$ is a right-angled triangle, right angled at $A$. Suppose $P$ is a moving point on $AB$ with uniform velocity $v$. Find the angular velocity of $P$ with respect to $O$.

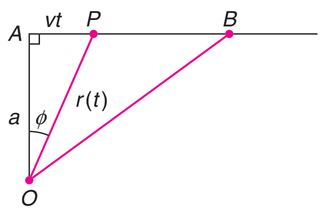



**FIGURE 3.4** Example 3.6.

***Solution:*** Since the point $P$ is moving with uniform velocity $v$ we have $AP = vt$ (since $ds = vdt$ and $v$ is constant, $s = vt$). Let $\theta(t)$ be the angle made by $OP$ with $OA$ at time $t$. Let $OA = a$, $OP = rt$ and $\angle AOP = \theta(t)$. Then

$$a = r(t)\cos(\theta(t))$$

and
$$[r(t)]^2 = OP^2 = OA^2 + AP^2 = a^2 + v^2t^2 \tag{3.1}$$

Differentiating both sides of Eq. (3.1) with respect to $t$, we get

$$2r(t)\frac{dr}{dt} = 2v^2t \tag{3.2}$$

Differentiating both sides of the equation $a = r(t)\cos(\theta(t))$, w.r.t. $t$ we have

$$\begin{aligned}
0 &= \frac{dr}{dt}\cos(\theta(t)) + r(t)(-\sin(\theta(t)))\frac{d\theta}{dt} \\
&= \frac{dr}{dt}\cdot\frac{a}{r(t)} - r(t)\left(\frac{vt}{r(t)}\right)\frac{d\theta}{dt} \\
&= \frac{dr}{dt}\cdot\frac{a}{r(t)} - (vt)\frac{d\theta}{dt} \\
&= \frac{a}{r(t)}\cdot\frac{v^2t}{r(t)} - (vt)\frac{d\theta}{dt} \quad \text{[from Eq. (3.2)]}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{a}{[r(t)]^2}\left(\frac{v^2t}{vt}\right) \\
&= \frac{av}{(r(t))^2} \\
&= \frac{av}{(OP)^2}
\end{aligned}$$

Thus the angular velocity of $P$ w.r.t. $O$ is $av/(OP)^2$.

## 3.3 | Mean Value Theorems

Mean value theorem in differential calculus connects the values of a function to values of its derivative. It is one of the most useful tools in real analysis. Especially, a special case of mean value theorem, viz. *Rolle's Theorem* established in 1690 by Michel Rolle (1652–1719) – a French mathematician, is useful for finding points on a curve where the tangents are horizontal. This theorem gives the position of roots of derivatives of polynomial functions. Mean value theorem is also useful in deciding the intervals of monotonicity of functions and finally, in proving the most famous L'Hospital's Rule. First we begin with Rolle's theorem.

### *3.3.1 Rolle's Theorem*

**Theorem 3.3** Suppose $a < b$, $f:[a, b] \to \mathbb{R}$ is continuous such that $f(a) = f(b)$. Further assume that $f$ is differentiable in the open interval $(a, b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.

**Proof** If $f$ is a constant function, then $f'(c) = 0$ for any $c \in (a, b)$. Hence, we may assume that $f$ is a non-constant function so that $f$ is bounded (see Theorem 1.31). Let

$$m = \underset{x\in[a,b]}{\text{g.l.b.}}\, f(x) \quad \text{and} \quad M = \underset{x\in[a,b]}{\text{l.u.b.}}\, f(x)$$

so that $m < M$. Then either $f(a) \neq m$ or $f(a) \neq M$. Suppose $f(a) \neq M$. Then

$$f(b) \neq M \quad [\because f(a) = f(b)]$$

Hence, by Theorem 1.33, there exists $c \in (a, b)$ such that $f(c) = M$, so that

$$f(c) = M \geq f(x)\ \forall\, x \in [a, b]$$

Now, for $x \in (a, c)$,

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

so that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0$$

Also, for $x \in (c, b)$,

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

so that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \leq 0$$

Hence, $f'(c) = 0$. ■

### Example

Let $f(x) = x^2$ for $x \in [-1, 1]$. Then $f$ is continuous on $[-1, 1]$ and

$$f(-1) = (-1)^2 = 1 = 1^2 = f(1)$$

Also $f$ is differentiable in $(-1, 1)$ and

$$f'(x) = 2x\ \forall\ x \in (-1, 1)$$

Further $f'(0) = 0$ and $0 \in (-1, 1)$.

### Example

Consider the function $f(x) = \sin x$ for $x \in [0, \pi]$. Obviously $f$ is continuous on $[0, \pi]$, differentiable in $(0, \pi)$ and $f(0) = f(\pi) = 0$. Hence $f'(c) = 0$ for some $c \in (0, \pi)$. That is $\cos c = 0$ for some $c \in (0, \pi)$. Clearly $c = \pi/2$.

## *Geometric Interpretation of Rolle's Theorem*

See Fig. 3.5.

1. Let $y = f(x)$ be a function satisfying the conditions of Rolle's theorem on a closed interval $[a, b]$. Then $f'(c) = 0$ for at least one $c \in (a, b)$. But $f'(c)$ is the slope of the tangent to the curve $y = f(x)$. That is, at the point $(c, f(c))$ the tangent is parallel to the $x$-axis.

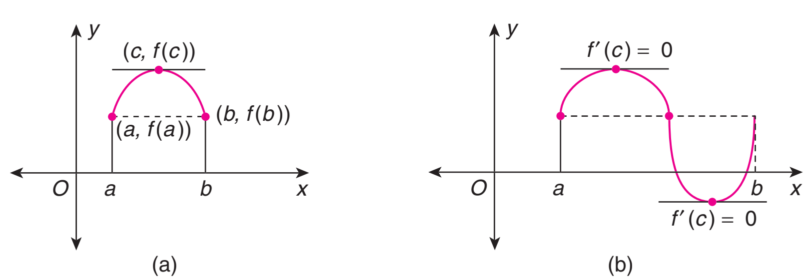



**FIGURE 3.5** Geometric meaning of Rolle's theorem.

2. According to Rolle's theorem, there exists a $c \in (a, b)$ such that $f'(c) = 0$. But there may be more than one $c$ at which $f'(c) = 0$. For example, consider $f(x) = x^2(x-1)^2$ on the interval $[-1, 2]$ so that $f(-1) = 4 = f(2)$.

Now

$$f'(x) = 0 \Rightarrow 2x(x-1)(2x-1) = 0$$
$$\Rightarrow x = 0, \frac{1}{2}, 1$$

In this case $c = 0, 1/2, 1$ (more than one value).

The following theorem which may be called **Rolle's theorem for polynomial functions** gives the position of roots of the derivative polynomial of a given polynomial.

**THEOREM 3.4** Let $P(x)$ be a polynomial of degree $n$ ($\geq 2$). Then between any two real roots of $P(x) = 0$, there exists a root of $P'(x) = 0$.

**PROOF** Let $a, b$ be two roots of $P(x) = 0$. We may assume that $a < b$. Then, clearly $P(a) = 0 = P(b)$. Also $P(x)$ is continuous for all $x$ in $[a, b]$ and differentiable for all $x$ in $(a, b)$. Hence by Rolle's theorem, $P'(c) = 0$ for some $c \in (a, b)$. Thus $P'(x) = 0$ has a root $c \in (a, b)$. ■

***Note:*** In general, if $f(x)$ is continuous and differentiable and $\alpha, \beta$ are real roots of the equation $f(x) = 0$, then $f'(x) = 0$ has a root in between $\alpha$ and $\beta$.

**IMPORTANT NOTE**

1. A function need not satisfy any one condition of the Rolle's theorem on an interval, but the derivative may vanish at many points of the interval. For example consider the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < \dfrac{1}{2} \\ 2 & \text{for } \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

Then $f$ satisfies none of the conditions of Rolle's theorem, but $f'(x) = 0$ for all $x \neq 1/2$ in $[0, 1]$ and $f$ is discontinuous at 1/2.

2. In Theorem 3.4, if $\alpha$ repeats $t$ times as a root for $P(x) = 0$, then $\alpha$ repeats $t - 1$ times as a root for $P'(x) = 0$ because $(x - \alpha)^t$ is a factor of $P(x)$.

### 3.3.2 Lagrange's Mean Value Theorem

**THEOREM 3.5 (LAGRANGE'S MEAN VALUE THEOREM)** Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Further, suppose $f$ is differentiable in the open interval $(a, b)$. Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**PROOF** Write

$$g(x) = f(x) + \frac{b - x}{b - a}[f(b) - f(a)]$$

Then $g(x)$ is

1. Continuous on the closed interval $[a, b]$
2. Differentiable in the open interval $(a, b)$
3. $g(a) = g(b)$ $(= f(b))$

Hence by Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. But it is given that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Therefore

$$g'(c) = 0 \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

***Note:***

1. If $f(a) = f(b)$, then Rolle's theorem follows from Lagrange's mean value theorem. But, we use Rolle's theorem in proving Lagrange's mean value theorem. Hence, *Rolle's theorem cannot be considered as a corollary of Lagrange's mean value theorem.*
2. Lagrange's mean value theorem says that the mean or average value

$$\frac{f(b) - f(a)}{b - a}$$

of $f$ in $[a, b]$ is equal to the derivative $f'(c)$ for some point $c$ in $(a, b)$.

## *Geometric Interpretation of Lagrange's Mean Value Theorem*

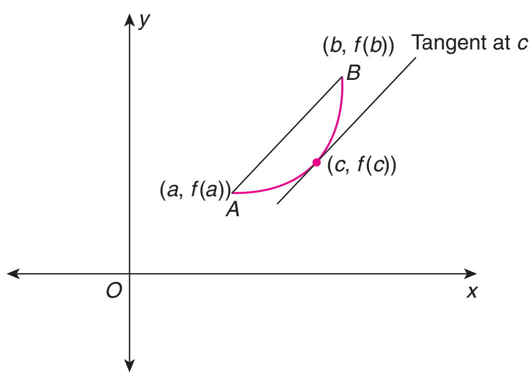



**FIGURE 3.6** Geometric meaning of Lagrange's mean value theorem.

See Fig. 3.6. Clearly,

$$\frac{f(b) - f(a)}{b - a}$$

is the gradient (slope) of the line joining the points $(a, f(a))$ and $(b, f(b))$ which lie on the curve of $y = f(x)$. Lagrange's mean value theorem says that, for some $c \in (a, b)$, the tangent at the point $(c, f(c))$ to the curve $y = f(x)$ is parallel to the line joining the points $(a, f(a))$ and $(b, f(b))$.

As immediate consequences of Lagrange's mean value theorem we have the following results stated as corollaries.

**Corollary 3.1** If $f$ is continuous on the closed interval $[a, b]$, differentiable in the open interval $(a, b)$ and $f'(x) = 0 \ \forall\, x \in (a, b)$, then $f$ is a constant on $[a, b]$.

**Proof** Suppose $a < \alpha \le b$. Then $f$ is continuous on $[a, \alpha]$ and differentiable in $(a, \alpha)$. Hence by Lagrange's mean value theorem, there exists $c \in (a, \alpha)$ such that

$$0 = f'(c) = \frac{f(\alpha) - f(a)}{\alpha - a}$$

so that $f(a) = f(\alpha)$. Thus

$$f(\alpha) = f(a) \ \forall\, \alpha \in [a, b]$$

Hence $f$ is a constant in $[a, b]$.

**COROLLARY 3.2** Suppose $f$ is differentiable in the open interval $(a, b)$ with derivative $f'(x) = 0\ \forall\, x \in (a, b)$. Then $f$ is a constant in $(a, b)$.

**PROOF** Suppose $a < \alpha < \beta < b$. Then, by hypothesis, $f$ is continuous on $[\alpha, \beta]$, differentiable in $(\alpha, \beta)$ and $f'(x) = 0\ \forall\, x \in (\alpha, \beta)$. Hence by Corollary 3.1, $f$ is a constant in $[\alpha, \beta]$. In particular, $f(\alpha) = f(\beta)$. This being true for all $\alpha, \beta \in (a, b)$, it follows that $f$ is a constant over $(a, b)$. ■

***Caution:*** Corollary 3.2 fails if the domain of definition of $f$ is not an interval.

## Example

Define $f(x) = \begin{cases} 0, & \text{if } x \in (0,1) \\ 1, & \text{if } x \in (2,3) \end{cases}$

Then $f$ is differentiable in $(0,1) \cup (2,3)$ with $f'(x) = 0$ $\forall\, x \in (0,1) \cup (2,3)$, but $f$ is not a constant over $(0,1) \cup (2,3)$.

Another nice and important corollary of Lagrange's mean value theorem is the following.

**COROLLARY 3.3** Suppose $f$ and $g$ are continuous on $[a, b]$ and differentiable in $(a, b)$. Further suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists $k$ such that $f(x) - g(x) = k$ for all $x \in [a, b]$.

**PROOF** Write

$$h(x) = f(x) - g(x)\ \forall\, x \in [a, b]$$

Then, by hypothesis, $h$ is continuous on $[a, b]$, differentiable in $(a, b)$ and

$$h'(x) = f'(x) - g'(x) = 0\ \forall\, x \in (a, b)$$

Hence by Corollary 3.1, $h(x)$ is a constant say $k\ \forall\, x \in [a, b]$. Hence,

$$f(x) - g(x) = h(x) = k\ \forall\, x \in [a, b].$$ ■

Before going to some more consequences of Lagrange's mean value theorem, we consider some examples on Lagrange's mean value theorem.

## Example 3.7

Consider $f(x) = x(x-1)(x-2)$ on the interval $[0, 1/2]$. Show that it satisfies Lagrange's mean value theorem and find the value of $c$.

***Solution:*** The function $f(x)$ being a polynomial function, is continuous and differentiable for all real values of $x$ and hence in $[0, 1/2]$. Also

$$f(0) = 0 \quad \text{and} \quad f\left(\frac{1}{2}\right) = \frac{3}{8}$$

Now,

$$f(x) = x^3 - 3x^2 + 2x$$
$$\Rightarrow f'(c) = 3c^2 - 6c + 2$$

But

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = \frac{\left(\frac{3}{8}\right)}{\left(\frac{1}{2}\right)} = \frac{3}{4}$$

Therefore

$$3c^2 - 6c + 2 = \frac{3}{4}$$
$$\Rightarrow 12c^2 - 24c + 5 = 0$$
$$\Rightarrow c = 1 \pm \frac{\sqrt{21}}{6}$$

But

$$c = 1 - \frac{\sqrt{21}}{6} \in \left(0, \frac{1}{2}\right)$$

Thus the value of $c$ in the Lagrange's mean value theorem is $1-(\sqrt{21}/6)$.

## Example 3.8

Consider the function $f(x) = Ax^2 + Bx + C$. Show that it satisfies the Lagrange's mean value theorem.

***Solution:*** It is known that from the school stage that $y = Ax^2 + Bx + C$ represents a parabola. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the curve $y = Ax^2 + Bx + C$. Now consider

$$f(x) = Ax^2 + Bx + C$$

for $x \in [x_1, x_2]$. Since $f$ is continuous on $[x_1, x_2]$ and differentiable in $(x_1, x_2)$, by Lagrange's mean value theorem, there exists $x_0 \in (x_1, x_2)$ such that

$$f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This means

$$\begin{aligned} 2Ax_0 + B &= \frac{(Ax_2^2 + Bx_2 + C) - (Ax_1^2 + Bx_1 + C)}{x_2 - x_1} \\ &= \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1} \\ &= A(x_2 + x_1) + B \end{aligned}$$

Therefore

$$x_0 = \frac{x_1 + x_2}{2}$$

Thus the abscissa of the point on the parabola where the tangent is parallel to the chord $PQ$ is the abscissa of the midpoint of the chord joining $P$ and $Q$.

The following theorem relates to the *monotonic nature* (i.e., increasing or decreasing nature) of a function which is also a consequence of Lagrange's mean value theorem. We recall that (Definition 2.3, Chapter 2) a function $f$ defined on a subset $A$ of $\mathbb{R}$ is said to be increasing if $f(x_1) \le f(x_2)$ whenever $x_1, x_2$ are in $A$ and $x_1 < x_2$.

**THEOREM 3.6** Let $f : [a, b] \to \mathbb{R}$ be a function differentiable in $(a, b)$. Then

**(i)** $f$ is *increasing* on $[a, b]$ if and only if $f'(x) \ge 0$ for all $x \in (a, b)$.
**(ii)** $f$ is *decreasing* on $[a, b]$ if and only if $f'(x) \le 0$ for all $x \in (a, b)$.

**PROOF** **(i)** Suppose $f'(x) \ge 0 \; \forall \, x \in [a, b]$. Let $x_1, x_2$ belong to $(a, b)$ and $x_1 < x_2$. Applying Lagrange's mean value theorem for $f$ on the interval $[x_1, x_2]$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

for some $c \in (x_1, x_2)$. But $f'(c) \ge 0$ (by hypothesis). Since $f'(c) \ge 0$ and $x_2 - x_1 > 0$, it follows that

$$\begin{aligned} & f(x_2) - f(x_1) \ge 0 \\ \Rightarrow\; & f(x_2) \ge f(x_1) \end{aligned}$$

This being true for any arbitrary $x_1$ and $x_2$ $(x_1 < x_2)$, it follows that $f$ is increasing on $[a, b]$.

Conversely, suppose $f$ is increasing on $[a, b]$ and differentiable. Let $c \in (a, b)$. If $x \in (a, b)$ and $x \ne c$, then either $x > c$ or $x < c$ which implies that either $f(x) \ge f(c)$ or $f(x) \le f(c)$ because $f$ is increasing. In any case

$$\frac{f(x) - f(c)}{x - c} \ge 0$$

Hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \ge 0$$

That is $f'(c) \ge 0$.

**(ii)** We observe that

$$
\begin{aligned}
f \text{ is decreasing} &\Leftrightarrow -f \text{ is increasing} \\
&\Leftrightarrow (-f')(x) \geq 0 \ \forall\, x \in (a, b) \\
&\Leftrightarrow -f'(x) \geq 0 \ \forall\, x \in (a, b) \\
&\Leftrightarrow f'(x) \leq 0 \ \forall\, x \in (a, b)
\end{aligned}
$$

■

**IMPORTANT NOTE**

The argument given along the same lines of the proof of Theorem 3.6 can be used to prove that a function having positive derivative on an interval is strictly increasing and that having a negative derivative is strictly decreasing. However the converse parts are not true. For example, consider the functions $f(x) = x^3$ and $g(x) = -x^3$ for $x \in \mathbb{R}$. The function $f(x)$ is strictly increasing with $f'(0) = 0$ and the function $g(x)$ is strictly decreasing in $(-1, 1)$ with $g'(0) = 0$.

This implies the following:

1. For a strictly increasing function which is differentiable, the derivative need not be positive, but it may be non-negative.
2. For a strictly decreasing function, the derivative need not be negative, but it may be non-positive.

The following concept is related to the nature of the function which is strictly increasing (strictly decreasing) at a point of its domain; however, this is different from the concept of strictly increasing (decreasing) in a neighbourhood of the point.

**DEFINITION 3.8** **Strictly Locally Increasing or Decreasing** Let $f$ be a function defined in a neighbourhood of a point $c$. If there exists $\delta > 0$ such that

1. $c < x < c + \delta \Rightarrow f(x) > f(c)$
2. $c - \delta < x < c \Rightarrow f(x) < f(c)$

then we say that $f$ is strictly locally increasing at $c$. Similarly, strictly locally decreasing at $c$ can be defined.

**IMPORTANT NOTE**

Strictly locally increasing at $c$ need not imply that the function is increasing in a neighbourhood of $c$. For example, consider

1. $f(x) = \begin{cases} x + 2x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

It is strictly locally increasing at 0 but in any neighbourhood of 0, the function is not increasing.

2. $g(x) = \begin{cases} x \left| \sin \dfrac{1}{x} \right|, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Here $g$ is locally increasing (strictly) at 0, but not increasing in any neighbourhood of 0.

## 3.4 | Maxima–Minima

**DEFINITION 3.9** Let $f : [a, b] \to \mathbb{R}$ be a function and $a < c < b$.

1. Suppose there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and

$$f(x) \leq f(c) \ \forall\, x \in (c - \delta, c + \delta)$$

Then we say that $f$ has *local maximum* at $c$. The point $c$ is called *point of local maximum* and $f(c)$ is called the *local maximum value*.

2. Suppose there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and $f(x) \geq f(c)\ \forall\ x \in (c - \delta, c + \delta)$. Then we say that $f$ has *local minimum* at $c$. The point $c$ is called *point of local minimum* and $f(c)$ is called the *local minimum value*.
3. If $f$ has either a local maximum or a local minimum at $c$, then we say that $f$ has *local extremum* at $c$. In this case, $c$ is called *point of local extremum* and $f(c)$ is called the *local extremum value*.

**THEOREM 3.7 (NECESSARY CONDITION FOR EXTREMUM)** Suppose $f:[a,b] \to \mathbb{R}$ is a function and $c \in (a,b)$. If $f$ is differentiable at $c$ and $c$ is a point of local extremum, then $f'(c) = 0$.

**PROOF** **Case I:** Suppose $f$ has local maximum at $c$. Let $f'(c) > 0$. Then

$$0 < f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Let $\varepsilon = (1/2)f'(c)$. Since

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

there exists $\delta > 0$ such that $c - \delta < x < c + \delta$, which implies

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \frac{1}{2} f'(c)$$

Therefore

$$\frac{-1}{2} f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < \frac{1}{2} f'(c)$$

Taking the left-hand side inequality we get

$$\frac{f(x) - f(c)}{x - c} > f'(c) - \frac{1}{2} f'(c) = \frac{1}{2} f'(c) > 0$$

So in the neighbourhood $(c - \delta, c + \delta)$,

$$\frac{f(x) - f(c)}{x - c} > 0$$

$$f(x) - f(c) > 0\ \forall\, x \in (c, c + \delta) \quad (\because x - c > 0)$$

$$f(x) \geq f(c)\ \forall\, x \in (c, c + \delta)$$

This contradicts the local maximum nature of $c$. Therefore $f'(c) \not> 0$. Similarly $f'(c) \not< 0$. Hence $f'(c) = 0$.

**Case II:** Suppose $f$ has local minimum at $c$. Then

$$\begin{aligned} &-f \text{ has local maximum at } c \\ &\Rightarrow (-f)'(c) = 0 \quad \text{(by Case I)} \\ &\Rightarrow -f'(c) = 0 \\ &\Rightarrow f'(c) = 0 \end{aligned}$$

■

**IMPORTANT NOTE**

From Theorem 3.7, we conclude that if a function is differentiable at $c$ of local extrema, then $f'(c) = 0$. The following examples show that

1. A function having a local extremum at a point need not be differentiable at that point.
2. A function whose derivative is zero at a point need not have local extremum at that point.

## Examples

1. Let $f(x) = |x|$ for $x \in [-1, -1]$. Then $f$ has local minimum at $x = 0$, but $f$ is not differentiable at 0.
2. Let $f(x) = x^3$ for $x \in [-1, -1]$. Then $f'(0) = 0$, but zero is not a point of extremum of $f$, because $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$.

The following two theorems are on sufficient condition for a function to have a local extrema at an interior point of an interval.

**THEOREM 3.8 (FIRST DERIVATIVE TEST)** Suppose $f : [a, b] \to \mathbb{R}$ is continuous, $a < c < b$ and $f$ is differentiable in $(a, c)$ and $(c, b)$. Let $\delta > 0$ be such that $(c - \delta, c + \delta) \subset (a, b)$. Then

**(i)** $f$ has local maximum at $c$ if $f'(x) \geq 0\ \forall\, x \in (c - \delta, c)$ and $f'(x) \leq 0\ \forall\, x \in (c, c + \delta)$.
**(ii)** $f$ has local minimum at $c$ if $f'(x) \leq 0\ \forall\, x \in (c - \delta, c)$ and $f'(x) \geq 0\ \forall\, x \in (c, c + \delta)$.

**PROOF**

**(i)** Let $c - \delta < x_0 < c$. Using Lagrange's mean value theorem for $f(x)$ on the interval $[x_0, c]$, there exists $c_{x_0} \in (x_0, c)$ such that

$$f(c) - f(x_0) = (c - x_0) f'(c_{x_0})$$

Since $f'(c_{x_0}) \geq 0$, we have $f(x_0) \leq f(c)$. Similarly, if $c < y_0 < c + \delta$, then again we have $c_{y_0} \in (c, y_0)$ such that

$$f(y_0) - f(c) = (y_0 - c) f'(c_{y_0})$$

Now $f'(c_{y_0}) \leq 0$ implies that $f(c) \geq f(y_0)$. Therefore

$$f(x) \leq f(c)\ \forall\, x \in (c - \delta, c + \delta)$$

so that $f$ has local maximum at $c$.

**(ii)** Proof is similar to (i). ■

**QUICK LOOK 1**

1. $f$ has local maximum at $c$ if $f'$ changes sign from *positive to negative* at $c$.
2. $f$ has local minimum at $c$ if $f'$ changes sign from *negative to positive.*
3. If $f'$ keeps the same sign at $c$, then $c$ is not a point of extremum.

The information in the following sub-section permits to formulate a rule in testing a differentiable function $y = f(x)$ (may not be differentiable at a point or $f'$ is discontinuous) for a local extremum. Before, that we give the following definition.

**DEFINITION 3.10** **Critical Point** A point $x_0$ in the domain of a function $f$ is called a *critical point* of $f$ if either $f'(x_0)$ exists and is equal to zero or $f'(x_0)$ does not exist [but $f'(x)$ exists at nearby points, so that at this point $x_0$, $f'$ is discontinuous].

## 3.4.1 Testing a Differentiable Function for a Local Extremum with First Derivative

**PROCEDURE**

**Step 1:** Find the first derivative. That is $f'(x)$.

**Step 2:**
**(a)** Equate $f'(x)$ to zero and find the values of $x$.
**(b)** Find the values of $x$ at which $f'(x)$ is discontinuous.

The values of $x$ obtained by Steps 2($a$) and 2($b$) are the critical points.

**Step 3:** Now investigate the change of sign of the derivative $f'$ at a critical point.

From Quick Look 2, we can draw a table which gives point of extremum. Keep it in the mind so that *not every critical point is a point of extremum*.

| *Signs of derivative when passing critical point $x = x_0$* | | | |
|---|---|---|---|
| *$x < x_0$* | *$x = x_0$* | *$x > x_0$* | *Character of critical point* |
| + | $f'(x_0) = 0$ or $f'$ is discontinuous at $x_0$ | – | Local maximum point |
| – | $f'(x_0) = 0$ or $f'$ is discontinuous at $x_0$ | + | Local minimum point |
| + | $f'(x_0) = 0$ or $f'(x)$ is discontinuous at $x_0$ | + | Neither maximum nor minimum (actually function increases) |
| – | $f'(x_0) = 0$ or $f'$ is discontinuous at $x_0$ | – | Neither maximum nor minimum (function decreases) |

### Example 3.9

Let $f(x) = x^3 - 9x^2 + 15x + 3$. Find its critical points and mention its character at those points.

***Solution:*** We have

$$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5)$$

Now

$$f'(x) = 0 \Rightarrow x = 1, 5$$

Since $f'(x)$ is continuous for all real $x$, the only critical points of $f$ are 1, 5 only.

1. For $x < 1$ we have $f'(x) > 0$ and for $x > 1$ we have $f'(x) < 0$. That is at $x = 1$, $f'(x)$ changes sign from + to –. Hence, at $x = 1$, $f$ has local maximum value and the maximum value of $f$ is $f(1) = 10$.
2. For $x < 5, f'(x) = (+)\,(-) < 0$

   For $x > 5, f'(x) = (+)\,(+) > 0$

   Therefore at $x = 5$, $f$ has local minimum and the local minimum value is $f(5) = -22$.

### Example 3.10

Let $f(x) = (x-1)x^{2/3}$. Find its critical points and mention its character at those points.

***Solution:*** Differentiating the given function we get

$$f'(x) = x^{2/3} + (x-1)\frac{2}{3}x^{-1/3} = \frac{5x-2}{3x^{1/3}}$$

Now

$$f'(x) = 0 \Rightarrow x = \frac{2}{5}$$

Also $f'(0)$ does not exist, but $f$ is continuous at $x = 0$. Therefore, the critical points of $f$ are 2/5 and 0. We now investigate the character of these points.

**1.** For $x < 2/5$, $f'(x)$ sign $= \frac{-}{+} < 0$.

For $x > 2/5$, $f'(x)$ sign $= \frac{+}{+} > 0$.

Therefore $f$ is minimum at $x = 2/5$ and the minimum value of $f$ is

$$f\left(\frac{2}{5}\right) = \frac{-3}{5}\left(\frac{4}{25}\right)^{1/3}$$

**2.** For $x < 0$, $f'(x)$ sign $= \frac{-}{-} > 0$.

For $x > 0$, $f'(x)$ sign $= \frac{-}{+} < 0$.

Therefore $f$ is maximum at $x = 0$ and the maximum value of $f$ is $f(0) = 0$.

***Note:*** The maximum value at a point may be less than the minimum value at another point (see Fig. 3.7).

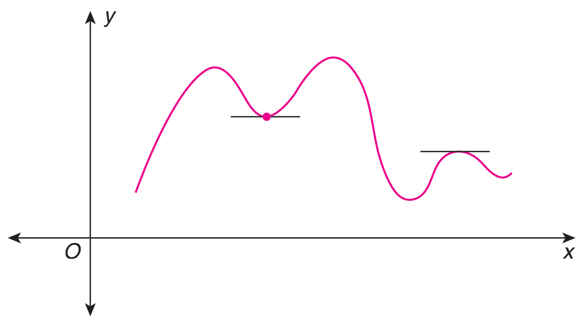



FIGURE 3.7

**THEOREM 3.9 (SECOND DERIVATIVE TEST)** Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and $a < x_0 < b$. Suppose that $f'(x_0) = 0$, $f''(x_0)$ exists and $f''(x_0) \neq 0$. Then

**(i)** $f$ has local maximum at $x_0$ if $f''(x_0) < 0$.

**(ii)** $f$ has local minimum at $x_0$ if $f''(x_0) > 0$

[Assuming that $f''(x_0)$ is continuous in some small neighbourhood of $x_0$.]

**PROOF** **(i)** Suppose $f''(x_0) < 0$. Since $f''(x)$ is continuous in some small neighbourhood of $x_0$, we can choose a small closed interval containing $x_0$ and that is also contained in the small neighbourhood of $x_0$ in which $f''(x) < 0$ at all points of this closed interval.

So, $f'(x)$ decreases in this closed interval (Theorem 3.6). But $f'(x_0) = 0$. Therefore, $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$. Thus, the derivative $f'(x)$ changes sign for *plus to minus* at $x_0$. Hence by the first derivative test, $f$ is maximum at $x_0$.

**(ii)** Similarly, we can prove that $f$ has minimum at $x_0$ if $f''(x_0) > 0$. ■

**QUICK LOOK 2**

1. $f''(x_0) < 0 \Rightarrow f$ has local maximum at $x_0$.
2. $f''(x_0) > 0 \Rightarrow f$ has local minimum at $x_0$.
3. If $f''(x_0) = 0$, then nothing can be said about the character of $x_0$.

## Example 3.11

Let $f(x) = \frac{x^3}{3} - 2x^2 + 3x + 1$. Find its points of maximum and minimum.

***Solution:*** Differentiating the given function we get

$$f'(x) = x^2 - 4x + 3 = (x-1)(x-3)$$

Now

$$f'(x) = 0 \Rightarrow x = 1, 3.$$

Again

$$f''(x) = 2x - 4$$

Therefore

**1.** $f''(1) = 2 - 4 = -2 < 0 \Rightarrow f$ has maximum at $x = 1$.
**2.** $f''(3) = 6 - 4 = 2 > 0 \Rightarrow f$ has minimum at $x = 3$.

## Example 3.12

Find the points of maximum and minimum for $f(x) = x \log x$.

***Solution:*** Note that $f$ is defined for all $x > 0$. Differentiating we get

$$f'(x) = \log x + 1$$

Now

$$f'(x) = 0 \Rightarrow x = \frac{1}{e}$$

$$f''(x) = \frac{1}{x} \Rightarrow f'' = \left(\frac{1}{e}\right) = e > 0$$

Therefore, $f$ has minimum at $x = 1/e$ and the minimum value of $f$ is

$$f\left(\frac{1}{e}\right) = \frac{1}{e}\left(\log\frac{1}{e}\right) = -\frac{1}{e}$$

## Example 3.13

Consider the function

$$f(x) = \begin{cases} 2^x + 1 & \text{if } -1 \le x < 0 \\ 2^x & \text{if } x = 0 \\ 2^x - 1 & \text{if } 0 < x \le 1 \end{cases}$$

Show that $f$ has neither maximum nor minimum.

***Solution:*** Note that $f$ is discontinuous at $x = 0$, because

$$\lim_{x \to 0-0} f(x) = 2^0 + 1 = 2 \ne f(0)$$

Now, for $-1 \le x < 0$, we have $f'(x) = 2^x \log 2 > 0$ so that $f$ is strictly increasing in $(-1, 0)$. For $0 < x < 1$,

$$f'(x) = 2^x \log 2 > 0$$

so that $f$ is also strictly increasing in $(-1, 0)$. Hence $f$ has no extremum value in $(-1, 1)$.

## Example 3.14

Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers. Let $f(x) = \sum_{i=1}^{n} (x - \alpha_i)^2$ for all $x \in \mathbb{R}$.

Show that the only point of extremum of $f$ is

$$\alpha = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}$$

***Solution:*** Differentiating the given function we get

$$f'(x) = \sum_{i=1}^{n} 2(x - \alpha_i)$$

so that

$$f'(x) = 0 \Rightarrow x = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}$$

Let

$$\alpha = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}$$

It is easy to see that

1. $x < \alpha \Rightarrow f'(x) < 0$
2. $x > \alpha \Rightarrow f'(x) > 0$
3. $f'(\alpha) = 0$

This shows that $f$ decreases in $(-\infty, 0)$ and increases in $(0, \infty)$. Thus $f$ is minimum at $x = \alpha$.

**DEFINITION 3.11** **Absolute Maximum and Absolute Minimum**

1. If a function $f$ has a point $\alpha$ in its domain such that $f(x) \le f(\alpha)\ \forall\, x$ in the domain, then $f(\alpha)$ is called the *absolute maximum value* or greatest value of $f$.
2. If $f$ has a point $\beta$ in its domain such that $f(x) \ge f(\beta)\ \forall\, x$ in the domain, then $f(\beta)$ is called the *absolute minimum value* or the least value of $f$.

**THEOREM 3.10 (DARBOUX THEOREM OR INTERMEDIATE VALUE THEOREM FOR THE DERIVATIVE)**

If $f$ is differentiable on closed interval $[a, b]$, then $f'$ assumes every value between $f'(a)$ and $f'(b)$. Here we consider right derivative $f'(a+0)$ at $a$ and left derivative $f'(b-0)$ at $b$.

**PROOF** We may suppose that $f'(a) < f'(b)$. Let $f'(a) < \lambda < f'(b)$. Define $g(x) = \lambda x - f(x)$ for all $x \in [a, b]$. Since $g$ is continuous on $[a, b]$, by Theorem 1.33, $g$ assumes its maximum value at some $c \in [a, b]$. Further,

$$g'(x) = \lambda - f'(x)$$

so that

$$g\,(a) = \lambda - f'(a) > 0$$

and

$$g'(b) = \lambda - f'(b) < 0$$

Since

$$0 < g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

there exists $\delta > 0$ such that $(a, a+\delta) \subset (a, b)$ and

$$x \in (a, a+\delta) \Rightarrow \frac{g(x) - g(a)}{x - a} > 0$$
$$\Rightarrow g(x) > g(a)$$

Similarly, $(b-\delta, b) \subset (a, b)$ such that $x \in (b-\delta, b) \Rightarrow g(x) > g(b)$. Hence neither $a$ nor $b$ is a point of maximum for $g$. Therefore, there exists $c \in (a, b)$ at which $g$ is maximum. Therefore

$$g'(c) = 0 \Rightarrow \lambda - f'(c) = 0 \quad \text{or} \quad \lambda = f'(c)$$

***Note:*** Darboux theorem can be used to find functions which are not derivatives of any functions. For this consider the following example.

## Example

Define

$$g(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/2 \\ 1 & \text{if } 1/2 \le x \le 1 \end{cases}$$

We show that *there exists no function* $f$ on $[0, 1]$ such that $f'(x) = g\,(x)$ for all $x \in [0, 1]$. For this, suppose $f$ is a function defined on $[0, 1]$ such that $f' = g$. Since $0 < 1/4 < 1$, by Darboux theorem

$$f'(x) = \frac{1}{4} \quad \text{for some } x \in (0, 1)$$

But

$$f'(x) = g(x) = 0 \text{ or } 1$$

which is a contradiction.

**Corollary 3.4** If $f$ is differentiable on a closed interval $[a, b]$ and its derivative $f'$ changes sign in $[a, b]$, then $f'(x) = 0$ for some $x \in [a, b]$.

**Proof** Suppose $\alpha, \beta \in [a, b]$ and $f'(\alpha) \cdot f'(\beta) < 0$. Since zero lies between $f'(\alpha)$ and $f'(\beta)$, by Darboux theorem, there exists $x$ between $\alpha$ and $\beta$ such that $f'(x) = 0$. ■

***Note:*** In an interval $[a, b]$, the derivative of a function may not vanish at any point and hence the derivative keeps the same sign throughout the interval. Thus the function is either increasing or decreasing in the interval, so that maximum and minimum values occur at the end points of the interval $[a, b]$.

Keeping the above note in mind the preceding information about the extremum values of a function, in the following subsection we formulate, how to find the greatest and least values of a function on a given interval.

## 3.4.2 Deriving the Greatest and Least Values of a Function on a Given Interval

**PROCEDURE**

Suppose a function $f(x)$ is defined and continuous on a closed interval $[a, b]$ where $a$ and $b$ are finite real numbers. To find the greatest and least values of the function, it is necessary to find all local maxima (local minima) of the function $f(x)$ in the open interval $(a, b)$. Let $x_1, x_2, x_3, \ldots, x_n$ be all the critical points in $(a, b)$ at which $f(x)$ has *extremum value*. To this list add $a$ and $b$. Now let

$$S = \{f(a), f(x_1), f(x_2), \ldots, f(x_n), f(b)\}$$

Then the *greatest element* of $S$ is precisely the greatest value of $f(x)$ on $[a, b]$ and the *least element* of $S$ is precisely the least value of $f(x)$ on $[a, b]$.

### Example 3.15

Let $f(x) = 3x^4 + 4x^3 + 1, x \in [-2, 1]$. Find its greatest and least values.

***Solution:*** Since $f$ is continuous and differentiable for all real $x$, it is continuous and differentiable on $[-2, 1]$. Therefore, the only critical points of $f$ are the roots of $f'(x) = 0$. So

$$\begin{aligned} f'(x) = 0 &\Rightarrow 12x^3 + 12x^2 = 0 \\ &\Rightarrow x = 0, -1 \end{aligned}$$

Now at $x = 0$, $f'(x) = 12x^2(x+1)$ keeps the same sign +. Hence $x = 0$ is not a point of extremum.

Again at $x = -1$, $f'(x) = x^2(x+1)$ changes sign from minus to plus. Hence $f$ is minimum at $x = -1$ and $f(-1) = 0$. Also $f(-2) = 17$ and $f(1) = 8$. Now

$$S = \{f(-2) = 17, f(-1) = 0, f(1) = 8\}$$

Therefore, the greatest value of $f$ is 17 and the least value is zero on $[-2, 1]$.

### Example 3.16

Let $f(x) = 4x^3 - x|x-2|, x \in [0, 3]$. Find the points of maxima and minima as well as the maximum and minimum values.

***Solution:*** We can write the given function as

$$f(x) = \begin{cases} 4x^3 - x(2-x) = 4x^3 + x^2 - 2x & \text{for } 0 \le x < 2 \\ 4x^3 - x(x-2) = 4x^3 - x^2 + 2x & \text{for } 2 \le x \le 3 \end{cases}$$

So, $x = 2$ is a point of continuity of $f$. Now

$$f'(x) = \begin{cases} 12x^2 + 2x - 2 & \text{for } 0 \le x < 2 \\ 12x^2 - 2x + 2 & \text{for } 2 \le x \le 3 \end{cases}$$

Note that at $x = 2$, $f$ is not differentiable but in every neighbourhood of 2, contained in [0, 3], $f$ is differentiable. Thus, 2 is a critical point of $f$. Also for $0 \le x < 2$,

$$f'(x) = 0 \Rightarrow x = \frac{-1}{2}, \frac{1}{3}$$

But $-1/2 \notin [0, 3]$. Therefore $x = 1/3$ is a critical point and

$$f''\left(\frac{1}{3}\right) = 24\left(\frac{1}{3}\right) + 2 = 10 > 0$$

Hence $f$ is minimum at $x = 1/3$ and

$$f\left(\frac{1}{3}\right) = \frac{4}{27} + \frac{1}{9} - \frac{2}{3}$$

$$= \frac{4+3-18}{27}$$

$$= \frac{-11}{27}$$

Also for $x > 1/3$, $f'(x) > 0$ which means $f$ increasing from 1/3 to 3. Now

$$S = \left\{f(0) = 0,\ f\left(\frac{1}{3}\right) = \frac{-11}{27},\ f(3) = 105\right\}$$

Therefore $f$ is minimum at $x = 1/3$ and the minimum value is $-11/27$ and $f$ is maximum at $x = 3$ and the maximum value is 105.

***Note:*** Suppose $F(x) = g(f(x))$ where $f(x)$ and $g(y)$ are continuous functions on $[a, b]$ and $[c, d]$, respectively. Let $c = \underset{x\in[a,b]}{\text{Min}}\ f(x),\ d = \underset{x\in[c,d]}{\text{Max}}\ f(x)$, then

$$\underset{x\in[a,b]}{\text{Min}}\, F(x) = \underset{y\in[c,d]}{\text{Min}}\ g(y)$$

and

$$\underset{x\in[a,b]}{\text{Max}}\ F(x) = \underset{y\in[c,d]}{\text{Max}}\ g(y)$$

That is we can put substitution $f(x) = y$. See the following example.

## Example 3.17

Let $F(x) = \dfrac{\sin 2x}{\sin[(\pi/4) + x]}$ for $x \in \left[0, \dfrac{\pi}{2}\right]$. Find the maximum and minimum values.

***Solution:*** We have

$$F(x) = \frac{2\sin x \cos x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)}$$

Now put $t = \sin x + \cos x$, so that

$$2 \sin x \cos x = t^2 - 1$$

Here $f(x) = \sin x + \cos x$ so that

$$x \in \left[0, \frac{\pi}{2}\right] \Rightarrow t \in [1, \sqrt{2}]$$

Therefore

$$F(x) = \frac{\sqrt{2}(t^2 - 1)}{t}$$

where $t = \sin x + \cos x = f(x)$. Hence $F(x)$ is of the form $g(f(x))$ where

$$f(x) = t = \sin x + \cos x$$

and

$$g(t) = \frac{\sqrt{2}(t^2 - 1)}{t}$$

Now

$$g'(t) = \sqrt{2}\left(1 + \frac{1}{t^2}\right) > 0$$

so that $g$ is increasing for $t \in [1, \sqrt{2}]$. Hence

$$\underset{x\in\left[0,\frac{\pi}{2}\right]}{\text{Max}}\ F(x) = \underset{t\in[1,\sqrt{2}]}{\text{Max}}\ g(t)$$

$$= g(\sqrt{2})$$

$$= \frac{\sqrt{2}(2-1)}{\sqrt{2}}$$

$$= 1$$

and

$$\underset{x\in\left[0,\frac{\pi}{2}\right]}{\text{Min}}\ F(x) = \underset{t\in[1,\sqrt{2}]}{\text{Min}}\ g(t)$$

$$= g(1)$$

$$= \frac{\sqrt{2}(1-1)}{1}$$

$$= 0$$

### Example 3.18

Let $f(x) = \tan x + \cot x,\ x \in [\pi/6, \pi/3]$. Find its least and greatest values.

***Solution:*** The given function can be written as

$$f(x) = \frac{1}{\sin x \cos x} = 2 \operatorname{cosec} 2x$$

Now,

$$f'(x) = -4 \operatorname{cosec} 2x \cot 2x = 0$$

at $x = \pi/4$. Also

$$f'(x) < 0 \text{ for } \frac{\pi}{6} < x < \frac{\pi}{4}$$

and

$$f'(x) > 0 \text{ for } \frac{\pi}{4} < x < \frac{\pi}{3}$$

Therefore $f$ is locally minimum at $x = \pi/4$ and $f(\pi/4) = 2$. Also

$$f\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} + \sqrt{3} = \frac{4}{\sqrt{3}} = f\left(\frac{\pi}{3}\right)$$

Therefore $f$ is maximum at $x = \pi/6$ and $\pi/3$ and maximum of $f(x) = 4/\sqrt{3}$.

Hence on the interval $[\pi/6, \pi/3]$, the least value of $f(x)$ is 2 and the greatest value of $f(x)$ is $4/\sqrt{3}$.

***Note:*** Sometimes, if the functional values are positive, the minimum value of a sum can be determined by using AM–GM inequality. In the above case, since both $\tan x$ and $\cot x$ are positive in $[\pi/6, \pi/3]$, we have

$$\tan x + \cot x \ge 2\sqrt{\tan x \cot x} = 2$$

and the equality occurs when $\tan x = \cot x$, which implies that $x = \pi/4$.

## 3.5 | Convexity, Concavity and Points of Inflection

The following concepts and theorems will be useful in drawing the curves of continuous and differentiable functions. The validity of the theorems is to be assumed.

**DEFINITION 3.12** **Convex** Let $f(x)$ be a continuous function defined on an interval $[a, b]$. If the graph of $f(x)$ lies below the line segment joining the points $(x, f(x))$ and $(y, f(y))$ for all $a \le x < y \le b$, then $f$ is said to be *convex* function on the interval $[a, b]$ and the curve said to be convex. See Fig. 3.8.

**DEFINITION 3.13** **Concave** If the curve always lies above the line segment joining $(x, f(x))$ and $(y, f(y))$, then $f$ is said to be a *concave* function on the interval $[a, b]$ and curve is said to be concave. See Fig. 3.9.

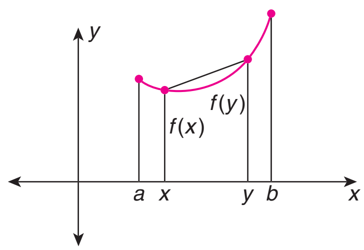



**FIGURE 3.8** Convex function and curve.

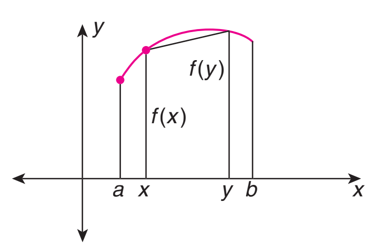



**FIGURE 3.9** Concave function and curve.

**THEOREM 3.11 (DERIVATIVE TEST FOR CONVEXITY)** Suppose $f$ is continuous on $[a, b]$ and differentiable in $(a, b)$. If $f'$ is increasing on $(a, b)$, then $f$ is convex on $[a, b]$. In particular, $f''$ exists and is positive (possibly at some points $f''$ may be zero) in $(a, b)$.

**THEOREM 3.12** Suppose $f$ is continuous on $[a, b]$ and is differentiable in $(a, b)$. If $f'$ is decreasing in $(a, b)$, then $f$ is concave in $(a, b)$. In particular, $f''$ is negative (possibly at some points $f''$ may be zero) in $(a, b)$.

***Note:*** If $f$ is convex, then $-f$ is concave.

**DEFINITION 3.14** **Point of Inflection** If $f''(a) = 0$ or $f''(a)$ does not exist and the second derivative $f''(x)$ changes sign at $x = a$, then $a$ is called a *point of inflection* of $f$. Geometrically, point of inflection means, the point which separates convex and concave parts of the curve (see Fig. 3.10).

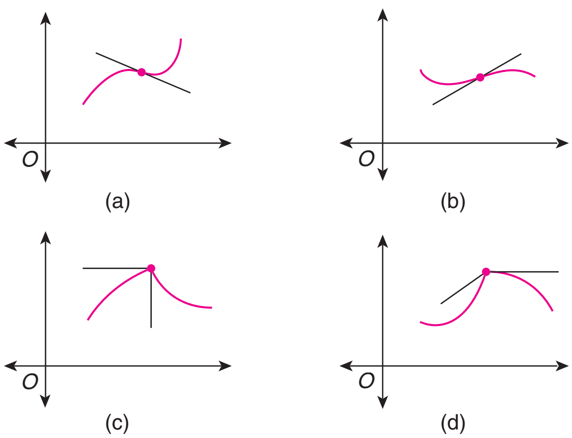



**FIGURE 3.10** Various graphs to show points of inflection.

## Example 3.19

Find out whether the following curves are convex or concave or both.

1. $f(x) = 3 - x^2$
2. $f(x) = x^3$
3. $f(x) = \cos x$
4. $f(x) = x^4$

***Solution:***

1. The given function is $f(x) = 3 - x^2$. The second de rivative is
$$f''(x) = -2 < 0$$
Therefore, $f$ is concave everywhere.

2. The given function is $f(x) = x^3$. The second derivative is
$$f''(x) = 6x$$
This implies
$$f''(x) < 0 \quad \text{for } x < 0$$
$$f''(x) > 0 \quad \text{for } x > 0$$
Thus $f$ is concave for $x < 0$ and convex for $x > 0$. See Fig. 3.11. Here, observe that $f''(0) = 0$ and $f''(x)$ changes sign at $x = 0$ and hence $x = 0$ is a point of inflection also.

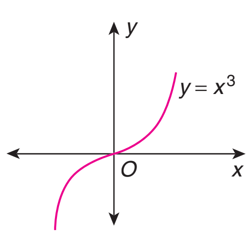



**FIGURE 3.11** Part (2), Example 3.19.

3. The given function is $f(x) = \cos x$. The second derivative is
$$f''(x) = -\cos < 0 \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$
Therefore $f(x)$ is concave in $[-\pi/2, \pi/2]$. Now
$$f''(x) = -\cos x > 0 \quad \text{for } \frac{\pi}{2} < x < \frac{3\pi}{2}$$
This implies $f$ is convex in $[\pi/2, 3\pi/2]$.

4. The given function is $f(x) = x^4$. (See Fig. 3.12.) The second derivative is
$$f''(x) = 12x^2 > 0 \ \forall\, x \neq 0$$
Hence the curve is convex on any interval. Further $f''(x) = 0$, but $f''(x)$ does not change its sign at $x = 0$. Hence $x = 0$ is not a point of inflection.

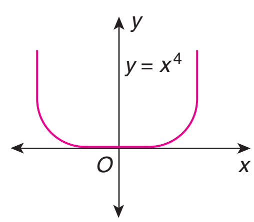



**FIGURE 3.12** Part (4), Example 3.19.

## 3.6 | Cauchy's Mean Value Theorem and L'Hospital's Rule

In this section we prove Cauchy's mean value theorem which is an extension of Lagrange's mean value theorem for two functions and prove the most famous theorem – L'Hospital's rule.

### 3.6.1 Cauchy's Mean Value Theorem

**Theorem 3.13 (Cauchy's Mean Value Theorem)** Suppose $f$ and $g$ are two functions such that

**(i)** both are continuous on $[a, b]$ and differentiable in $(a, b)$
**(ii)** $g(a) \neq g(b)$
**(iii)** $|f'(x)| + |g'(x)| \neq 0 \ \forall x \in (a, b)$

Then, there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

**Proof** Let

$$h(x) = f(x) + \frac{f(b) - f(a)}{g(b) - g(a)}(g(b) - g(x))$$

Then $h$ is continuous on $[a, b]$ and differentiable in $(a, b)$. Also $h(a) = f(b) = h(b)$. Hence, by Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. Therefore

$$f'(c) = -\left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)[-g'(c)]$$

$$= \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(c)$$

If $g'(c) = 0$, then $f'(c) = 0$ which contradicts hypotheses (iii). Therefore $g'(c) \neq 0$. Hence

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$ ■

***Note:*** Lagrange's mean value theorem is a special case of Cauchy's mean value theorem, if we take $g(x) = x \ \forall x \in [a, b]$.

### 3.6.2 L'Hospital's Rule

Suppose functions $f$ and $g$ are defined in a neighbourhood of "$c$". If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ are both either 0 or $\pm\infty$, then $\lim_{x \to c} f(x)/g(x)$ is called an *indeterminate form*. This limit of the ratio may exist or may not exist.

**DEFINITION 3.15** Suppose we write $\alpha = \lim_{x \to c} f(x)$ and $\beta = \lim_{x \to c} g(x)$.

1. If $\alpha = 0 = \beta$ or $\alpha = \pm\infty$ and $\beta = \pm\infty$, then we say that $\lim_{x \to c} \frac{f(x)}{g(x)}$ is the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
2. If $\alpha = \beta = \pm\infty$, then we say that $\lim_{x \to c} (f(x) - g(x))$ is the indeterminate $\infty - \infty$.

In the following theorem, we prove the most primitive form of L'Hospital's rule which will be frequently used to find the limit of the indeterminate $\frac{0}{0}$.

**THEOREM 3.14** Suppose $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$. Then $\lim_{x\to a} f(x)/g(x)$ exists and is equal to $f'(a)/g'(a)$.

**PROOF** Since $f'(a)$ and $g'(a)$ exist, $f(x)$ and $g(x)$ are continuous at $a$. Hence, by hypothesis

$$f(a) = g(a) = 0$$

Also

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

and

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a}$$

Therefore

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f(x)}{x-a} \cdot \frac{x-a}{g(x)} \\ &= \frac{f(x)-f(a)}{x-a} \cdot \frac{x-a}{g(x)-g(a)} \qquad [\because f(a) = g(a) = 0]\end{aligned}$$

This implies

$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} \frac{x-a}{g(x)-g(a)} \\ &= \frac{f'(a)}{g'(a)}\end{aligned}$$

One has to observe that

$$g'(a) = \lim_{x\to a} \frac{g(x)}{x-a}$$

and $g'(a) \neq 0$ implies that $g(x) \neq 0$ for all $x$ in a neighbourhood of $a$. ■

The following theorem is another form of L'Hospital's rule which will be also useful in practical problems. In the first reading, the student may skip the proof and assume its validity.

**THEOREM 3.15 (ANOTHER FORM OF L'HOSPITAL'S RULE)** If $f$ and $g$ are differentiable in a deleted neighbourhood of $a$, are continuous at $a$, $f(a) = g(a) = 0$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and is equal to $l$.

**PROOF** Since $f'(x)/g'(x) \to l$ as $x \to a+0$, there exists a positive number $\lambda$ such that $0 < h < \lambda$, both $f$ and $g$ are differentiable and $g'(x) \neq 0$ for all $x$ in $(a, a+h)$. If $g(x) = g(a) (= 0)$ for some $x$ in $(a, a+h)$, then by Rolle's theorem $g'(x_0) = 0$ for some $x_0$ in $(a, a+h)$ which is a contradiction to the fact that $g'(x) \neq 0$ in any $(a, a+h)$. Hence $g(x) \neq g(a)$ for any $x \in (a, a+h)$. Thus, $f$ and $g$ satisfy the conditions of Cauchy's mean value theorem on $[a, a+h]$ so that there exists $c$ in $(a, a+h)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f(a+h)}{g(a+h)} \tag{3.4}$$

Let $\varepsilon > 0$. Since

$$\frac{f'(x)}{g'(x)} \to l \quad \text{as } x \to a+0$$

there exists a positive number $\delta(\varepsilon)$ such that $x \in (a, a+\delta(\varepsilon))$ and

$$\left|\frac{f'(x)}{g'(x)} - l\right| < \varepsilon$$

Choose $\delta = \text{Min}(\lambda, \delta(\varepsilon))$. Therefore for any $x \in (a, \delta)$, by Eq. (3.4) we have

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

for some $c$ in $(a, x)$. Hence

$$\left|\frac{f(x)}{g(x)} - l\right| = \left|\frac{f'(c)}{g'(c)} - l\right| < \varepsilon$$

because $0 < c - a < x - a < \delta$. Therefore

$$\frac{f(x)}{g(x)} \to l \quad \text{as } x \to a + 0$$

Similarly

$$\frac{f(x)}{g(x)} \to l \quad \text{as } x \to a - 0$$

Hence

$$\lim_{x \to a} \frac{f(x)}{g(x)} = l$$

■

Now we state (without proofs) two theorems which comprise all forms of L'Hospital's rule.

**Theorem 3.16 (L'Hospital's Rule – I)**

**(i)** Suppose $f$ and $g$ are differentiable in a right neighbourhood $V = (c, c + \delta)$ of $c$, $g'(x) \neq 0$ in $V$,

$$\lim_{x \to c+0} f(x) = 0 = \lim_{x \to c+0} g(x)$$

Then

$$\lim_{x \to c+0} \frac{f'(x)}{g'(x)} = \lim_{x \to c+0} \frac{f(x)}{g(x)}$$

(Here the limit may be real number or $+\infty$ or $-\infty$.)

***Note:*** The above result is equally valid, if the right neighbourhood is replaced by left neighbourhood.

**(ii)** Suppose $f$ and $g$ are defined and differentiable in $(-\infty, b)$, $g'(x) \neq 0$ in $(-\infty, b)$ and

$$\lim_{x \to -\infty} f(x) = 0 = \lim_{x \to -\infty} g(x)$$

Then

$$\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to -\infty} \frac{f(x)}{g(x)}$$

(Here the limit may be a real number or $+\infty$ or $-\infty$.)

**(iii)** Suppose $f$ and $g$ are defined and differentiable in $(a, \infty)$, $g'(x) \neq 0$ in $(a, \infty)$ and

$$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x)$$

Then

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

(Here the limit may be a real number or $+\infty$ or $-\infty$.)

**Theorem 3.17 (L'Hospital's Rule – II)**

**(i)** Suppose $f$ and $g$ are differentiable in a neighbourhood $V = (a-\delta, a+\delta)$ of "$a$", $g'(x) \neq 0\ \forall\, x \in V$ and $\lim\limits_{x \to a} g(x) = \pm\infty$. Then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

(Here the limit may be a real number or $+\infty$ or $-\infty$.)

**(ii)** Suppose $f$ and $g$ are defined and differentiable in $(-\infty, b)$, $g'(x) \neq 0$ in $(-\infty, b)$ and $\lim\limits_{x \to \infty} g(x) = \pm\infty$. Then

$$\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to -\infty} \frac{f(x)}{g(x)}$$

(Here the limit may be a real number or $+\infty$ or $-\infty$.)

**(iii)** Suppose $f$ and $g$ are defined and differentiable in $(a, \infty)$, $g'(x) \neq 0$ in $(a, \infty)$ and $\lim\limits_{x \to \infty} g(x) = \pm\infty$. Then

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)}$$

(Here the limit may be a real number or $+\infty$ or $-\infty$.)

***Note:*** The indeterminate forms of types $1^\infty, 0^0, \infty^0$, etc. can be reduced to the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ by using logarithms, exponentials, etc.

The following are few examples of the above results.

## Example 3.20

Let $f(x) = e^x - 1$ and $g(x) = x$ in $[0, \infty]$. Find $\lim\limits_{x \to 0+} f(x)/g(x)$.

***Solution:*** We have by L'Hospital's rule – I, part (i),

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \left(\frac{e^x}{1}\right) = 1$$

Note that $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$ is in the indeterminate form $\dfrac{0}{0}$.

## Example 3.21

Let $f(x) = \log x$ and $g(x) = x - 1$ in $(0, \infty)$. Find $\lim\limits_{x \to 1} f(x)/g(x)$.

***Solution:*** By L'Hospital's rule – I we have

$$\lim_{x \to 1} \left(\frac{\log x}{x - 1}\right)\left(\frac{0}{0}\text{form}\right) = \lim_{x \to 1} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 1} \frac{1/x}{1} = 1$$

## Example 3.22

Let $f(x) = \log x$ and $g(x) = x$ for $x \in (1, \infty)$. Find $\lim_{x\to\infty} f(x)/g(x)$.

***Solution:*** By L'Hospital's rule – II, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to\infty} \frac{1/x}{1} = 0$$

## Example 3.23

Let $f(x) = x^2$ and $g(x) = e^x$. Find $\lim_{x\to\infty} f(x)/g(x)$.

***Solution:*** By L'Hospital's rule – II, we have

$$\lim_{x\to\infty}\left(\frac{x^2}{e^x}\right) = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to\infty} \frac{2x}{e^x}$$

$$= \lim_{x\to\infty} \frac{2}{e^x} \quad \text{(By L'Hospital's rule)}$$

$$= 0$$

## Example 3.24

Find $\lim_{x\to 0+0} (x \log x)$

***Solution:*** This limit is of the form $0 \times (-\infty)$. Take $f(x) = \log x$ and $g(x) = 1/x$ so that

$$x \log x = \frac{f(x)}{g(x)}$$

Then,

$$\lim_{x\to 0+0} (x\log x) = \lim_{x\to 0+0} \frac{f(x)}{g(x)}$$

$$= \lim_{x\to 0+0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0+0} \frac{1/x}{-1/x^2}$$

$$= 0$$

## Example 3.25

Find $\lim_{x\to 0+0} (x^x)$.

***Solution:*** This of the form $0^0$. We have

$$\lim_{x\to 0+0} (x^x) = \lim_{x\to 0+0} e^{x\log x}$$

$$= \exp\{\lim_{x\to 0+0} (x\log x)\}$$

$$= e^0 \quad \text{(by Example 3.24)}$$

$$= 1$$

Thus $\lim_{x\to 0+0} x^x = 1$.

## Example 3.26

Find $\lim_{x\to 0+0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$ for $x \in \left(0, \frac{\pi}{4}\right)$.

***Solution:*** This limit is of the form $\infty - \infty$. Now

$$\lim_{x\to 0+0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0+0} \frac{\sin x - x}{x \sin x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0+0} \frac{(\cos x - 1)}{\sin x + x\cos x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0+0}\left(\frac{-\sin x}{2\cos x - x\sin x}\right)$$

$$= \frac{0}{2} = 0$$

## Example 3.27

Show that $\lim\limits_{x\to 0+0} \dfrac{\sin x}{\sqrt{x}} = 0.$

***Solution:*** Write $f(x) = \sin x$ and $g(x) = \sqrt{x}$, $x \in (0, \infty)$. By L'Hospital's rule – II,

$$\lim_{x\to 0+0} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0+0} \frac{f(x)}{g(x)}$$
$$= \lim_{x\to 0+0} \frac{\cos x}{1/2\sqrt{x}}$$
$$= \lim_{x\to 0+0} (2\sqrt{x}\cos x)$$
$$= 0$$

Note

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

while

$$\lim_{x\to 0+0} \frac{\sin x}{\sqrt{x}} = 0$$

## Example 3.28

Show that $\lim\limits_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$.

***Solution:*** We have

$$\lim_{x\to 0} (\cos x)^{1/x^2} = (1^{\infty})$$
$$= \exp\{\lim_{x\to 0}(1/x^2)\log(\cos x)\}$$
$$= \exp\left(\lim_{x\to 0} \frac{\log(\cos x)}{x^2}\right)\left(\frac{0}{0}\right)$$
$$= \exp\left(\lim_{x\to 0}\left(\frac{-\tan x}{2x}\right)\right)\left(\frac{0}{0}\right)$$
$$= \exp\left((-1/2)\lim_{x\to 0}\left(\frac{\tan x}{x}\right)\right)$$
$$= e^{-1/2}$$

## Example 3.29

Show that $\lim\limits_{x\to 0}\left(\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right)(\infty - \infty) = \dfrac{-1}{2}$

***Solution:*** We have

$$\lim_{x\to 0}\left(\frac{1}{e^x-1} - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x - e^x + 1}{xe^x - x}\right)\left(\frac{0}{0}\right)$$
$$= \lim_{x\to 0}\left(\frac{1 - e^x}{e^x + xe^x - 1}\right)\left(\frac{0}{0}\right)$$
$$= \lim_{x\to 0}\left(\frac{-e^x}{2e^x + xe^x}\right)$$
$$= \frac{-1}{2}$$

Note that $2e^x + xe^x \neq 0$ for $x > -2$.

# WORKED-OUT PROBLEMS

## *Single Correct Choice Type Questions*

## Tangents and Normals

**1.** If the line $lx + my = 1$ is normal to the curve $y^2 = 4ax$, then $l^3 a + 2alm^2$ is equal to

(A) $m^2$ (B) $m$

(C) $2m$ (D) $-m^2$

**Solution:** Suppose the line $lx + my = 1$ is normal to the curve at $(x_1, y_1)$. Then

$$\left.\begin{aligned} lx_1 + my_1 &= 1 \\ y_1^2 &= 4ax_1 \end{aligned}\right\} \quad (3.5)$$

Differentiating $y^2 = 4ax$, we get

$$\frac{dy}{dx} = \frac{2a}{y}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2a}{y_1}$$

Equation of normal at $(x_1, y_1)$ to $y^2 = 4ax$ is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

that is

$$xy_1 + 2ay - (2ay_1 + x_1y_1) = 0$$

is normal at $(x_1, y_1)$. But $lx + my = 1$ is normal at $(x_1, y_1)$. Therefore

$$\frac{y_1}{l} = \frac{2a}{m} = \frac{2ay_1 + x_1y_1}{1}$$

Solving we get

$$y_1 = \frac{2al}{m} \quad \text{and} \quad x_1 = \frac{1}{l} - 2a = \frac{1-2al}{l}$$

From Eq. (3.5) we have $y_1^2 = 4ax_1$. Therefore

$$\frac{4a^2l^2}{m^2} = 4a\frac{(1-2al)}{l}$$

$$\Rightarrow l^3a = m^2 - 2alm^2$$

$$\Rightarrow l^3a + 2alm^2 = m^2$$

**Answer: (A)**

**2.** The distance of any normal to the curve represented parametrically by the equations $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, $a > 0$ from the origin is

(A) $\frac{a}{2}$ (B) $a^2$
(C) $a$ (D) $\frac{a^2}{2}$

**Solution:** Differentiating the given equations we get

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$$

We can suppose that $\cos\theta \neq 0$ so that

$$\left(\frac{dy}{dx}\right)_\theta = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \tan\theta$$

Therefore the equation of the normal $\theta$ is

$$y - a(\sin\theta - \theta\cos\theta) = -\cot\theta[x - a(\cos\theta + \theta\sin\theta)]$$

On simplification, we have $x\cos\theta + y\sin\theta = a$ whose distance from $(0, 0)$ is $a$.

**Answer: (C)**

**3.** The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = 2$ on the axes of coordinates is

(A) 2 (B) 4
(C) 1 (D) $2\sqrt{2}$

**Solution:** Observe that both $x$ and $y$ are positive. Differentiating the given equation w.r.t. x, we have

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\sqrt{\frac{y_1}{x_1}}$$

Now equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = -\sqrt{\frac{y_1}{x_1}}(x - x_1)$$

$$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = 2$$

Therefore the sum of the intercepts is

$$2\sqrt{x_1} + 2\sqrt{y_1} = 2(\sqrt{x_1} + \sqrt{y_1}) = 2 \times 2 = 4$$

**Answer: (B)**

**4.** The number of points on the curve $y^2 = x^3$ at which the normal makes intercepts on coordinates whose lengths are numerically equal, is

(A) 4 (B) 6
(C) 3 (D) 2

**Solution:** Suppose $(x_1, y_1)$ is a point on the curve at which the normal makes intercepts on the axes that are numerically equal. Differentiating $y^2 = x^3$ w.r.t. $x$ we get

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

so that

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1^2}{2y_1}$$

Therefore

$$\text{Slope of the normal at } (x_1, y_1) = \frac{-2y_1}{3x_1^2}$$

Since the intercepts of the normal are numerically equal, we have slope $= \pm 1$. Therefore

$$\frac{2y_1}{3x_1^2} = \pm 1$$

$$\Rightarrow 2y_1 = \pm 3x_1^2$$

$$\Rightarrow 2x_1^{3/2} = \pm 3x_1^2$$

$$\Rightarrow x_1 = \frac{4}{9}$$

This gives

$$y_1 = \pm\frac{8}{27}$$

Since the curve is symmetric about $x$-axis, the points are $\left(\frac{4}{9}, \frac{8}{27}\right)$ or $\left(\frac{4}{9}, \frac{-8}{27}\right)$.

**Answer: (D)**

**5.** Let $P$ be a point on the curve $y = x^2 - x + 1$ and $Q$ be a point on the curve $y = x^3 - x^2 - 2x + 1$. Suppose the tangent at $P$ to curve $y = x^2 - x + 1$ is parallel to the tangent at $Q$ to the curve $y = x^3 - x^2 - 2x + 1$. Then, the number of such ordered pair of points $(P, Q)$ is

(A) 2 (B) 4
(C) 3 (D) infinite

**Solution:** Differentiating $y = x^2 - x + 1$ w.r.t. $x$, we get

$$\frac{dy}{dx} = 2x - 1$$

Therefore

$$\left(\frac{dy}{dx}\right)_{P(x_1, y_1)} = 2x_1 - 1$$

From $y = x^3 - x^2 - 2x + 1$,

$$\left(\frac{dy}{dx}\right)_{Q(x_2, y_2)} = 3x_2^2 - 2x_2 - 2$$

Therefore

$$\left(\frac{dy}{dx}\right)_P = \left(\frac{dy}{dx}\right)_Q$$

$$\Rightarrow 2x_1 = 3x_2^2 - 2x_2 - 1$$

$$\Rightarrow 3x_2^2 - 2x_2 - (1 + 2x_1) = 0$$

So, there will be infinitely many values for $x_1$ such that the above quadratic equation in $x_2$ has real solutions, because its discriminant

$$4 + 12(1 + 2x_1) > 0 \text{ for all } x_1 > \frac{-2}{3}$$

**Answer: (D)**

**6.** The number of common points to the curves $y = \frac{1}{1+x}$ and $y = \frac{1}{1+x^2}$ at which the tangent to the second curve is horizontal is

(A) 1 (B) 2
(C) infinite (D) 0

**Solution:** We have

$$\frac{1}{1+x} = \frac{1}{1+x^2}$$

$$\Rightarrow x^2 - x = 0$$

$$\Rightarrow x = 0, 1$$

Therefore the common points are $(0, 1)$ and $(1, 1/2)$. Now

$$y = \frac{1}{1+x} \Rightarrow \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(0, 1)} = -1$$

Again

$$y = \frac{1}{1+x^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(0, 1)} = 0$$

Therefore, at $(0, 1)$ the tangent to the second curve is horizontal.

**Answer: (A)**

**7.** The length of the normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \pi/2$ is

(A) $a$ (B) $a\sqrt{2}$
(C) $2a$ (D) $a^2$

**Solution:** Differentiating the given equations we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

So

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = 1$$

and ordinate of the point $\theta = \pi/2$ is $a$. Therefore, length of the normal is

$$a\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{\theta = \frac{\pi}{2}}} = a\sqrt{1+1} = a\sqrt{2}$$

**Answer: (B)**

**8.** The tangent at $(x_0, y_0)$ to the curve $x^3 + y^3 = a^3$ meets the curve again at $(x_1, y_1)$, then

$$\frac{x_1}{x_0} + \frac{y_1}{y_0} =$$

(A) 1 (B) $-1$

(C) $a$ (D) $-a$

**Solution:** The given equation is $x^3 + y^3 = a^3$. Differentiating both sides w.r.t. $x$, we have

$$\frac{dy}{dx} = \frac{-x^2}{y^2}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)} = \frac{-x_0{}^2}{y_0{}^2} \tag{3.6}$$

Equation of the tangent at $(x_0, y_0)$ is

$$y - y_0 = \frac{-x_0^2}{y_0^2}(x - x_0)$$

$$\Rightarrow x_0^2 x + y_0^2 y = x_0^3 + y_0^3$$

This passes through $(x_1, y_1)$. Therefore

$$\begin{aligned} x_0^2 x_1 + y_0^2 y_1 &= x_0^3 + y_0^3 \\ &= a^3 \\ &= x_1^3 + y_1^3 \end{aligned}$$

as $(x_1, y_1)$ lies on the curve. Hence

$$x_1(x_0^2 - x_1^2) = -y_1(y_0^2 - y_1^2)$$

$$-\frac{x_1(x_0 + x_1)}{y_1(y_0 + y_1)} = \frac{y_0 - y_1}{x_0 - x_1}$$

= Slope of the line joining $(x_0, y_0)$ and $(x_1, y_1)$

which is the slope of the tangent at $(x_0, y_0)$. So from Eq. (3.6)

$$\frac{-x_1(x_0 + x_1)}{y_1(y_0 + y_1)} = \frac{-x_0{}^2}{y_0{}^2}$$

$$x_1 x_0 y_0^2 + x_1^2 y_0^2 = y_1 y_0 x_0^2 + y_1^2 x_0^2$$

$$x_0\, y_0(x_1 y_0 - x_0 y_1) = -(x_1 y_0 - x_0 y_1)\,(x_1 y_0 - x_0 y_1) \tag{3.7}$$

Suppose $x_1 y_0 - x_0 y_1 = 0$ so that

$$\frac{x_0}{x_1} = \frac{y_0}{y_1} = \lambda \quad \text{(say)}$$

Hence, $x_0 = \lambda x_1, y_0 = \lambda y_1$ so that

$$a^3 = x_0^3 + y_0^3 = \lambda^3(x_1^3 + y_1^3) = \lambda^3 a^3$$

which implies $\lambda = 1$ and hence $x_0 = x_1$ and $y_0 = y_1$, a contradiction. Therefore $x_1 y_0 - x_0 y_1 \neq 0$. Hence from Eq. (3.7),

$$x_0\, y_0 = -(x_1 y_0 + x_0 y_1)$$

Dividing both sides with $x_0 y_0$, we have

$$\frac{x_1}{x_0} + \frac{y_1}{y_0} = -1$$

**Answer: (B)**

**9.** The distance of the point on the curve $3x^2 - 4y^2 = 72$ nearest to the line $3x + 2y + 1 = 0$ is

(A) $\dfrac{11}{\sqrt{13}}$ (B) $\dfrac{9}{\sqrt{13}}$

(C) $\sqrt{13}$ (D) $\dfrac{\sqrt{13}}{11}$

**Solution:** We have to find points on the curve at which the tangents are parallel to the line $3x + 2y + 1 = 0$. Let $(x_1, y_1)$ be a point on the curve at which the tangent is parallel to the given line. Differentiating the curve equation w.r.t. $x$ we get

$$6x - 8y\frac{dy}{dx} = 0$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{3x_1}{4y_1}$$

The tangent is parallel to the line $3x + 2y + 1 = 0$. This implies

$$\frac{3x_1}{4y_1} = \frac{-3}{2}$$

$$\Rightarrow x_1 = -2y_1$$

$(x_1, y_1)$ lies on the curve. This implies

$$3x_1^2 - 4y_1^2 = 72$$

$$\Rightarrow 12y_1^2 - 4y_1^2 = 72$$

$$\Rightarrow y_1 = \pm 3$$

Therefore the points on the curve are $(6, -3)$ and $(-6, 3)$ whose distances from the line $3x + 2y + 1 = 0$, respectively, are $\sqrt{13}$ and $11/\sqrt{13}$. Therefore, the nearest point is $(-6, 3)$ and its distance from the given line is $11/\sqrt{13}$.

**Answer: (A)**

**10.** If the tangent at $(4t^2, 8t^3)$ to the curve $y^2 = x^3$ is also normal to the curve at some other point, then $t$ is equal to

(A) $\dfrac{2}{3}$ (B) $\dfrac{3}{2}$

(C) $\dfrac{\sqrt{2}}{3}$ (D) $\dfrac{\sqrt{3}}{2}$

**Solution:** Observe that $x = t^2, y = t^3$, $t \in \mathbb{R}$ are parametric equations of the curve $y^2 = x^3$. Now

$$2y\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(t^2, t^3)} = \frac{3t^4}{2t^3} = \frac{3}{2}t \tag{3.8}$$

and
$$\left(\frac{dy}{dx}\right)_{(4t^2, 8t^3)} = \frac{3(4t^2)^2}{2(8t^3)} = 3t \tag{3.9}$$

Therefore, equation of the tangent at $(4t^2, 8t^3)$ is

$$y - 8t^3 = 3t\,(x - 4t^2)$$

$$\Rightarrow y = 3tx - 4t^3$$

Putting this value of $y$ in the curve equation, we have

$$(3tx - 4t^3)^2 = x^3$$

$$x^3 - 9t^2x^2 + 24t^4x - 16t^6 = 0$$

for which $x - 4t^2$ is a factor. Therefore

$$(x - 4t^2)\,(x^2 - 5t^2x + 4t^4) = 0$$

$$(x - 4t^2)\,(x - 4t^2)\,(x - t^2) = 0$$

So, $x = 4t^2$ is a repeated root and $x = t^2$ is the other root. Therefore the tangent at $P(4t^2, 8t^3)$ meets the curve again at $(t^2, t^3)$ and $(t^2, -t^3)$. Now,

$$\text{Slope of the normal at } (t^2, t^3) = \frac{-2}{3t} \tag{3.10}$$

Equations (3.9) and (3.10) imply $3t = -2/3t$ which is not true. Therefore the point $(t^2, -t^3)$ is the point on the curve at which the normal is the tangent at $(4t^2, 8t^3)$. So

$$3t = \frac{2}{3t} \Rightarrow t^2 = 2/9 \Rightarrow t = \frac{\sqrt{2}}{3}$$

**Answer: (C)**

**11.** The equation of the normal to the curve

$$y = (1 + x)^y + \mathrm{Sin}^{-1}(\sin^2 x)$$

at $x = 0$ is

(A) $2x + y = 1$ (B) $2x - y + 1 = 0$
(C) $x - y + 1 = 0$ (D) $x + y - 1 = 0$

**Solution:** $x = 0 \Rightarrow y = 1$ so that the given point on the curve is $P(0, 1)$. Differentiating the curve equation w.r.t. $x$, we have

$$\frac{dy}{dx} = (1+x)^y\left[\frac{dy}{dx}\log(1+x) + \frac{y}{1+x}\right] + \frac{2\sin x\cos x}{\sqrt{1+\sin^4 x}}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(0,1)} = 1(0+1) + 0 = 1$$

So normal equation at $(0, 1)$ is

$$y - 1 = -1\,(x - 0)$$

or
$$x + y = 1$$

**Answer: (D)**

**12.** The sum of the ordinates of the points on the curve $3x^2 + y^2 + x + 2y = 0$ at which the tangents are perpendicular to the line $4x - 2y - 1 = 0$ is

(A) 2 (B) $-2$
(C) 1 (D) $-1$

**Solution:** Differentiating the curve equation we get

$$6x + 2y\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(6x+1)}{2(y+1)}$$

But the tangent is perpendicular to the line $4x - 2y - 1 = 0$. Therefore

$$-\frac{(6x+1)}{2(y+1)} = \frac{-1}{2} \Rightarrow y = 6x$$

Substituting the value of $y = 6x$ in the curve equation, we get

$$3x^2 + 36x^2 + x + 12x = 0$$

$$\Rightarrow 39x^2 + 13x = 0$$

$$\Rightarrow x = 0, \frac{-1}{3}$$

Thus, the points are $(0, 0)$ and $(-1/3, -2)$. Sum of the ordinates $= 0 - 2 = -2$.

**Answer: (B)**

**13.** The angle of intersection of the curves $x^2 + 4y^2 = 32$ and $x^2 - y^2 = 12$ at any point of their intersection is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

**Solution:** The given curves are

$$x^2 + 4y^2 = 32 \tag{3.11}$$

and $$x^2 - y^2 = 12 \tag{3.12}$$

Subtraction Eq. (3.12) from Eq. (3.11), we get

$$5y^2 = 20 \Rightarrow y = \pm 2$$

The points of intersection are (±4, ±2) (four points). Since both curves are symmetric about both axes, the angle of intersection is same at any of these points. Now from Eq. (3.11),

$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(4,2)} = \frac{-4}{8} = -\frac{1}{2}$$

From Eq. (3.12),

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(4,2)} = \frac{4}{2} = 2$$

Therefore product of the slopes of the tangents to the curves at (4, 2) is –1. Hence the angle of intersection of the curves is $\pi/2$.

**Answer: (D)**

**14.** The area of the triangle formed by the coordinate axes and the tangent to the curve $y = \log_e x$ at $(1, 0)$ is (in square unit)

(A) 1 (B) $\frac{1}{2}$

(C) 2 (D) $\frac{3}{2}$

**Solution:** We have

$$y = \log_e x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1$$

Therefore equation of the tangent at $(1, 0)$ is

$$y = x - 1$$

or $$x + y = 1$$

Area of the given triangle is

$$\frac{1}{2}(1)(1) = \frac{1}{2}$$

**Answer: (B)**

**15.** If the two curves $ax^2 + bx = 1$ and $a'x^2 + b'y^2 = 1$ intersect orthogonally, then

(A) $\frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'}$ (B) $\frac{1}{a} + \frac{1}{a'} = \frac{1}{b} + \frac{1}{b'}$

(C) $\frac{1}{a} - \frac{1}{b'} = \frac{1}{b} - \frac{1}{a'}$ (D) $\frac{1}{a} + \frac{1}{b} = \frac{1}{b'} + \frac{1}{a'}$

**Solution:** Suppose the curves

$$ax^2 + by^2 = 1 \tag{3.13}$$

and $$a'x^2 + b'y^2 = 1 \tag{3.14}$$

intersect orthogonally at $(x_1, y_1)$. Now from Eq. (3.13),

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-ax_1}{by_1}$$

and from Eq. (3.14)

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-a'x_1}{b'y_1}$$

By hypothesis

$$\left(\frac{-ax_1}{by_1}\right)\left(\frac{-a'x_1}{b'y_1}\right) = -1$$

and hence

$$\frac{aa'}{bb'}\left(\frac{x_1^2}{y_1^2}\right) = -1 \tag{3.15}$$

Also from Eqs. (3.13) and (3.14)

$$x_1^2(a - a') + y_1^2(b - b') = 0$$

so that

$$\frac{x_1^2}{y_1^2} = -\frac{(b - b')}{a - a'}$$

From Eq. (3.15) we have

$$\frac{aa'}{bb'}\left(\frac{b - b'}{a - a'}\right) = 1$$

$$\Rightarrow aa'b - aa'b' = abb' - bb'a'$$

Dividing by $aa'bb'$, we get

$$\frac{1}{b'} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{a}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'}$$

**Answer: (A)**

**16.** The slope of the tangent at $(\pi/4, 0)$ to the curve $1 + 16x^2y = \tan(x - 2y)$ is

(A) $\frac{2}{\pi + 4}$ (B) $\frac{1}{\pi^2 + 4}$

(C) $\frac{2}{\pi^2 + 4}$ (D) $\frac{1}{\pi + 4}$

**Solution:** Differentiating the curve equation w.r.t. $x$ we get

$$32xy + 16x^2 \frac{dy}{dx} = \sec^2(x-2y)\left(1 - 2\frac{dy}{dx}\right)$$

Put $x = \pi/4$ and $y = 0$ so that

$$16 \cdot \frac{\pi^2}{16}\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},0\right)} = \sec^2\left(\frac{\pi}{4} - 0\right)\left(1 - 2\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},0\right)}\right)$$

$$\Rightarrow \pi^2\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},0\right)} = 2 - 4\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},0\right)}$$

$$\Rightarrow (\pi^2 + 4)\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},0\right)} = 2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},0\right)} = \frac{2}{\pi^2 + 4}$$

**Answer: (C)**

**17.** The angle of intersection of the two curves $4x^2 + 9y^2 = 45$ and $x^2 - 4y^2 = 5$ at any of their common points is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\mathrm{Tan}^{-1}(2)$ (D) $\frac{\pi}{3}$

**Solution:** Since the two curves are symmetric about both axes, the angle of intersection is same at any of their common points. We have

$$4x^2 + 9y^2 = 45 \tag{3.16}$$

$$x^2 - 4y^2 = 5 \tag{3.17}$$

Solving Eqs. (3.16) and (3.17), the points of intersection are $(\pm 3, \pm 1)$. From Eq. (3.16)

$$\frac{dy}{dx} = \frac{-4x}{9y} \Rightarrow \left(\frac{dy}{dx}\right)_{(3,1)} = \frac{-4}{3}$$

From Eq. (3.17),

$$\frac{dy}{dx} = \frac{2x}{8y} \Rightarrow \left(\frac{dy}{dx}\right)_{(3,1)} = \frac{3}{4}$$

Product of the slopes of the tangents at $(3, 1) = -1$. This implies that the curves cut each other orthogonally.

**Answer: (A)**

**18.** The area of the triangle formed by the tangent to the curve $\sin y = x^3 - x^5$ at the point $(1, 0)$ and the coordinate axes is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) $\frac{3}{2}$

**Solution:** Differentiating the given equation

$$(\cos y)\frac{dy}{dx} = 3x^2 - 5x^4$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 3 - 5 = -2$$

Equation of the tangent at $(1, 0)$ is

$$y = -2(x-1)$$

$$\Rightarrow 2x + y = 2 \quad \text{or} \quad \frac{x}{1} + \frac{y}{2} = 1$$

So area of the triangle is

$$\frac{1}{2}(1)(2) = 1$$

**Answer: (B)**

**19.** The number of points on the curve $y = x \sin x$ at which the line $y = x$ is a tangent is

(A) 2 (B) 4
(C) 0 (D) infinite

**Solution:** We know that $x \sin x = x$ whenever $x = x = (4n+1)(\pi/2), n \in \mathbb{Z}$. Therefore the line $y = x$ meets the curve $y = x \sin x$ at infinite number of points. Also

$$y = x\sin x \Rightarrow \frac{dy}{dx} = \sin x + x\cos x$$

When $x = (4n+1)(\pi/2)$, we have $dy/dx = 1$ which is also the slope of the line $y = x$. So $y = x$ touches the curve $y = x \sin x$ at infinite number of points.

**Answer: (D)**

**20.** The number of values of $x$ at which the graph of $y = \sec x$ is horizontal are

(A) 2 (B) 0
(C) infinite (D) 4

**Solution:** We have

$$y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \tan x$$

Horizontal tangent implies

$$\frac{dy}{dx} = 0$$

Now

$$\sec x \neq 0 \Rightarrow \tan x = 0 \Rightarrow x = n\pi, n \in \mathbb{Z}$$

Therefore $y = \sec x$ has infinite number of horizontal tangents.

**Answer: (C)**

**21.** The angle at which the curve $3y = \sin 3x$ crosses $x$-axis is

(A) $\mathrm{Tan}^{-1}(\pm 2)$ (B) $\mathrm{Tan}^{-1}(\pm 1)$

(C) $\mathrm{Tan}^{-1}(\pm\sqrt{3})$ (D) $\mathrm{Tan}^{-1}\left(\pm\dfrac{1}{\sqrt{3}}\right)$

**Solution:** The curve crosses the $x$-axis when

$$\frac{1}{3}\sin 3x = y = 0 \Rightarrow x = \frac{n\pi}{3}, n \in \mathbb{Z}$$

Slope of the tangent at line is

$$\frac{dy}{dx} = \cos 3x$$

and at $x = n\pi/3$ is

$$\frac{dy}{dx} = \cos n\pi = \pm 1$$

Therefore the slopes of the tangents at $x = n\pi$ is $\pm$ 1. The tangents at these points make $\pm\pi/4$ angle with $x$-axis

**Answer: (B)**

**22.** Let $c$ be the curve $y^3 - 3xy + 2 = 0$. Let $m$ be the number of points on $c$ at which tangents are horizontal and $n$ be the number of points on $c$ at which the tangent is vertical. Then $m + n$ equals

(A) 1 (B) 2

(C) 3 (D) 4

**Solution:** From the curve equation we have

$$3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$

$$\Rightarrow (y^2 - x)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{y^2 - x}$$

But $y \neq 0$, otherwise $2 = 0$. Therefore $y^2 = x$. This implies

$$y^3 - 3y^2 + 2 = 0$$

so that $y = 1$ and $x = 1$. Therefore $m = 0$ and $n = 1$. Thus, $m + n = 1$.

**Answer: (A)**

**23.** If the normal to the curve $y = f(x)$ at the point $(3, 4)$ makes an angle $3\pi/4$ with the positive direction of the $x$-axis, then $f'(3)$ is equal to

(A) $-1$ (B) $\dfrac{-3}{4}$

(C) $\dfrac{4}{3}$ (D) 1

**Solution:** We know that

$$\frac{dy}{dx} = f'(x)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(3,4)} = f'(3)$$

By hypothesis the slope of the normal at $(3, 4)$ is

$$\frac{-1}{f'(3)} = \tan\frac{3\pi}{4}$$

$$\Rightarrow f'(3) = 1$$

**Answer: (D)**

**24.** Which one of the following curves cuts the curve $y^2 = 4ax$ at right angles?

(A) $x^2 + y^2 = a^2$ (B) $y = e^{-x/2a}$

(C) $y = ax$ (D) $x^2 = 4ay$

**Solution:** We have

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y} = m \quad \text{(suppose)}$$

$$x^2 + y^2 = a^2 \Rightarrow \frac{dy}{dx} = \frac{-x}{y} = m_1 \quad \text{(suppose)}$$

$$y = e^{-x/2a} \Rightarrow \frac{dy}{dx} = \frac{-1}{2a}e^{-x/2a} = m_2 \quad \text{(suppose)}$$

$$y = ax \Rightarrow \frac{dy}{dx} = a = m_3 \quad \text{(suppose)}$$

$$x^2 = 4ay \Rightarrow \frac{dy}{dx} = \frac{x}{2a} = m_4 \quad \text{(suppose)}$$

Now

$$mm_4 = -1$$

$$\Rightarrow \frac{x}{2a} \times \frac{2a}{y} = -1$$

$$\Rightarrow y = -x$$

$$\Leftrightarrow x = y = 0$$

[Since point lies on both the curves $y^2 = 4ax$ and $x^2 = 4ay$, and the points are $(0, 0)$ and $(4a, 4a)$.] Therefore $x^2 = 4ay$ cuts $y^2 = 4ax$ orthogonally at $(0, 0)$.

We can check that $mm_1 \neq -1, mm_2 \neq -1$ and $mm_3 \neq -1$ at the points where the curves $x^2 + y^2 = a^2, y = e^{-x/2a}$ and $y = ax$ meet.

**Answer: (D)**

***Note:*** For the curve $y^2 = 4ax$, the $y$-axis is tangent at $(0, 0)$ and for the curve $x^2 = 4ay$, the $x$-axis is the tangent at $(0, 0)$.

**25.** The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent(s) is vertical is (are)

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$

(C) $(0, 0)$ (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

**Solution:** Differentiating the given equation we get

$$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\Rightarrow (y^2 - 4)\frac{dy}{dx} = -2x$$

Now

$$\text{Tangent is vertical} \Leftrightarrow y^2 - 4 = 0$$
$$\Leftrightarrow y = \pm 2$$

Therefore

$$y = 2 \Rightarrow x = \pm\frac{4}{\sqrt{3}}$$

$$y = -2 \Rightarrow x^2 = \frac{-24 + 8}{3} = \frac{-16}{3} < 0$$

Therefore the points are $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$.

**Answer: (D)**

**26.** The number of points on the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ at which the tangent has slope $-1/2$ is

(A) 1 (B) 2
(C) 4 (D) 8

**Solution:** Differentiating the given equation, $y = \cos(x + y)$, we have

$$\frac{dy}{dx} = -\sin(x + y)\left(\frac{dy}{dx} + 1\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)} = \frac{-1}{2}$$

$$\Rightarrow \sin(x + y) = 1$$

Now $\sin(x + y) = 1 \Rightarrow \cos(x + y) = 0$. Therefore

$$x + y = (4n + 1)\frac{\pi}{2}$$

and $$x + y = (2n + 1)\frac{\pi}{2}, n \in \mathbb{Z}$$

Now

$$-2\pi \le x \le 2\pi \Rightarrow x = \frac{\pi}{2}, \frac{-3\pi}{2}$$

Therefore, the number of points on the curve at which the tangent slope equals $-1/2$ is two.

**Answer: (B)**

**27.** A curve in $xy$-plane is parametrically represented by the equations $x = t^2 + t + 1$, $y = t^2 - t + 1$ where $t \ge 0$. The number of straight lines passing through the point $(1, 1)$ which are tangent to the curve is

(A) 0 (B) 1
(C) 2 (D) 3

**Solution:** $t = 0 \Leftrightarrow x = 1$ and $y = 1$. Therefore $(1, 1)$ is a point on the given curve. Also

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t - 1}{2t + 1}$$

Now

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \left(\frac{dy}{dx}\right)_{t=0} = \frac{-1}{1} = -1$$

Equation of the tangent at $(1, 1)$ is

$$y - 1 = -1(x - 1)$$
$$\Rightarrow x + y = 2$$

**Answer: (B)**

**28.** The number of points belonging to the set $\{(x, y) | -10 \le x \le 10 \text{ and } -3 \le y \le 3\}$ which lie on the curve $y^2 = x + \sin x$ at which the tangent to the curve is horizontal is

(A) 4 (B) 8
(C) 2 (D) 0

**Solution:** The given equation is $y^2 = x + \sin x$. Differentiating both sides w.r.t. $x$, we get

$$2y\frac{dy}{dx} = 1 + \cos x$$

Horizontal tangent implies

$$y \ne 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Therefore, $\cos x = -1$, which implies that $x = (2n + 1)\pi$, $n \in \mathbb{Z}$. Now

$$-10 \le x \le 10 \Rightarrow \frac{\frac{-10}{\pi} - 1}{2} \le n \le \frac{\frac{10}{\pi} - 1}{2}$$

$$\Rightarrow \frac{-92}{44} \le n \le \frac{48}{44}$$

Therefore $n = -2, -1, 0, 1$. Now

$$n = -2 \Rightarrow x = \cos(-3\pi) = -1$$

but $y^2 = -3\pi + 0$, which is not true. So $n \neq -2$. Similarly $n \neq -1$. This implies $n = 0, 1$. But $n = 1$ implies $x = 3\pi$ so that $y^2 = 3\pi \Rightarrow y > 3$. Hence $n \neq 1$. Therefore, the points in the set are $(\pi, \pm\sqrt{\pi})$.

**Answer: (C)**

**29.** The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point $(1, 1)$ and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of $b$ is

(A) $-1$  (B) 3
(C) $-3$  (D) 1

**Solution:** The triangle lies in the first quadrant implies the tangent at $(1, 1)$ makes obtuse angle with the positive direction of the $x$-axis. The slope of the tangent at $(1, 1)$ is negative. From the curve equation,

$$f'(x) = 2x + b$$

Therefore

$$f'(1) = 2 + b$$

Now $f'(1)$ = slope of the tangent at $(1, 1) < 0$. This implies

$$b + 2 < 0 \tag{3.18}$$

Equation of the tangent at $(1, 1)$ is

$$y - 1 = (b + 2)(x + 1)$$

Now

$$y = 0 \Rightarrow x = 1 - \frac{1}{b+2} = \frac{1+b}{2+b}$$

$$x = 0 \Rightarrow y = -(b + 1)$$

Therefore

$$\begin{aligned} 2 &= \text{Area of the triangle} \\ &= \frac{1}{2}|b+1|\left|\frac{b+1}{b+2}\right| \\ &= -\frac{1}{2}\frac{(b+1)^2}{b+2} \quad \text{[from Eq. (3.18)]} \end{aligned}$$

Solving we get

$$\begin{aligned} & 4(b + 2) = -(b + 1)^2 \\ & \Rightarrow b^2 + 6b + 9 = 0 \\ & \Rightarrow (b + 3)^2 = 0 \\ & \Rightarrow b = -3 \end{aligned}$$

**Answer: (C)**

**30.** The angle of intersection of the curves $y = x^2$, $6y = 7 - x^2$ at $(1, 1)$ is

(A) $\dfrac{\pi}{4}$  (B) $\dfrac{\pi}{2}$

(C) $\dfrac{\pi}{6}$  (D) $\mathrm{Tan}^{-1}(7)$

**Solution:** Differentiating $y = x^2$ we get

$$\frac{dy}{dx} = 2x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 2 = m_1 \quad \text{(suppose)}$$

Differentiating $6y = 7 - x^2$ we get

$$\frac{dy}{dx} = \frac{-x}{3}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{1}{3} = m_2 \quad \text{(suppose)}$$

Suppose the acute angle of intersection is $\theta$. Therefore

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{2 + \frac{1}{3}}{1 - \frac{2}{3}}\right| = 7$$

$$\Rightarrow \theta = \mathrm{Tan}^{-1}(7)$$

**Answer: (D)**

***Note:*** In the above problem, if the second curve is $6y = 7 - x^3$, then $\theta = \pi/2$.

**31.** If $\theta$ is the angle of intersection of the curves $x^2 + 2xy - y^2 + 2ax = 0$ and $3y^3 - 2a^2 x - 4a^2 y + a^3 = 0$ at the point $(a, -a)$, then $\tan\theta$ is equal to

(A) $\dfrac{9}{8}$  (B) $\dfrac{8}{9}$

(C) $\dfrac{3}{8}$  (D) $\dfrac{4}{9}$

**Solution:** Differentiating the first equation, we get

$$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} + 2a = 0$$

$$\Rightarrow \frac{dy}{dx}(x - y) = -(x + y + a)$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(a,-a)} = \frac{-a}{2a} = \frac{-1}{2} = m_1 \quad \text{(say)}$$

Again, differentiating the second equation we get

$$9y^2\frac{dy}{dx}-2a^2-4a^2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}(9y^2-4a^2)=2a^2$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(a,-a)}=\frac{2a^2}{5a^2}=\frac{2}{5}=m_2 \quad \text{(say)}$$

Now

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$
$$=\left|\frac{-\frac{1}{2}-\frac{2}{5}}{1+\left(-\frac{1}{2}\times\frac{2}{5}\right)}\right|$$
$$=\frac{9}{10}\times\frac{5}{4}=\frac{9}{8}$$

**Answer: (A)**

**32.** The acute angle of intersection of curves $x^2 - y^2 = a^2$ and $x^2+y^2=a^2\sqrt{2}$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

**Solution:** Suppose the two curves intersect in $(x_1, y_1)$ and let $\theta$ be their acute angle of intersection. Differentiating $x^2 - y^2 = a^2$, we get

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{x_1}{y_1}$$

Differentiating $x^2+y^2=a^2\sqrt{2}$, we get

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-\frac{x_1}{y_1}$$

Therefore

$$\tan\theta=\left|\frac{\frac{x_1}{y_1}+\frac{x_1}{y_1}}{1-\frac{x_1^2}{y_1^2}}\right|=\frac{2|x_1y_1|}{|y_1^2-x_1^2|}$$

Solving the given equations, we have

$$x_1=\pm\frac{a}{\sqrt{2}}\sqrt{\sqrt{2}+1} \quad \text{and} \quad y_1=\pm\frac{a}{\sqrt{2}}\sqrt{\sqrt{2}-1}$$

Therefore, at the point

$$x_1=\frac{a}{\sqrt{2}}\sqrt{\sqrt{2}+1} \text{ and } y_1=\frac{a}{\sqrt{2}}\sqrt{\sqrt{2}-1}$$

we have

$$\tan\theta=\frac{2\left(\frac{a^2}{2}\right)\sqrt{2-1}}{|-a^2|}=1$$

Therefore $\theta=\pi/4$. This angle is same, because the curves are symmetric about the axes.

**Answer: (C)**

**33.** The angle of intersection of the curves $y = \sin x$ and $y = \cos x$ in the first quadrant is

(A) $\text{Tan}^{-1}\sqrt{2}$ (B) $\text{Tan}^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(C) $\text{Tan}^{-1}\left(\frac{3}{\sqrt{2}}\right)$ (D) $\text{Tan}^{-1}(2\sqrt{2})$

**Solution:** The given curves intersect at $(\pi/4, 1/\sqrt{2})$ in the first quadrant. Now

$$y=\sin x \Rightarrow \frac{dy}{dx}=\cos x$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},\frac{1}{\sqrt{2}}\right)}=\frac{1}{\sqrt{2}}$$

$$y=\cos x \Rightarrow \frac{dy}{dx}=-\sin x$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},\frac{1}{\sqrt{2}}\right)}=-\frac{1}{\sqrt{2}}$$

Therefore if $\theta$ is the acute angle of intersection of the curves, then

$$\tan\theta=\left|\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{1-\frac{1}{2}}\right|=\frac{4}{\sqrt{2}}=2\sqrt{2}$$
$$\Rightarrow \theta=\text{Tan}^{-1}2\sqrt{2}$$

**Answer: (D)**

**34.** The sub-normal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies inversely as

(A) cube of the abscissa of the point

(B) square of the abscissa

(C) $\frac{3}{2}$th power of abscissa

(D) $\frac{2}{3}$rd power of abscissa

**Solution:** Differentiating the given equation we get

$$2xy^2 + 2x^2y\frac{dy}{dx} = 2a^2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2 - y^2}{xy}$$

Now the sub-normal is given by

$$\left|y\cdot\frac{dy}{dx}\right| = \left|\frac{a^2-y^2}{x}\right|$$

$$= \left|\frac{a^2 - \dfrac{a^2(x^2-a^2)}{x^2}}{x}\right|$$

$$= \frac{a^4}{|x^3|}$$

**Answer: (A)**

**35.** The tangent at any point on the curve $x^3 + y^3 = 2a^3$ cuts off lengths $p$ and $q$ on the coordinate axes. Then $p^{-3/2} + q^{-3/2} =$

(A) $2^{-1/2}a^{-3/2}$ (B) $2^{-1/2}a^{-1/2}$

(C) $2^{1/2}\,a^{1/2}$ (D) $2^{1/2}a^{3/2}$

**Solution:** Differentiating the curve equation, we get

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = \frac{-x_1^2}{y_1^2}$$

Equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = \frac{-x_1^2}{y_1^2}(x - x_1)$$

Since $(x_1, y_1)$ lies on the curve we have

$$x_1^2x + y_1^2y = x_1^3 + y_1^3 = 2a^3$$

Therefore

$$p = \frac{2a^3}{x_1^2} \quad\text{and}\quad q = \frac{2a^3}{y_1^2}$$

So

$$p^{-3/2} + q^{-3/2} = 2^{-3/2}a^{-9/2}\left(\frac{1}{x_1^{-3}} + \frac{1}{y_1^{-3}}\right)$$

$$= 2^{-3/2}a^{-9/2}(x_1^3 + y_1^3)$$

$$= 2^{-3/2}a^{-9/2}(2a^3)$$

$$= 2^{-1/2}a^{-3/2}$$

**Answer: (A)**

**36.** The acute angle of intersection of the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ in the first quadrant is

(A) $\text{Tan}^{-1}\left(\frac{2\sqrt{2}}{7}\right)$ (B) $\text{Tan}^{-1}\left(\frac{\sqrt{2}}{7}\right)$

(C) $\text{Tan}^{-1}\left(\frac{3\sqrt{2}}{7}\right)$ (D) $\text{Tan}^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

**Solution:** We have

$$y = |x^2-1| = \begin{cases} x^2 - 1 & \text{if } x \le -1 \\ 1 - x^2 & \text{if } -1 < x < 1 \\ x^2 - 1 & \text{if } x \ge 1 \end{cases}$$

$$y = |x^2-3| = \begin{cases} x^2 - 3 & \text{if } x \le -\sqrt{3} \\ 3 - x^2 & \text{if } -\sqrt{3} < x < \sqrt{3} \\ x^2 - 3 & \text{if } x \ge \sqrt{3} \end{cases}$$

Therefore, the two curves intersect at $(\sqrt{2}, 1)$ in the first quadrant. Now $x \ge 1$ implies

$$y = x^2 - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=\sqrt{2}} = 2\sqrt{2}$$

and

$$y = 3 - x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{x=\sqrt{2}} = -2\sqrt{2}$$

Now, if $\theta$ is the acute angle of intersection of the curves at $(\sqrt{2}, 1)$, then

$$\tan\theta = \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 + (2\sqrt{2})(-2\sqrt{2})}\right| = \frac{4\sqrt{2}}{7}$$

$$\Rightarrow \theta = \text{Tan}^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

**Answer: (D)**

***Note:*** The two curves also intersect at $(-\sqrt{2}, 1)$.

**37.** The acute angle of intersection of the curves $x^2 + y^2 = 5$ and $y = [|\sin x| + |\cos x|]$ where $[\cdot]$ is the greatest integer function is

(A) $\text{Tan}^{-1}(2\sqrt{2})$ (B) $\text{Tan}^{-1}2$

(C) $\text{Tan}^{-1}\sqrt{2}$ (D) $\text{Tan}^{-1}\left(\frac{1}{\sqrt{2}}\right)$

**Solution:** Since $0 \le |\sin x| \le 1$ and $0 \le |\cos x| \le 1$,

$$\sin x = 0 \Leftrightarrow \cos x = \pm 1$$

and vice versa. Also

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

We have

$$1 \le |\sin x| + |\cos x| \le \sqrt{2}$$

Therefore

$$y = [|\sin x| + |\cos x|] = 1$$

So, the points of intersection of the two curves are (2, 1) and (–2, 1). Differentiating $x^2 + y^2 = 4$ we have

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,-1)} = 2$$

The angle made by the tangent at (2, –1) to the circle with the line $y = 1$ is $\mathrm{Tan}^{-1}2$. Also

$$\left(\frac{dy}{dx}\right)_{(2,1)} = -2$$

This implies that the acute angle between the tangent to the circle with the line $y = 1$ is $\mathrm{Tan}^{-1}2$.

**Answer: (B)**

**38.** The tangent at any point of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets coordinate axes in $A$ and $B$. Then, the locus of the mid-point of the segment $AB$ is

(A) $x^2 + y^2 = a^2$ (B) $x^2 + y^2 = \dfrac{a^2}{2}$

(C) $x^2 + y^2 = \dfrac{a^2}{4}$ (D) $x^2 + y^2 = 2a^2$

**Solution:** Differentiating both the equations we get

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

and

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = -\tan\theta$$

Equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$ is

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a(\cos^2\theta + \sin^2\theta) = a$$

Therefore $A = (a\cos\theta, 0)$ and $B = (0, a\sin\theta)$. Let $(x_1, y_1)$ be the mid-point of $\overline{AB}$. Then

$$x_1 = \frac{a\cos\theta}{2} \quad \text{and} \quad y_1 = \frac{a\sin\theta}{2}$$

So,

$$\left(\frac{2x_1}{a}\right)^2 + \left(\frac{2y_1}{a}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow x_1^2 + y_1^2 = \frac{a^2}{4}$$

Therefore the locus of $(x_1, y_1)$ is

$$x^2 + y^2 = \frac{a^2}{4}$$

**Answer: (C)**

**39.** If the algebraic sum of the intercepts on the axes cut off by tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$ at $(a/8, a/8)$ is 2, then the value of $a$ is

(A) 8 (B) 4

(C) 2 (D) $4\sqrt{2}$

**Solution:** Differentiating the curve equation, we get

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{2/3}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(a/8, a/8)} = -1$$

Equation of the tangent at $(a/8, a/8)$ is

$$y - \frac{a}{8} = -1\left(x - \frac{a}{8}\right)$$

$$\Rightarrow x + y = \frac{a}{4}$$

The intercepts of the tangent on the axes are $a/4$ and $a/4$. By hypothesis,

$$\frac{a}{4} + \frac{a}{4} = 2 \Rightarrow a = 4$$

**Answer: (B)**

**40.** If tangents are drawn from the origin to the curve $y = \sin x$, then their point of contact lies on the curve whose equation is

(A) $x^2 - y^2 = 1$ (B) $x^2 + y^2 = 1$

(C) $\dfrac{1}{x^2} - \dfrac{1}{y^2} = -1$ (D) $x^2 - y^2 = x^2y^2$

**Solution:** Let the tangent at $(x_1, y_1)$ to the curve pass through (0, 0). Therefore the tangent at $(x_1, y_1)$ is

$$y - y_1 = \cos x_1(x - x_1)$$

This passes through (0, 0) which implies

$$x_1 \cos x_1 - y_1 = 0$$

$$\Rightarrow \frac{y_1}{x_1} = \cos x_1 = \sqrt{1-\sin^2 x_1} = \sqrt{1-y_1^2}$$

$$\Rightarrow \frac{y_1^2}{x_1^2} = 1 - y_1^2 \quad \text{(squaring both sides)}$$

$$\Rightarrow \frac{1}{x_1^2} = \frac{1}{y_1^2} - 1$$

$$\Rightarrow \frac{1}{x_1^2} - \frac{1}{y_1^2} = -1$$

Therefore $(x_1, y_1)$ lies on the curve

$$\frac{1}{x^2} - \frac{1}{y^2} = -1$$

**Answer: (C)**

**41.** If $p_1$ and $p_2$ are the lengths of the perpendiculars drawn from origin onto the tangent and normal, respectively, to the curve $x^{2/3} + y^{2/3} = a^{2/3}$, then $4p_1^2 + p_2^2$ is equal to

(A) $a^2$ (B) $2a^2$
(C) $a$ (D) $2a$

**Solution:** Differentiating $x^{2/3} + y^{2/3} = a^{2/3}$ we get

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Let $P(x_1, y_1)$ be a point on the curve. Then

$$x_1{}^{2/3} + y_1{}^{2/3} = a^{2/3} \tag{3.19}$$

Therefore the equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = -\left(\frac{y_1}{x_1}\right)^{1/3}(x - x_1)$$

$$\frac{x}{x_1^{1/3}} + \frac{y}{y_1^{1/3}} = x_1^{2/3} + y_1^{2/3} = a^{2/3} \quad \text{[By Eq. (3.19)]}$$

So

$$p_1 = \left|\frac{-a^{2/3}}{\sqrt{\dfrac{1}{x_1^{2/3}} + \dfrac{1}{y_1^{2/3}}}}\right| = \left|x_1^{1/3} y_1^{1/3}\right| \frac{a^{2/3}}{a^{1/3}} = \left|x_1^{1/3} y_1^{1/3}\right| a^{1/3}$$

$$\Rightarrow 4p_1^2 = 4x_1^{2/3} y_1^{2/3} a^{2/3} \tag{3.20}$$

Equation of the normal at $(x_1, y_1)$ is

$$y - y_1 = \left(\frac{x_1}{y_1}\right)^{1/3}(x - x_1)$$

$$\Rightarrow x_1^{1/3} x - y_1^{1/3} y + x_1^{4/3} - y_1^{4/3} = 0$$

$$\Rightarrow x_1^{1/3} x - y_1^{1/3} y + (x_1^{2/3} - y_1^{2/3})(x_1^{2/3} + y_1^{2/3}) = 0$$

$$\Rightarrow x_1^{1/3} x - y_1^{1/3} y + a^{2/3}(x_1^{2/3} - y_1^{2/3}) = 0$$

Therefore

$$p_2 = \left|\frac{a^{2/3}(x_1^{2/3} - y_1^{2/3})}{\sqrt{x_1^{2/3} + y_1^{2/3}}}\right| = \left|a^{1/3}(x_1^{2/3} - y_1^{2/3})\right|$$

$$\Rightarrow p_2^2 = a^{2/3}(x_1^{2/3} - y_1^{2/3})^2 \tag{3.21}$$

From Eqs. (3.20) and (3.21), we have

$$\begin{aligned} 4p_1^2 + p_2^2 &= a^{2/3}[4x_1^{2/3} y_1^{2/3} + (x_1^{2/3} - y_1^{2/3})^2] \\ &= a^{2/3}(x_1^{2/3} + y_1^{2/3})^2 \\ &= a^{2/3} \cdot a^{4/3} = a^2 \end{aligned}$$

**Answer: (A)**

***Advice:*** The curve is parametrically represented by the equations $x = a\cos^3\theta$, $y = a\sin^3\theta$ so that $dy/dx = -\tan\theta$. Now proceed.

**42.** A tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ meets $x$-axis in $A$ and $y$-axis is $B$. Then the point which divides the segment $\overline{AB}$ internally in the ratio 2:1 lies on the curve whose equation is

(A) $x^2 + \dfrac{y^2}{4} = \dfrac{a^2}{9}$ (B) $\dfrac{x^2}{4} + \dfrac{y^2}{1} = \dfrac{a^2}{9}$

(C) $\dfrac{x^2}{4} - \dfrac{y^2}{1} = \dfrac{a^2}{9}$ (D) $x^2 + y^2 = 36a^2$

**Solution:** The parametric equations of the curve are $x = a\cos^3\theta$, $y = a\sin^3\theta$ so that $dy/dx$ at any $\theta$ is $-\tan\theta$. Therefore the equation of the tangent is

$$y - a\sin^3\theta = -\tan\theta(x - a\sin^3\theta)$$

$$\Rightarrow \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a(\sin^2\theta + \cos^2\theta) = a$$

Therefore

$$A = (a\cos\theta, 0) \quad \text{and} \quad B = (0, a\sin\theta)$$

Suppose $P(x_1, y_1)$ divides $\overline{AB}$ in the ratio 2:1. then

$$x_1 = \frac{a\cos\theta}{3}, \; y_1 = \frac{2a\sin\theta}{3}$$

So,

$$\frac{9x_1^2}{a^2} + \frac{9y_1^2}{4a^2} = 1$$

Therefore $P(x_1, y_1)$ lies on the curve $x^2 + \frac{y^2}{4} = \frac{a^2}{9}$

**Answer: (A)**

**43.** Tangent to the curve $xy = a^2$ at point $P$ meets the $x$-axis in $A$ and $y$-axis in $B$. Then the ratio $AP{:}PB$ is

(A) 1:2 (B) 2:1

(C) 1:1 (D) 2:3

**Solution:** $xy = a^2$ is represented parametrically. So,

$$x = at \quad \text{and} \quad y = \frac{a}{t}, 0 \neq t \in \mathbb{R}$$

Now,

$$\frac{dy}{dx} = -\frac{1}{t^2}$$

Therefore, the tangent at $(at, a/t)$ is

$$y - \frac{a}{t} = -\frac{1}{t^2}(x - at)$$

$$\Rightarrow x + t^2 y = 2at$$

so that

$$A = (2at, 0) \quad \text{and} \quad B = \left(0, \frac{2a}{t}\right)$$

$$\text{Mid-point of } \overline{AB} = \left(at, \frac{a}{t}\right) = P$$

Therefore, the ratio $AP{:}PB$ is 1:1.

**Answer: (C)**

**44.** Angle of intersection of the curves $x^3 - 3xy^2 = -2$ and $3x^2y - y^3 = 2$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

**Solution:** Suppose the two curves intersect in $(x_1, y_1)$. Differentiating the first curve equation, we get

$$3x^2 - 3y^2 - 6xy\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1^2 - y_1^2}{2x_1 y_1} \tag{3.22}$$

Differentiating the second equation, we get

$$6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2x_1 y_1}{x_1^2 - y_1^2} \tag{3.23}$$

From Eqs. (3.22) and (3.23) it follows that the product of slopes of the tangents to the curves at $(x_1, y_1)$ is $-1$. Hence the two curves intersect orthogonally.

**Answer: (A)**

**45.** The curve $y = be^{-x/a}$ crosses the $y$-axis at $P$. The equation of the tangent at $P$ is

(A) $\frac{x}{a} + \frac{y}{b} = 2$ (B) $\frac{x}{a} - \frac{y}{b} = 2$

(C) $\frac{x}{a} + \frac{y}{b} = 1$ (D) $\frac{x}{a} - \frac{y}{b} = 1$

**Solution:** $x = 0 \Rightarrow y = b$. Therefore $P = (0, b)$. Now

$$\frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}$$

Equation of the tangent at $P(0, b)$ is

$$y - b = \frac{-b}{a}(x - 0)$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

**Answer: (C)**

**46.** The curves $y^2 = 4ax$ and $ay^2 = 4x^3$ where $a > 0$ intersect at a point $P$ (not the origin). If the normals to the curves at $P$ meet the $x$-axis in $A$ and $B$, then the distance $AB$ is

(A) $2a$ (B) $3a$

(C) $4a$ (D) $5a$

**Solution:** It can be easily seen that the two curves intersect in the points (0, 0), $(a, 2a)$ and $(a, -2a)$. Take $P = (a, 2a)$. Then

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P = \frac{2a}{2a} = 1$$

$$ay^2 = 4x^3 \Rightarrow \frac{dy}{dx} = \frac{12x^2}{2ay}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P = \frac{12a^2}{4a^2} = 3$$

Therefore, the equations of the normals to the curves at $P$ are, respectively,

$$y - 2a = -1(x - a)$$

and

$$y - 2a = -\frac{1}{3}(x - a)$$

Therefore, $A = (3a, 0)$ and $B = (7a, 0)$. So

$$AB = 7a - 3a = 4a$$

**Answer: (C)**

***Note:*** At $Q(a, -2a)$ also, you can see that $A = (-a, 0)$ and $B = (-5a, 0)$ so that $AB = 4a$.

**47.** For the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$, the equation of the normal at $\theta = \pi/4$ is

(A) $x + y = 0$
(B) $2x - y = 0$
(C) $x - 2y = 0$
(D) $x - y = 0$

**Solution:** Differentiating both the given equations w.r.t. $\theta$, we get

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$

$$\frac{dx}{d\theta} = 3\cos^2\theta\sin\theta - 3\sin\theta$$

Therefore

$$\frac{dy}{dx} = \frac{3\cos\theta - 3\sin^2\theta\cos\theta}{3\cos^2\theta\sin\theta - 3\sin\theta} = -\frac{\cos^3\theta}{\sin^3\theta}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -1$$

Now $\theta = \pi/4$ implies

$$x = \frac{3}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{5}{2\sqrt{2}} \quad \text{and} \quad y = \frac{5}{2\sqrt{2}}$$

Therefore the equation of the normal at $\left(\frac{5}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}\right)$ is

$$y - \frac{5}{2\sqrt{2}} = 1\left(x - \frac{5}{2\sqrt{2}}\right)$$

$$\Rightarrow x - y = 0$$

**Answer: (D)**

**48.** For the curve $y = a\log(x^2 - a^2)$ where $a$ is positive, the algebraic sum of the tangent and sub-tangent at any point on the curve is

(A) $\dfrac{xy}{a}$
(B) $\dfrac{2xy}{a}$
(C) $\dfrac{xy}{2a}$
(D) $\dfrac{x+y}{a}$

**Solution:** $y$ is defined for $|x| > a$. Differentiating the given equation we get

$$\frac{dy}{dx} = \frac{2ax}{x^2 - a^2}$$

The tangent at $(x, y)$ is

$$y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{y\sqrt{4a^2x^2 + (x^2 - a^2)^2}}{2ax}$$

$$= \frac{y(x^2 + a^2)}{2ax}$$

Sub-tangent is

$$y\left(\frac{dx}{dy}\right) = \frac{y(x^2 - a^2)}{2ax}$$

Therefore the sum of tangent and sub-tangent is

$$\frac{y(2x^2)}{2ax} = \frac{xy}{a}$$

**Answer: (A)**

**49.** For the curve $y^n = a^{n-1}x$, the sub-normal at any point is constant. Then the value of $n$ equals

(A) 0
(B) 1
(C) 2
(D) $-1$

**Solution:** Differentiating the given equation we get

$$ny^{n-1}\frac{dy}{dx} = a^{n-1}$$

Therefore

$$y\left(\frac{dy}{dx}\right) = \frac{a^{n-1}}{ny^{n-2}} = \text{Constant if } n - 2 = 0$$

So, $n = 2$.

**Answer: (C)**

**50.** Let $f(x) = x - x^3$ for $-2 \le x \le 2$. If the line $y = mx + b$ is a tangent to the curve $y = x - x^3$ at the point $(-1, 0)$, then $(m, b)$ is

(A) $(2, 2)$
(B) $(-2, -2)$
(C) $(-2, 2)$
(D) $(2, -2)$

**Solution:** Let $y = x - x^3$. Differentiating we get

$$\frac{dy}{dx} = 1 - 3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(-1,0)} = 1 - 3 = -2$$

So, $m = -2$. The point $(-1, 0)$ lies on the line. Now

$$y = mx + b \Rightarrow 0 = (-2)(-1) + b$$

$$\Rightarrow b = -2$$

Therefore, $(m, b) = (-2, -2)$

**Answer: (B)**

## Rate Measure

**51.** Each edge of a cube is expanding at the rate of 1 cm/sec. Then, the rate of change of its volume, when each edge is length 5 cm, is

(A) 75 $cm^3$/sec (B) 125 $cm^3$/sec
(C) 25 $cm^3$/sec (D) 175 $cm^3$/sec

**Solution:** Let $x$ be the edge of the cube. It is given that $dx/dt = 1$ where $t$ is the time. Let $V$ be the volume of the cube. Then

$$V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

When $x = 5$ and $dx/dt = 1$, then

$$\frac{dV}{dt} = 3(5)^2(1) = 75$$

**Answer: (A)**

**52.** A particle moves along the curve $6y = x^3 + 2$. The number of points on the curve at which the $y$-coordinate is changing eight times the $x$-coordinate is

(A) 4 (B) 3
(C) 2 (D) 1

**Solution:** Differentiating the given equation we have

$$\frac{dy}{dt} = \frac{3}{6}x^2\frac{dx}{dt} \tag{3.24}$$

But by hypothesis

$$\frac{dy}{dt} = 8\frac{dx}{dt} \tag{3.25}$$

From Eqs. (3.24) and (3.25) we have

$$8\frac{dx}{dt} = \frac{1}{2}x^2\frac{dx}{dt}$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

Now

$$x = 4 \Rightarrow y = 11$$

and

$$x = -4 \Rightarrow y = \frac{-31}{3}$$

Therefore the points on the curve are (4, 11) and (–4, –31/3).

**Answer: (C)**

**53.** A particle moves along the curve $3y = 2x^3 + 3$. If the rate of change of the ordinate of a point is twice the rate of change of its abscissa, then the sum of the ordinates of the points is

(A) 2 (B) 3
(C) 0 (D) 4

**Solution:** By hypothesis

$$\frac{dy}{dt} = 2\frac{dx}{dt} \tag{3.26}$$

Differentiating the curve equation, we have

$$3\frac{dy}{dt} = 6x^2\frac{dx}{dt} \tag{3.27}$$

Substituing the value obtained in Eq. (3.26) in Eq. (3.27) gives

$$6\frac{dx}{dt} = 6x^2\frac{dx}{dt}$$

$$\Rightarrow x^2 = 1 \quad \text{or} \quad x = \pm 1$$

Now

$$x = 1 \Rightarrow y = \frac{5}{3}$$

$$x = -1 \Rightarrow y = \frac{1}{3}$$

Therefore the sum of the ordinates is

$$\frac{5}{3} + \frac{1}{3} = 2$$

**Answer: (A)**

**54.** The length and width of a rectangle are respectively decreasing 5 cm/sec and increasing 4 cm/sec. When the length is 8 cm and width is 6 cm, the rate of change of the area is

(A) 4 $cm^2$/sec (B) 8 $cm^2$/sec
(C) 2 $cm^2$/sec (D) 1 $cm^2$/sec

**Solution:** Let $x$ and $y$ be the length and width of the rectangle and $A$ its area. By hypothesis

$$\frac{dx}{dt} = -5 \text{ cm/sec}$$

and

$$\frac{dy}{dt} = 4 \text{ cm/sec}$$

Now $A = xy$ implies

$$\frac{dA}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$

When $x = 8$ and $y = 6$,

$$\frac{dA}{dt} = 6(-5) + 8(4) = 2$$

Therefore

$$\frac{dA}{dt} = 2 \text{ cm}^2/\text{sec}$$

**Answer: (C)**

**55.** A spherical ball of napthlene loosing its volume at time $t$ is proportional to its surface area. (The constant of proportionality is $k < 0$.) Then the radius of the ball decreases at the rate of

(A) $\frac{-k}{2}$ (B) $\frac{-k}{3}$

(C) $\frac{-2k}{3}$ (D) $-k$

**Solution:** Let $r$ be the radious of the ball, $V$ the volume and $S$ the surface area. Then

$$S = 4\pi r^2 \quad \text{and} \quad V = \frac{4}{3}\pi r^3$$

It is given that

$$\frac{dV}{dt} = kS = 4k\pi r^2$$

But

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Therefore

$$4k\pi r^2 = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = k$$

Since $k$ is negative, the radius of the ball is decreasing constantly at the rate of $-k$.

**Answer: (D)**

**56.** A balloon which is always spherical is being inflated by pumping in gas at the rate of 900 cm$^3$/sec. When the radius of the balloon is 15 cm, the rate of change of the radius is

(A) $\frac{4}{\pi}$ cm/sec (B) $\frac{3}{\pi}$ cm/sec

(C) $\frac{2}{\pi}$ cm/sec (D) $\frac{1}{\pi}$ cm/sec

**Solution:** Let $r$ be the radius and $V$ be the volume. Then

$$V = \frac{4}{3}\pi r^3$$

Differentiating this we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now $dV/dt = 900$ and $r = 15$. Substituting these in the above equation, we get

$$900 = 4\pi(225)\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}$$

**Answer: (D)**

**57.** A small stone is dropped into a quiet lake and the waves move in a circle at a rate of 3.5 cm/sec. At the instant when the radius of circular wave is 7.5 cm/sec, the rate of change of the area of the nearest circle is

(A) $\left(\frac{105}{2}\right)\pi$ cm$^2$/sec (B) $(50)\pi$ cm$^2$/sec

(C) $(51)\pi$ cm$^2$/sec (D) $(52)\pi$ cm$^2$/sec

**Solution:** Let $r$ be the radius of the circle and $A$ its area. Then

$$A = \pi r^2$$

Differentiating we get

$$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$$

When $r = 15/2$ we have

$$\frac{dA}{dt} = 2\pi\left(\frac{15}{2}\right)\left(\frac{7}{2}\right)\left(\because \frac{dr}{dt} = 3.5\right)$$

$$= \left(\frac{105}{2}\right)\pi$$

**Answer: (A)**

**58.** A ladder of 5 m length is leaning against a wall. The bottom of the ladder is moving on the ground away from the edge of the wall at a rate of 2 cm/sec. When the foot of the ladder is 4 m away from the wall, the height of the top of the ladder on the wall decreases at the rate of

(A) $\frac{4}{3}$ cm/sec (B) 2 cm/sec

(C) $\frac{8}{3}$ cm/sec (D) 3 cm/sec

**Solution:** $PQ = 5$ (ladder), $Q$ being the foot of the ladder. Let $OQ = x$ and $OP = y$ (see Fig. 3.13).

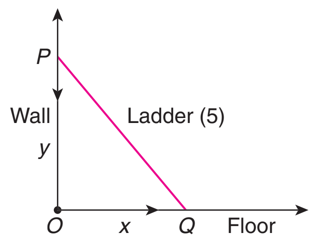



**FIGURE 3.13** Single correct choice type question 58.

Now by Pythagoras theorem,

$$x^2 + y^2 = 5^2$$

Differentiating we get

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

Now $x = 4 \Rightarrow y = 3$ and $dx/dt$ is given to be 2 cm/sec. Therefore

$$\frac{dy}{dt} = -\frac{4}{3}(2) = -\frac{8}{3} \text{ cm/sec}$$

Hence $y$ decreases at the rate of (8/3) cm/sec.

**Answer: (C)**

**59.** A man of height 2 m walks at a uniform speed of 5 km/hour away from a lamp post of 6 m height. His shadow length increases at the rate of

(A) 2.5 km/hour (B) 2 km/hour
(C) 3 km/hour (D) 3.5 km/hour

**Solution:** In Fig. 3.14, $AB = 6$ (lamp post), $PQ = 2$ (man), $BP = x$ (distance from the lamp post) and $PS = y$ (length of the shadow). From the similar triangles property, we have

$$\frac{6}{2} = \frac{y+x}{y} \Rightarrow y = \frac{x}{2}$$

Differentiating we have

$$\frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{5}{2} \quad \left(\because \frac{dx}{dt} = 5\right)$$

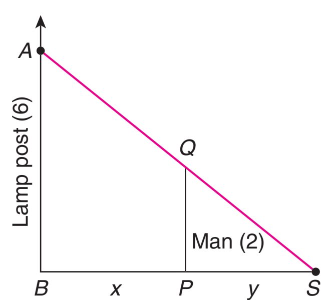



**FIGURE 3.14** Single correct choice type question 59.

**Answer: (A)**

**60.** A water tank is in the shape of a right circular cone with vertex down. The radius of the base is 15 feet and height is 10 feet. Water is poured into the tank at a constant rate of $c$ cubic feet per second. Water leakes out from the bottom at a constant rate of one cubic foot per second. The value of $c$ for which the water level is rising at the rate of 4 feet per second at the time when the water level is 2 feet deep, is given by

(A) $c = 1 + 9\pi$ (B) $c = 1 + 4\pi$
(C) $c = 1 + 18\pi$ (D) $c = 1 + 36\pi$

**Solution:** See. Fig. 3.15. At time $t$, let $r$ be the radius of the water surface and $h$ be the depth of the water level. Let $V$ be the volume of the water at time $t$. By the hypothesis

$$\frac{dV}{dt} = c$$

But

$$V = \frac{\pi}{3}r^2 h \tag{3.28}$$

From the similar triangles property

$$\frac{15}{r} = \frac{10}{h} \Rightarrow r = \frac{3}{2}h$$

Substituting this value of $r$ in Eq. (3.28), we get

$$V = \frac{\pi}{3}\left(\frac{9}{4}\right)h^3$$

Therefore

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2\frac{dh}{dt}$$

When $h = 2$, $dh/dt = 4$ and $dv/dt = 1$, we get

$$1 = 36\pi$$

Therefore

$$c - 1 = 36\pi$$

$$\Rightarrow c = 1 + 36\pi$$

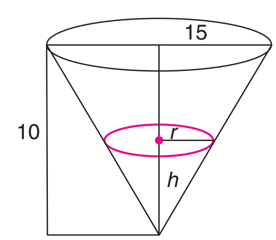



**FIGURE 3.15** Single correct choice type question 60.

**Answer: (D)**

**61.** A particle moves along the curve $y = 2x^3 - 3x^2 + 4$. If $x$-coordinate is increasing at the rate of 0.5/sec, the rate of charge of the $y$-coordinate when $x = 2$ is

(A) 4/sec (B) 6/sec
(C) 8/sec (D) 2/sec

**Solution:** Differentiating the given equation we get

$$\frac{dy}{dt}=(6x^2-6x)\frac{dx}{dt}$$

When $x = 2$ and $dx/dt = 0.5$, we get

$$\frac{dy}{dt}=(24-12)\frac{1}{2}=6$$

**Answer: (B)**

**62.** A plane is flying parallel to the ground at a height of 4 km/hour over a radar station. A short time later, the radar station staff announces that the distance between the plane and the station is 5 km. They also announced that the distance between the plane and the station is increasing at a rate of 300 km/hour. At that moment, the rate at which the plane is moving parallel to the ground is

(A) 300 km/hour (B) 400 km/hour
(C) 500 km/hour (D) 600 km/hour

**Solution:** See Fig. 3.16. Let $S$ be the radar station. At a time $t$, $P$ is the position of the plane. Let $x$ be the distance horizontal to the ground and $y$ be the distance $SP$. Then

$$y^2 = 4^2 + x^2$$

Therefore

$$y\frac{dy}{dt}=x\frac{dx}{dt}$$

When $y = 5$, then $x=\sqrt{5^2-4^2}=3$.

When $y = 5$ and $\frac{dy}{dt}=300$, we get

$$5\times 300 = 3\times\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt}=500$$

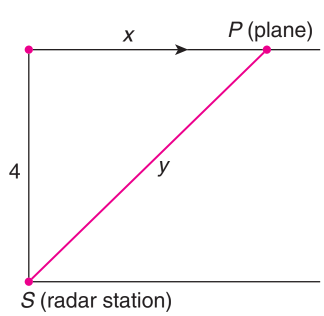



**FIGURE 3.16** Single correct choice type question 62.

**Answer: (C)**

**63.** Water is pouring into a conical vessel with vertex upwards at the rate of 3 cubicmeters per minute. The radius of the base is 5 m and the height is 10 m. When the water level is 7 m from the base, the rate at which water level increases is

(A) $\frac{18}{7\pi}$ (B) $\frac{9\pi}{7}$
(C) $\frac{4}{3\pi}$ (D) $\frac{9}{7\pi}$

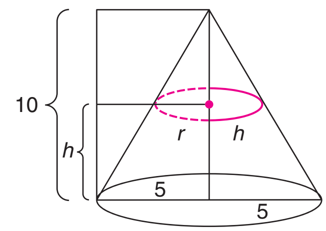



**FIGURE 3.17** Single correct choice type question 63.

**Solution:** Let $r$ be the radious of the water surface and $h$ its height from the base at time $t$ (Fig. 3.17). Let $V_1$ be the volume whole cone. Then

$$V_1=\frac{1}{3}\pi(5)^2(10)=250\frac{\pi}{3}$$

Let $V_2$ be the volume of the cone above the water surface at time $t$. Then

$$V_2=\frac{1}{3}\pi r^2(10-h)$$

Also we have

$$\frac{10}{10-h}=\frac{5}{r}$$
$$\Rightarrow 2r = 10-h$$

Therefore

$$V_2=\frac{\pi}{3}\left(\frac{10-h}{2}\right)^2(10-h)=\frac{\pi}{12}(10-h)^3$$

Let $V$ be the volume of the water at time $t$. Then

$$V = V_1 - V_2$$
$$=\frac{250\pi}{3}-\frac{\pi}{12}(10-h)^3$$

Differentiating we get

$$\frac{dV}{dt}=\frac{3\pi}{12}(10-h)^2\frac{dh}{dt}$$

When $h = 7$, we have

$$3=\frac{dV}{dt}$$
$$=\frac{\pi}{4}(10-7)^2\frac{dh}{dt}$$

$$= \frac{\pi}{4} \times 9 \times \frac{dh}{dt}$$

This implies

$$\frac{dh}{dt} = \frac{4}{3\pi}$$

**Answer: (C)**

**64.** Consider Fig. 3.18. $ABCD$ is a square. $A$ runner starts at $C$ and is running towards $D$ at a rate of 20 ft/sec. The rate of change of the distance of the runner from the point $A$ when he is at a distance of 60 ft from the point $C$ is

(A) $-2\sqrt{10}$ ft/sec (B) $-\sqrt{10}$ ft/sec

(C) $-\frac{1}{2}\sqrt{10}$ ft/sec (D) $-\frac{3}{2}\sqrt{10}$ ft/sec

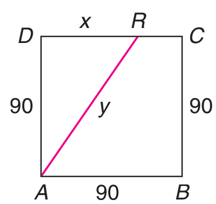



**FIGURE 3.18** Single correct choice type question 64.

**Solution:** In Fig. 3.18 $x$ is the distance of the runner from $D$ and $y$ is the distance between $A$ and the runner at a time $t$. Using Pythagoras theorem for triangle $ADR$ we get

$$y^2 = x^2 + 90^2$$

Differentiating we have

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \quad (3.29)$$

When $x = 90 - 60 = 30$, then $y = \sqrt{90^2 + 30^2} = 30\sqrt{10}$. From Eq. (3.29) we have

$$\frac{dy}{dt} = \frac{30}{30\sqrt{10}}(-20) \quad (\because x \text{ is decreasing})$$
$$= -2\sqrt{10}$$

**Answer: (A)**

**65.** A ladder of 26 ft length is leaning against a wall. The bottom foot of the ladder is moving away from the base of the wall at a rate of 3 feet per second. When the bottom of the ladder is 10 ft away from the wall, then the angle $\theta$ made by the ladder with the ground decreases at the rate of

(A) $\frac{1}{4}$ radian/sec (B) $\frac{1}{6}$ radian/sec

(C) $\frac{1}{8}$ radian/sec (D) 1 radian/sec

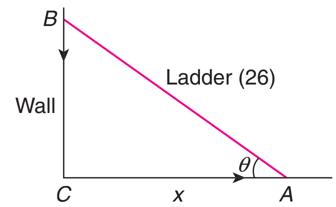



**FIGURE 3.19** Single correct choice type question 65.

**Solution:** See Fig. 3.19. $AB = 26$ (ladder), $BC$ is wall, $CA = x$ is the distance of the bottom of the ladder at time $t$ from the base of the wall. Now,

$$\cos\theta = \frac{x}{26}$$

Differentiating we get

$$(-\sin\theta)\frac{d\theta}{dt} = \frac{1}{26}\frac{dx}{dt}$$

When $x = 10$, $\sin\theta = \frac{BC}{AC} = \frac{24}{26} = \frac{12}{13}$ and $\frac{dx}{dt} = 3$ we get

$$\frac{d\theta}{dt} = -\frac{26}{24} \times \frac{1}{26} \times 3 = -\frac{1}{8}$$

**Answer: (C)**

## Mean Value Theorems

**66.** The value of $c$ in Rolle's theorem for the function $f(x) = x^{2/3} - 2x^{1/3}$ for $x \in [0, 8]$ is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) $\frac{3}{2}$

**Solution:** Clearly $f$ is continuous on $[0, 8]$ and

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3}$$

exists for $x > 0$. Therefore, $f$ is differentiable in (0, 8). Also

$$f(0) = 0 \quad \text{and} \quad f(8) = 4 - 2(2) = 0$$

Thus, $f(0) = f(8)$. Hence by Rolle's theorem, $f'(c) = 0$ for some $c \in (0, 8)$. So

$$f'(c) = 0 \Rightarrow \frac{2}{3}c^{-1/3} - \frac{2}{3}c^{-2/3} = 0$$
$$\Rightarrow c^{-1/3} - c^{-2/3} = 0$$
$$\Rightarrow c^{2/3} - c^{1/3} = 0$$
$$\Rightarrow c^{1/3}(c^{1/3} - 1) = 0$$
$$\Rightarrow c = 1 \quad (\because c \neq 0)$$

**Answer: (B)**

**67.** Let

$$f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1} & \text{for } x \neq 1 \text{ and } x \in [-2, 3] \\ -6 & \text{for } x = 1 \end{cases}$$

Then, value of $c$ in the Rolle's theorem for the function $f(x)$ on $[-2, 3]$ is

(A) 1 (B) 2

(C) $-\frac{1}{2}$ (D) $\frac{1}{2}$

**Solution:** We have

$$f(x) = \begin{cases} x^2 - x - 6 & \text{if } x \neq 1 \text{ and } x \in [-2, 3] \\ -6 & \text{if } x = 1 \end{cases}$$

Clearly $\lim\limits_{x \to 1} f(x) = -6 = f(1) \Rightarrow f$ is continuous at $x = 1$ and hence $f$ is continuous on $[-2, 3]$. $f$ is also differentiable in $(-2, 3)$. Further,

$$f(-2) = 4 + 2 - 6 = 0$$

and
$$f(3) = 9 - 3 - 6 = 0$$

Thus $f(-2) = 0 = f(3)$. Hence $f'(c) = 0$ for some $c \in (-2, 3)$. Therefore $f'(c) = 0$ implies

$$2c - 1 = 0 \quad \text{or} \quad c = \frac{1}{2} \in (-2, 3)$$

**Answer: (D)**

**68.** Let $f(x) = x^3 - 3x^2 + x + 1, x \in [1, 1+\sqrt{2}]$. The value of $c$ in the Rolle's theorem for $f(x)$ is

(A) $1 + \frac{1}{3}\sqrt{6}$ (B) $1 + \frac{1}{4}\sqrt{6}$

(C) $1 + \frac{2}{5}\sqrt{6}$ (D) $\frac{3}{\sqrt{2}}$

**Solution:** Since $f(x)$ is a polynomial, it is differentiable for all $x$ and in particular on $[1+1+\sqrt{2}]$. Also $f(1) = 0$ and

$$f(1+\sqrt{2}) = (1+\sqrt{2})^3 - 3(1+\sqrt{2})^2 + (1+\sqrt{2}) + 1$$
$$= (1 + 3\sqrt{2} + 3 \times 2 + 2\sqrt{2}) - 3(1 + 2\sqrt{2} + 2) + 1 + \sqrt{2} + 1$$
$$= (7 - 9 + 2) + 6\sqrt{2} - 6\sqrt{2}$$
$$= 0$$

Thus, $f(1) = f(1+\sqrt{2})$. Hence by Rolle's theorem, $f'(c) = 0$ for some $c \in (1, 1+\sqrt{2})$. That is

$$3c^2 - 6c + 1 = 0$$
$$\Rightarrow c = \frac{6 \pm \sqrt{24}}{6} = 1 + \frac{1}{3}\sqrt{6}$$

Now

$$1 + \frac{1}{3}\sqrt{6} < 1 + \sqrt{2}$$
$$\Rightarrow c = 1 + \frac{1}{3}\sqrt{6}$$

**Answer: (A)**

**69.** Let

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$$

Then the value of $c$ in Rolle's theorem for $f(x)$ in $(0, 2)$ is

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{3}{2}$ (D) does not exist

**Solution:** Obviously

$$f(1-0) = 1 \quad \text{and} \quad f(1+0) = 2 - 1 = 1$$

This implies $f$ is continuous at $x = 1$ and hence continuous on $[0, 2]$. Therefore

$$f'(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ -1, & 1 < x \leq 2 \end{cases}$$

which implies $f'(1)$ does not exists. Hence, Rolle's theorem cannot be applied.

**Answer: (D)**

**70.** Let

$$f(x) = \log\left(\frac{x^2 + ab}{(a+b)x}\right)$$

where $0 < a < b$. Then value of $c$ in Rolle's theorem for $f(x)$ on $[a, b]$ is

(A) $\dfrac{a+b}{2}$ (B) $\sqrt{ab}$

(C) $\dfrac{a+b}{ab}$ (D) $\dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$

**Solution:** Since $f(x)$ is a log function, it is continuous and differentiable for all $x > 0$ and in particular continuous in $[a, b]$ and differentiable in $(a, b)$. Also

$$f(a) = \log\left(\frac{a(a+b)}{a(a+b)}\right) = \log 1 = 0 = f(b)$$

Therefore, Rolle's theorem is applicable. Now

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$$

$$= \frac{x^2-ab}{x(x^2+ab)}$$

Therefore, $f'(x)=0$ for some $x \in (a, b) \Rightarrow x = \sqrt{ab}$. Hence $c = \sqrt{ab}$ and $a < \sqrt{ab} < b$.

**Answer: (B)**

**71.** If Rolle's theorem is applied for $f(x) = (x-2)\log x$ on the interval $[1, 2]$, then one of the following equations has solution in $(1, 2)$.

(A) $(x-2)\log x = 2$ (B) $(x-2)\log x = x$

(C) $(x-2)\log x = 2-x$ (D) $x\log x = 2-x$

**Solution:** Since $f$ is continuous on $[1, 2]$ and differentiable in $(1, 2)$ and $f(1) = 0 = f(2)$, by Rolle's theorem $f'(1) = 0$ for some $x \in (1, 2)$. That is

$$\log x + \frac{x-2}{x} = 0$$

Hence $x\log x = 2-x$ has a solution in $(1, 2)$.

**Answer: (D)**

**72.** The number of values of $c$ in Rolle's theorem for $f(x) = (x-1)(x-2)^2, 1 \le x \le 2$ is

(A) 2 (B) 1

(C) 0 (D) not applicable

**Solution:** Clearly $f$ is continuous on $[1, 2]$ and differentiable in $(1, 2)$. Further $f(1) = 0 = f(2)$. Therefore, by Rolle's theorem,

$$f'(x) = (x-2)^2 + 2(x-1)(x-2) = 0$$

for some $x \in (1, 2)$. That is

$$(x-2)(3x-4) = 0$$

Therefore $c = 2, 4/3$. Since $c \in (1, 2)$, we have $c = 4/3$.

**Answer: (B)**

**73.** In $[0, 1]$, Lagrange's mean value theorem is not applied to

(A) $f(x) = \begin{cases} \dfrac{1}{2} - x & \text{if } x < \dfrac{1}{2} \\ \left(\dfrac{1}{2} - x\right)^2 & \text{if } x \ge \dfrac{1}{2} \end{cases}$

(B) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

(C) $f(x) = x\,|x|$

(D) $f(x) = |x|$

**Solution:**

(A) Clearly $f$ is continuous on $[0, 1]$. Now

$$f'(x) = \begin{cases} -1 & \text{for } x < \dfrac{1}{2} \\ -2\left(\dfrac{1}{2} - x\right) & \text{for } x \ge \dfrac{1}{2} \end{cases}$$

Clearly

$$f'\left(\frac{1}{2} - 0\right) = -1 \quad \text{and} \quad f'\left(\frac{1}{2} + 0\right) = 0$$

That is, $f$ is not differentiable at $1/2 \in (0, 1)$. Therefore Lagrange's mean value theorem is **not applicable** to $f(x)$ on $[0, 1]$. Hence (A) is correct.

(B) We have

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = f(0)$$

This implies $f$ is continuous on $[0, 1]$. Also $(\sin x)/x$ is differentiable for all $x \ne 0$ and hence Lagrange's mean value theorem is **applicable**.

(C) We have $f(x) = x^2$ $(\because x \ge 0)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Hence Lagrange's mean value theorem is **applicable**.

(D) We have $f(x) = |x| = x$ $(\because x \ge 0)$ and hence Lagrange's mean value is **applicable**.

**Answer: (A)**

**74.** Let $f: [0, 4] \to \mathbb{R}$ be a differentiable function. Then there exist real numbers $a, b$ belonging to $(0, 4)$ such that

$$[f(4)]^2 - [f(0)]^2 = kf'(a)f(b)$$

where $k$ is

(A) 4 (B) 8

(C) $\dfrac{1}{12}$ (D) 2

**Solution:** By Lagrange's mean value theorem, there exists $a \in (0, 4)$ such that

$$\frac{f(4)-f(0)}{4-0} = f'(a)$$

Therefore

$$f(4)-f(0) = 4f'(a) \quad (3.30)$$

Since $[f(4) + f(0)]/2$ lies between $f(0)$ and $f(4)$, by the Intermediate value property of a continuous functions (Theorem 1.32, Chapter 1), there exists $b \in (0, 4)$ such that

$$\frac{f(4)+f(0)}{2} = f(b)$$
$$\Rightarrow f(4)+f(0) = 2f(b) \quad (3.31)$$

From Eqs. (3.30) and (3.31), we have

$$[f(4)]^2 - [f(0)]^2 = 8f'(a)f(b)$$

So, the value of $k$ is 8.

**Answer: (B)**

**75.** Let $a, b, c$ be real numbers and $a < b < c$. $f$ is continuous on $[a, c]$ and differentiale in $(a, c)$. If $f'(x)$ is strictly increasing, then $(c - b) f(a) + (b - a) f(c)$ is

(A) greater than $(c - a) f(b)$
(B) less than $(c - a) f(b)$
(C) equal to $(c - a) f(b)$
(D) greater than $2(c - a) f(b)$

**Solution:** Using Lagrange's mean value theorem for $f$ on $[a, b]$ and $[b, c]$, we have

$$\frac{f(b)-f(a)}{b-a} = f'(u)$$

and

$$\frac{f(c)-f(b)}{c-b} = f'(v)$$

for some $u \in (a, b)$ and $v \in (b, c)$. Since $a < u < b < v < c$ and $f'(x)$ is strictly increasing, we have that

$$\frac{f(b)-f(a)}{b-a} = f'(u) < f'(v) = \frac{f(c)-f(b)}{c-b}$$

Therefore ($\because b-a > 0$ and $c-b > 0$)

$$(c - b)[f(b) - f(a)] < (b - a)[f(c) - f(b)]$$
$$(c-b) f(a) + (b-a) f(c) > (c-b+b-a) f(b) = (c-a) f(b)$$

**Answer: (A)**

**76.** Let $f$ be continuous on $[0, 2]$, differentiable in $(0, 2)$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$. Then for some $x \in (0, 2)$, $f'(x)$ is equal to

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$

(C) $\frac{5}{3}$ (D) $\frac{1}{7}$

**Solution:** Using Lagrange's mean value theorem for $f$ on $[0, 2]$, there exists $x_0 \in (0, 2)$ such that

$$f'(x_0) = \frac{f(2)-f(0)}{2-0} = \frac{1-0}{2} = \frac{1}{2}$$

Now, using Rolle's theorem for $f$ on $[1, 2]$, there exists $y_0 \in (1, 2)$ such that $f'(y_0) = 0$. Observe that,

$$f'(y_0) = 0 < \frac{1}{7} < \frac{1}{2} = f'(x_0)$$

Hence, by Intermediate value theorem for derivative (Darboux theorem, Theorem 3.10), there exists $x_1$ lying between $x_0$ and $y_0$ and hence in $(0, 2)$ such that

$$f'(x_1) = \frac{1}{7}$$

**Answer: (D)**

**77.** $f$ is twice differentiable function on $[a, b]$ such that $f(a) = f(b) = 0$ and $f(x) > 0$ for all $x \in (a, b)$. Then

(A) $f''(x) > 0 \; \forall \; x \in (a, b)$

(B) $f''(x) < 0 \; \forall \; x \in (a, b)$

(C) $f''(x_0) < 0$ for some $x_0 \in (a, b)$

(D) $f''(x_0) = 0$ for some $x_0 \in (a, b)$

**Solution:** Using Rolle's theorem for $f$ on $[a, b]$, there exists $c \in (a, b)$ such that $f'(c) = 0$. Now, using Lagrange's mean value theorem for $f$ on the intervals $[a, c]$ and $[c, b]$, there exists $c_1 \in (a, c)$ and $c_2 \in (c, b)$ such that

$$\frac{f(c)-f(a)}{c-a} = f'(c_1)$$

and

$$\frac{f(b)-f(c)}{b-c} = f'(c_2)$$

Now, use Lagrange's mean value theorem for $f'$ on the interval $[c_1, c_2]$ so that there exists $x_0 \in (c_1, c_2)$ such that

$$\frac{f'(c_2)-f'(c_1)}{c_2-c_1} = f''(x_0)$$

$$\Rightarrow f''(x_0) = \frac{f'(c_2)-f'(c_1)}{c_2-c_1}$$
$$= \frac{1}{c_2-c_1}\left[\frac{-f(c)}{b-c} - \frac{f(c)}{c-a}\right] \quad [\because f(a) = f(b) = 0]$$
$$< 0 \quad [\because f(c) > 0]$$

**Answer: (C)**

**78.** Consider the following two statements:

**P:** Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f'(x) > f(x)\ \forall\ x\in\mathbb{R}$ and $f(x_0) = 0$ for some $x_0\in\mathbb{R}$. Then $f(x) > 0$ for all $x > x_0$.

**Q:** If $f$ is continuous on $[a, b]$, differentiable in $(a, b)$ and $f'(x)\neq 0$ for all $x$ in $(a, b)$, then for *at most one* value of $x$, $f(x)$ is zero.

Then,

(A) both P and Q are true
(B) both P and Q are not true
(C) P is true whereas Q is not true
(D) Q is true, but P is not true

**Solution:** **Statement P is true.** Define $h(x) = e^{-x} f(x)$ for $x\in\mathbb{R}$. Clearly $h$ is differentiable and

$$h'(x) = e^{-x}[f'(x) - f(x)] > 0\ \forall\ x\in\mathbb{R}$$

Therefore $h$ is increasing in $\mathbb{R}$ (see Theorem 3.6). So, $x > x_0$ implies

$$h(x) > h(x_0) = e^{-x_0} f(x_0) = 0 \text{ (given)}$$

$$\Rightarrow e^{-x}h(x) = h(x) > 0 \quad \text{for } x > x_0$$

$$\Rightarrow f(x) > 0 \quad \text{for } x > x_0$$

Thus P is true.

**Statement Q is true:** Suppose $f$ has two different zeros say $a<\alpha<\beta<b$. Hence by using Rolle's theorem for $f$ on $[\alpha,\beta]$, there exists some $x_0$ in $(\alpha,\beta)$ such that $f'(x_0) = 0$. But by hypothesis $f'(x)\neq 0\ \forall\ x\in(a, b)$. Hence $f$ cannot have more than one zero (if it has). Thus Q is also true.

**Answer: (A)**

**79.** Consider the following two statements:

**$P_1$:** Let $f$ and $g$ be continuous on $[a, b]$ and differentiable in $(a, b)$. There exists $c\in(a, b)$ such that

$$f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]$$

**$P_2$:** If $f$ is continuous on $[a, b]$, differentiable in $(a, b)$ and $f(a) = f(b) = 0$, then for any real $\lambda$, the equation $f'(x)+\lambda f(x) = 0$ has at least one solution in $(a, b)$.

Then,

(A) both $P_1$ and $P_2$ are false
(B) $P_1$ is true, but $P_2$ is false
(C) both $P_1$ and $P_2$ are true
(D) $P_1$ is false whereas $P_2$ is true

**Solution:** **$P_1$ is true.** Define

$$\phi(x) = f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)]$$

Clearly $\phi$ is continuous on $[a, b]$ and differentiable in $(a, b)$. Further

$$\phi(a) = f(a)[g(b)-g(a)] - g(a)[f(b)-f(a)]$$

$$= f(a)g(b) - f(b)g(a) = \phi(b)$$

Hence, by Rolle's theorem, there is $c\in(a, b)$ such that $\phi'(c) = 0$. Hence

$$f'(c)[g(b)-g(a)] - g'(c)[f(b)-f(a)] = 0$$

Thus $P_1$ is true.

**$P_2$ is true.** Define $\phi(x) = e^{\lambda x} f(x)$ so that $\phi(x)$ satisfies all the conditions of Rolle's theorem. Hence, there exists $c\in(a, b)$ such that $\phi'(c) = 0$. That is

$$e^{\lambda c}[f'(c)+\lambda f(c)] = 0$$

Hence

$$f'(c)+\lambda f(c) = 0 \quad (\because e^{\lambda c}\neq 0)$$

Thus, the equation $f'(x)+\lambda f(x) = 0$ has a solution in $(a, b)$.

**Answer: (C)**

**80.** If $f$ is continuous on $[a, b]$ and differentiable in $(a, b)$ $(ab > 0)$, then there exists $c\in(a, b)$ such that

$$\frac{f(b)-f(a)}{(1/b)-(1/a)} =$$

(A) $c^2 f'(c)$ (B) $cf'(c)$
(C) $-cf'(c)$ (D) $-c^2 f'(c)$

**Solution:** We have $ab > 0 \Rightarrow 0\notin(a, b)$. Define

$$F(x) = f\left(\frac{1}{x}\right)\ \forall\ x\in\left[\frac{1}{b},\frac{1}{a}\right]$$

Obviously $F$ is continuous on $[1/b, 1/a]$ and differentiable in $(1/b, 1/a)$. Hence, by Lagrange's mean value theorem, there $d\in(1/b, 1/a)$ such that

$$F'(d) = \frac{F(1/a)-F(1/b)}{(1/a)-(1/b)}$$

But

$$F'(d) = f'\left(\frac{1}{d}\right)\left(\frac{-1}{d^2}\right)$$

$$= -f'\left(\frac{1}{d}\right)\left(\frac{1}{d^2}\right)$$

Therefore

$$\frac{F(1/a)-F(1/b)}{(1/a)-(1/b)} = F'(d) = -f'\left(\frac{1}{d}\right)\left(\frac{1}{d^2}\right)$$

$$\Rightarrow \frac{f(a)-f(b)}{(1/a)-(1/b)} = -f'\left(\frac{1}{d}\right)\left(\frac{1}{d^2}\right)$$

Now, put $1/d = c$. Therefore

$$\frac{f(b)-f(a)}{(1/b)-(1/a)} = -c^2 f'(c)$$

**Answer: (D)**

## ALITER

Define

$$F(x)=\left(\frac{1}{b}-\frac{1}{a}\right)f(x)-\frac{1}{x}[f(b)-f(a)]$$

for $x\in[a,b]$ and use Rolle's theorem for $F(x)$ on $[a,b]$.

**81.** Consider the function $f(x) = 1 - x^{4/5}$ for $x \in [-1, 1]$. Then, the number of conditions of Rolle's theorem that are *not* satisfied by $f(x)$ is

(A) 1 (B) 2
(C) 3 (D) 0

**Solution:** Clearly $f(-1) = 0 = f(1)$ and $f$ is continuous on $[-1, 1]$. But

$$f'(x)=-\frac{4}{5}x^{-1/5}$$

implies that $f$ is not differentiable at $x = 0$ and $0 \in (-1, 1)$. That is, $f$ is not differentiable in $(-1, 1)$. Thus, only one condition is not satisfied.

**Answer: (A)**

**82.** $A(1, 0)$ and $B(e, 1)$ are points on the curve $y = \log_e x$. If $P$ is a point on the curve at which the tangent to the curve is parallel to the chord $AB$, then the abscissa of $P$ is

(A) $\frac{e-1}{2}$ (B) $e-1$

(C) $\frac{e+1}{\sqrt{2}}$ (D) $\frac{e-1}{\sqrt{2}}$

**Solution:** Using Lagrange's mean value theorem for $f(x) = \log_e x$ on the interval $[1, e]$, there exists $x_0 \in (1, e)$ such that

$$f'(x_0)=\frac{f(e)-f(1)}{e-1}$$
$$\Rightarrow \frac{1}{x_0}=\frac{1-0}{e-1}$$
$$\Rightarrow x_0=e-1$$

**Answer: (B)**

**83.** Let $f$ be continuous on $[a, b]$ and differentiable in $(a, b)$. If $f'(x)\neq 0\ \forall\ x\in[a,b]$, then there exists $\theta\in(a,b)$ such that

$$\frac{f'(\theta)}{f(\theta)}=$$

(A) $\frac{1}{a+\theta}+\frac{1}{b+\theta}$ (B) $\frac{1}{a-\theta}+\frac{1}{b-\theta}$

(C) $\frac{a+b}{2\theta}$ (D) $\frac{a+b}{\theta}$

**Solution:** Define

$$F(x) = (a - x)\,(b - x)\,f(x)$$

for $a \le x \le b$. Since both $f(x)$ and $(a - x)\,(b - x)$ are continuous and differentiable on $[a, b]$ and $(a, b)$, it follows that $F(x)$ is continuous on $[a, b]$ and differentiable in $(a, b)$. Further $F(a) = 0 = F(b)$. Hence by Rolle's theorem, there exists $\theta\in(a,b)$ such that $F'(\theta)=0$. Now

$$F'(x)=-(b-x)f(x)-(a-x)f(x)+(a-x)(b-x)f'(x)$$

So, $F'(\theta)=0$ implies

$$(a-\theta)(b-\theta)f'(\theta)=(b-\theta)f(\theta)+(a-\theta)f(\theta)$$
$$\Rightarrow \frac{f'(\theta)}{f(\theta)}=\frac{1}{a-\theta}+\frac{1}{b-\theta}$$

**Answer: (B)**

**84.** $f$ is continuous on $[0, 5]$ and differentiable in $(0, 5)$. Further $f(0) = 4$ and $f(5) = -1$. If

$$g(x)=\frac{f(x)}{x+1}$$

for $0 \le x \le 5$, then there exists $c\in(0, 5)$ such that $g'(c)$ is equal to

(A) $\frac{-4}{6}$ (B) $\frac{4}{6}$

(C) $\frac{5}{6}$ (D) $\frac{-5}{6}$

**Solution:** We have $0 \le x \le 5 \Rightarrow x + 1 \neq 0$. Obviously $g$ is continuous on $[0, 5]$ and differentiable in $(0, 5)$. Hence, by Lagrange's mean value theorem there exists $c \in (0, 5)$ such that

$$g'(c)=\frac{g(5)-g(0)}{5-0}$$

Now,

$$g(5)=\frac{f(5)}{6}=\frac{-1}{6}$$
$$g(0)=\frac{f(0)}{6}=\frac{4}{6}$$

Therefore

$$g'(c)=\frac{-(1/6)-4}{5}=\frac{-5}{6}$$

**Answer: (D)**

**85.** Let $f$ be continuous on $[a, b]$, differentiable in $(a, b)$ and $f(a) = f(b) = 0$. Further, suppose $g$ is continuous on $[a, b]$ and differentiable in $(a, b)$. Then for the function

$$H(x) = f'(x) + f(x)g'(x)$$

one of the following statements is definitely true.

(A) $H(x) = 0$ for infinitely many $x$ in $(a, b)$
(B) $H(x) > 0$ for infinitely many $x$ in $(a, b)$
(C) $H'(x) > 0$ for infinitely many $x$ in $(a, b)$
(D) $H(x) = 0$ for atleast one $x$ in $(a, b)$

**Solution:** Let $F(x) = e^{g(x)} f(x)$ for $x \in [a, b]$. Since $F(x)$ satisfies the conditions of Rolle's theorem, there exists $c \in [a, b]$ such that $F'(c) = 0$. Therefore

$$e^{g(c)}[g'(c)f(c) + f'(c)] = 0$$

$$f'(c) + f(c)g'(c) = 0 \quad (\because e^{g(c)} \neq 0)$$

Thus $H(x) = 0$ has a solution in $(a, b)$.

**Answer: (D)**

**86.** On the interval $[-\sqrt{2}, \sqrt{2}]$ one of the values of $c$ in Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ is

(A) $\dfrac{3}{5}$ (B) $\dfrac{4}{5}$

(C) $\dfrac{2}{3}$ (D) $\dfrac{-2}{3}$

**Solution:** $f(x)$ being a polynomial function, is continuous and differentiable for all real $x$. Also

$$f(x) \equiv (x^2 - 2)(2x + 1)$$

and $$f(-\sqrt{2}) = 0 = f(\sqrt{2})$$

Hence by Rolle's theorem for at least one $x \in (-\sqrt{2}, \sqrt{2})$

$$f'(x) = 6x^2 + 2x - 4 = 0$$
$$\Rightarrow 3x^2 + x - 2 = 0$$
$$\Rightarrow (3x - 2)(x + 1) = 0$$
$$\Rightarrow x = \frac{2}{3}, -1$$

Note that $2/3$ and $1 \in [-\sqrt{2}, \sqrt{2}]$.

**Answer: (C)**

**87.** The smallest positive root of the equation $\tan x - x = 0$ lies in

(A) $\left(0, \dfrac{\pi}{2}\right)$ (B) $\left(\dfrac{\pi}{2}, \pi\right)$

(C) $\left(\pi, \dfrac{3\pi}{2}\right)$ (D) $\left(\dfrac{3\pi}{2}, 2\pi\right)$

**Solution:** See Fig. 3.20. Let $f(x) = \tan x - x$. Clearly $f(x)$ is continuous at $x \neq$ odd multiple of $\pi/2$. Therefore

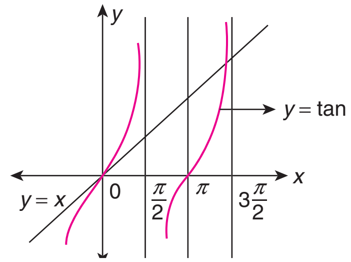



**FIGURE 3.20** Single correct choice type question 87.

$$f'(x) = \sec^2 x - 1 > 0 \quad \text{in} \quad \left(0, \frac{\pi}{2}\right)$$

Hence $f$ is strictly increasing in $(0, \pi/2)$. In $(\pi/2, \pi)$, $\tan x$ is negative so that $f(x) = \tan x - x < 0$.

We consider the interval $(\pi, 3\pi/2)$. Now

$$\lim_{x \to \pi+0} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(\pi + h)$$
$$= \lim_{\substack{h \to 0 \\ h > 0}} [\tan(\pi + h) - (\pi + h)]$$
$$= \lim_{h \to 0} (\tan h - \pi - h)$$
$$= -\pi < 0$$

Also,

$$\lim_{x \to \frac{3\pi}{2} - 0} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f\left(\frac{3\pi}{2} - h\right)$$
$$= \lim_{h \to 0} \left[\tan\left(\frac{3\pi}{2} - h\right) - \left(\frac{3\pi}{2} - h\right)\right]$$
$$= \lim_{h \to 0} \left(\cot h + h - \frac{3\pi}{2}\right)$$
$$= +\infty$$

Therefore, between $\pi$ and $3\pi/2$, $f$ changes sign. So $f$ must vanish in between the two values. Therefore $f(x) = 0$ has solution in $(\pi, 3\pi/2)$.

**Answer: (C)**

**88.** If $a + b + c = 0$ where $a, b, c$ are real, then the equation $3ax^2 + 2bx + c = 0$ has

(A) at least one root in $[0, 1]$
(B) one root in $[2, 3]$ and another root in $[-2, -1]$
(C) imaginary roots
(D) one root in $[1, 2]$ and another root in $(-1, 0)$

**Solution:** Let $f(x) \equiv ax^3 + bx^2 + cx$. So that $f$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$. Also $f(0) = 0$ and $f(1) = a + b + c = 0$ by hypothesis. Hence, by Rolle's theorem, $f'(x) = 0$ for some $x \in (0, 1)$. But

$$f'(x) = 3ax^2 + 2bx + c$$

Thus, $3ax^2 + 2bx + c = 0$ has a root in $(0, 1)$.

**Answer: (A)**

**89.** The function $f(x) = x^3 + bx^2 + ax + 5$ satisfies all the conditions of Rolle's theorem. Further it is given that the value of $c$ such that $f'(c) = 0$ is $2 + (1/\sqrt{3})$. Then,

(A) $a = 5, b = 11$ (B) $a = 11, b = -6$
(C) $a = -11, b = 6$ (D) $a = -11, b = -6$

**Solution:** We have $f(1) = f(3)$. This implies

$$b + a + 6 = 9b + 3a + 32$$
$$\Rightarrow a + 4b = -13 \tag{3.32}$$

Now $c = 2 + (1/\sqrt{3})$ and $f'(c) = 0$ implies

$$2b(2\sqrt{3} + 1) + \sqrt{3}a = -12 - 13\sqrt{3} \tag{3.33}$$

Solving Eqs. (3.32) and (3.33), we have $a = 11, b = -6$.

**Answer: (B)**

**90.** $\lim\limits_{x\to 1}\left(\dfrac{\cos(\pi/2)x}{\log(1/x)}\right) =$

(A) 0 (B) $\infty$ (C) $\dfrac{\pi}{2}$ (D) $\dfrac{\pi}{4}$

**Solution:** We use Cauchy's mean value theorem. Take

$$f(x) = \cos\frac{\pi}{2}x \quad \text{and} \quad g(x) = \log x$$

with $a = x > 0$ and $b = 1$. That is the interval is $[x, 1]$. By Cauchy's mean value theorem, there exists $\theta \in (x, 1)$ such that

$$\frac{f(1) - f(x)}{g(1) - g(x)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \frac{\cos\dfrac{\pi}{2} - \cos\dfrac{\pi}{2}x}{\log 1 - \log x} = \frac{-\dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}\theta\right)}{+\dfrac{1}{\theta}} \tag{3.34}$$

Since $x < \theta < 1; x \to 1 \Rightarrow \theta \to 1$. Taking limits $x \to 1$, on both sides of Eq. (3.34), we have

$$\lim_{x\to 1}\left(\frac{-\cos\dfrac{\pi}{2}x}{\log\dfrac{1}{x}}\right) = \lim_{\theta\to 1}\left(\frac{-\dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}\theta\right)}{\left(\dfrac{1}{\theta}\right)}\right) = -\frac{\pi}{2}$$

$$\Rightarrow \lim_{x\to 1}\left(\frac{\cos(\pi/2)x}{\log(1/x)}\right) = \frac{\pi}{2}$$

**Answer: (C)**

**91.** The number of real values of $k$ such that the equation $x^3 - 3x - k = 0$ has two distinct real roots in the interval $(0, 1)$ is

(A) 0 (B) 1 (C) 2 (D) 4

**Solution:** Let $f(x) = x^3 - 3x - k$ and suppose $0 < \alpha < \beta < 1$ are two distinct real roots of $f(x) = 0$. Using Rolle's theorem for $f(x)$ on the interval $[\alpha, \beta]$, there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$. That is,

$$3c^2 - 3 = 0$$
$$\Rightarrow c = \pm 1 \notin (\alpha, \beta)$$

Therefore, for no real $k$, $f(x) = 0$ has two distinct real roots in $(0, 1)$. If at all $f(x) = 0$ has real root in $(0, 1)$, it can have only one real root.

**Answer: (A)**

**92.** If $f(x) = (1 - x)^{5/2}$ satisfies the relation

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(\theta x)$$

then as $x \to 1$, the value of $\theta$ is

(A) $\dfrac{4}{25}$ (B) $\dfrac{25}{4}$ (C) $\dfrac{25}{9}$ (D) $\dfrac{9}{25}$

**Solution:** We have

$$f'(x) = -\frac{5}{2}(1-x)^{3/2}$$
$$f''(x) = \frac{15}{4}(1-x)^{1/2}$$

Also

$$f(0) = 1, f'(0) = -\frac{5}{2}$$

and $$f''(\theta x) = \frac{15}{4}(1 - \theta x)^{1/2}$$

Therefore

$$(1-x)^{5/2} = 1 - \frac{5}{2}x + \frac{x^2}{2}(1-\theta x)^{1/2} \times \frac{15}{4}$$

Taking limit $x \to 1$ on both sides, we get

$$0 = 1 - \frac{5}{2} + \frac{15}{8}(1-\theta)^{1/2}$$
$$\Rightarrow (1-\theta)^{1/2} = \frac{4}{5}$$
$$\Rightarrow \theta = 1 - \frac{16}{25} = \frac{9}{25}$$

**Answer: (D)**

**93.** Rolle's theorem holds for the function $f(x) = x^3 + mx^2 + nx$ on the interval $[1, 2]$ and the value of $c$ is 4/3. Then

(A) $m = 8, n = -5$ (B) $m = -5, n = 8$

(C) $m = 5, n = -8$ (D) $m = -5, n = -8$

**Solution:** We have

$$f(1) = f(2) \Rightarrow 1 + m + n = 8 + 4m + 2n$$
$$\Rightarrow 3m + n = -7 \qquad (3.35)$$

$$f'\left(\frac{4}{3}\right) = 0 \Rightarrow 3\left(\frac{4}{3}\right)^2 + 2m\left(\frac{4}{3}\right) + n = 0$$

$$\Rightarrow 8m + 3n = -16 \qquad (3.36)$$

Solving Eqs. (3.35) and (3.36), we obtain $m = -5, n = 8$.

**Answer: (B)**

**94.** Rolle's theorem is **not** applicable to one of the following functions:

(A) $f(x) = x^2$ on $[-1, 1]$
(B) $f(x) = x^2 - 3x + 2$ on $[1, 2]$
(C) $f(x) = \tan x$ on $[0, \pi]$
(D) $f(x) \sin x$ on $[0, \pi]$

**Solution:**

(A) $f(x) = x^2$ is continuous and differentiable for all real $x$. Further $f(-1) = 1 = f(1)$. Hence, Rolle's theorem is applicable.

(B) We have

$$f(x) \equiv x^2 - 3x + 2 = (x-1)(x-2)$$
$$f(1) = 0 = f(2)$$

Therefore, Rolle's theorem is applicable.

(C) $\tan x$ is not defined at $x = \pi/2$ and hence Rolle's theorem is **not** applicable.

(D) $\sin x$ is continuous on $[0, \pi]$, differentiable in $(0, \pi)$ and $\sin 0 = 0 = \sin \pi$. Hence, Rolle's theorem is applicable.

**Answer: (C)**

**95.** Suppose $f$ and $g$ are differentiable functions on $[0, 1]$ and $f(0) = 2, g(0) = 0, f(1) = 6, g(1) = 2$. Then, there exists $c \in (0, 1)$ such that $f'(c) = kg'(c)$ where the value of $k$ is

(A) 2 (B) $\frac{1}{2}$
(C) $\frac{1}{3}$ (D) 3

**Solution:** Consider $H(x) = f(x) - 2g(x)\ \forall\ x \in [0, 1]$. Clearly $H(x)$ is differentiable in $[0, 1]$. Also

$$H(0) = f(0) - 2g(0) = 2 - 0 = 2$$
$$H(1) = f(1) - 2g(1) = 6 - 2(2) = 2$$

Therefore

$$H(0) = H(1)$$

Hence by Rolle's theorem, there exists $c \in (0, 1)$ such that

$$H'(c) = 0.$$
$$\Rightarrow f'(c) - 2g'(c) = 0$$
$$\Rightarrow f'(c) = 2g'(c)$$

Thus $k = 2$.

**Answer: (A)**

**96.** Suppose $f(x)$ is twice differentiable for all real $x$. Further suppose $|f(x)| \le 1$ and $|f''(x)| \le 1$ for all real $x$. Then $|f'(x)|$ is

(A) less than or equal to 1
(B) greater than 2
(C) less than or equal to 3/2
(D) greater than 3/2

where

$$f(x+1) = f(x) + \frac{f'(x)}{\lfloor 1} + \frac{f'(\xi)}{\lfloor 2}$$

and $$f(x-1) = f(x) - \frac{f'(x)}{\lfloor 1} + \frac{f''(\eta)}{\lfloor 2}$$

for some $\xi$ and $\eta$.

**Solution:** By hypothesis

$$f(x+1) - f(x-1) = 2f'(x) + \frac{1}{2}[f''(\xi) - f''(\eta)]$$

Therefore

$$2|f'(x)| = \left| f(x+1) - f(x-1) + \frac{1}{2}(f''(\eta) - f''(\xi)) \right|$$
$$\le |f(x+1)| + |f(x-1)| + \frac{1}{2}|f''(\eta) - f''(\xi)|$$
$$\le |f(x+1)| + |f(x-1)| + \frac{1}{2}[|f''(\eta)| + |f''(\xi)|]$$
$$\le 1 + 1 + \frac{1}{2}(1+1) = 3$$

Therefore $|f'(x)| \le 3/2$.

**Answer: (C)**

**97.** Let $f$ be a twice differentiable function for all real $x$, $f(1) = 1, f(2) = 4$ and $f(3) = 9$. Then which one of the following statements is definitely true?

(A) $f''(x) = f'(x) = 5$ for some $x \in (1, 3)$.
(B) $f''(x) = 2$ for all $x \in (1, 3)$.

(C) $f''(x) = 3$ for all $x \in (1, 3)$.

(D) $f''(x)$ attains the value 2 for some $x \in (1, 3)$.

**Solution:** Consider the function

$$g(x) = f(x) - x^2 \text{ for } x \in [1, 3]$$

so that $g(1) = g(2) = g(3) = 0$. Therefore, by Rolle's theorem, $g'(\alpha) = 0$ for some $\alpha \in (1, 2)$ and $g'(\beta) = 0$ for some $\beta \in (2, 3)$. That is $f'(\alpha) = 2\alpha$ and $f'(\beta) = 2\beta$ where $1 < \alpha < 2 < \beta < 3$. Now, using Rolle's theorem for $h(x) = f'(x) - 2x$ on the interval $[\alpha, \beta]$, there exists $x_0 \in (\alpha, \beta)$ such that $h'(x_0) = 0$. That is

$$f''(x_0) = 2$$

**Answer: (D)**

**98.** Suppose $f$ is differentiable on $\mathbb{R}$, $f(0) = 0$ and $1 \le f'(x) \le 2$ for all $x \in \mathbb{R}$. Then

(A) $x \le f(x) \le 2x$ for all $x \ge 0$

(B) $2x < f(x) \le 3x$ for all $x \ge 0$

(C) $3x < f(x) \le 4x$ for all $x > 0$

(D) $4x < f(x) \le 5x$ for all $x > 0$

**Solution:** Let $x > 0$. Using Lagrange's mean value theorem on $[0, x]$, there exists $x_0 \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(x_0)$$

$$\Rightarrow f(x) = xf'(x_0) \qquad (3.37)$$

So

$$1 \le f'(x) \le 2 \Rightarrow x \le xf'(x_0) \le 2x$$

$$\Rightarrow x \le f(x) \le 2x \qquad \text{[By Eq. (3.37)]}$$

This is true for all $x > 0$. When $x = 0$, we have $f(x) = 0$. Thus, the equatity holds. Hence $x \le f(x) \le 2x \ \forall\ x \ge 0$.

**Answer: (A)**

**99.** Value of $c$ in the Lagrange's mean value theorem for the function

$$f(x) = x + \frac{1}{x} \quad \text{on} \left[\frac{1}{2}, 2\right]$$

is

(A) $\frac{3}{4}$ (B) $\frac{3}{2}$

(C) 1 (D) $\frac{5}{8}$

**Solution:** $f(x) = x + (1/x)$ is continuous and differentiable for all real $x \ne 0$. Hence by Lagrange's mean value theorem, there exists $c \in (1/2, 2)$ such that

$$f'(c) = \frac{f(2) - f(1/2)}{2 - (1/2)}$$

$$\Rightarrow 1 - \frac{1}{c^2} = \frac{(5/2) - (5/2)}{3/2} = 0$$

Therefore $c = \pm 1$. But $-1 \notin (1/2, 2)$. Therefore $c = 1$.

**Answer: (C)**

**100.** Let $f(x) = 3x^4 - 4x^2 + 5$. Then a value of $c$ in Lagrange's mean value theorem for $f(x)$ on the interval $[-1, 1]$ is

(A) $\frac{-1}{2}$ (B) $\frac{1}{2}$

(C) $\sqrt{\frac{2}{3}}$ (D) $-\frac{2}{3}$

**Solution:** We have $f'(c) = 12c^3 - 8c$. Now since the function satisfies Langrange's mean value theorem, we have

$$12c^3 - 8c = \frac{f(1) - f(-1)}{1 - (-1)} = 0$$

$$\Rightarrow c(3c^2 - 2) = 0$$

$$\Rightarrow c = 0, \pm\sqrt{\frac{2}{3}}$$

So, $c = \sqrt{2/3}$ is one of the values.

**Answer: (C)**

## Monotoncity; Maxima and Minima

**101.** The function $f(x) = 2x^2 - \log|x|$ is

(A) decreasing in $\left(\frac{1}{2}, \infty\right)$

(B) decreasing in $\left(0, \frac{1}{2}\right)$

(C) increasing in $\left(0, \frac{1}{2}\right)$

(D) increasing in $\left(-\infty, -\frac{1}{2}\right)$

**Solution:** The function $f(x) = 2x^2 - \log|x|$ is defined for all $x \ne 0$. Differentiating we get

$$f'(x) = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x} = \frac{(2x+1)(2x-1)x}{x^2}$$

Now

$$f'(x) > 0 \Leftrightarrow \text{either } -\frac{1}{2} < x < 0 \quad \text{or} \quad x > \frac{1}{2}$$

Therefore, $f$ increases in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$.

Again

$$f'(x) < 0 \Leftrightarrow \text{either } x < -\frac{1}{2} \text{ or } 0 < x < \frac{1}{2}$$

Therefore, $f$ is decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$.

Thus, (B) is correct.

**Answer: (B)**

**102.** If

$$f(x) = \frac{x}{\sin x} \quad \text{and} \quad g(x) = \frac{x}{\tan x}$$

where $0 < x \le 1$, then in this interval

(A) both $f(x)$ and $g(x)$ are increasing functions
(B) both $f(x)$ and $g(x)$ are decreasing functions
(C) $f(x)$ is an increasing function
(D) $g(x)$ is an increasing function

**Solution:** We have

$$f(x) = \frac{x}{\sin x}, 0 < x \le 1$$

Differentiating we get

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$$

Since $\sin^2 x > 0$, we have to determine the sign of $\sin x - x\cos x$. Now, let

$$F(x) = \sin x - x\cos x$$

so that

$$F'(x) = \cos x - \cos x + x\sin x = x\sin x > 0$$

for $0 < x \le 1$. Therefore $F(x)$ is increasing so that

$$0 < x \Rightarrow F(0) < F(x) \Rightarrow 0 < \sin x - x\cos x$$

Therefore

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} > 0$$

for $0 < x \le 1$. So $f$ is increasing in $(0, 1]$. Now

$$g(x) = \frac{x}{\tan x}$$

$$\Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$$

Let $H(x) = \tan x - x\sec^2 x$. Then

$$H'(x) = \sec^2 x - \sec^2 x - 2x\sec^2 x\tan x = -2x\sec^2 x\tan x < 0$$

for $0 < x \le 1$. Therefore $H(x)$ is decreasing in $(0, 1]$. Now

$$0 < x \Rightarrow H(0) > H(x) \Rightarrow 0 > \tan x - x\sec^2 x$$

Hence, $g'(x) < 0$ in $(0, 1]$ so that $g$ is decreasing. So, (C) is correct.

**Answer: (C)**

**103.** Let $f(x) = \cos x - 1 + (x^2/2)$. Then

(A) $f$ is increasing on $\mathbb{R}$
(B) $f$ is decreasing on $\mathbb{R}$
(C) $f$ increases in $(-\infty, 0]$ and decreases in $[0, \infty)$
(D) $f$ decreases in $(-\infty, 0]$ and increases in $[0, \infty)$

**Solution:** We have

$$f'(x) = -\sin x + x$$

and

$$f''(x) = 1 - \cos x \ge 0$$

for all real $x$. Therefore $f'(x)$ is increasing for all real $x$. So,

$$x < 0 \Rightarrow f'(x) < f'(0) = 0$$

So, for $x < 0$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x < 0$. Also

$$x > 0 \Rightarrow f'(x) > f'(0) = 0$$

Therefore, for $x > 0$, $f'(x) > 0 \Rightarrow f$ is increasing for $x > 0$.

**Answer: (D)**

**104.** Suppose $f(x)$ is a real-valued differentiable function such that $f(x)\,f'(x) < 0$ for all real $x$. Then

(A) $f(x)$ is an increasing function
(B) $|f(x)|$ is an increasing function
(C) $f(x)$ is a decreasing function
(D) $|f(x)|$ is a decreasing function

**Solution:** Let $g(x) = (f(x))^2$ so that $g'(x) = 2\,f(x)\,f'(x) < 0$ for all real $x$. Hence, $g(x)$ is decreasing so that $|f(x)|$ is decreasing.

**Answer: (D)**

**105.** Consider the following two statements.

**I:** If $f(x)$ and $g(x)$ are continuous and monotonic functions on the real line $\mathbb{R}$, then $f(x)+g(x)$ is also a monotonic function.

**II:** If $f(x)$ is a continuous decreasing function for all $x>0$ and $f(1)$ is positive, then $f(x)=0$ exactly at one value of $x$.

Then

(A) both I and II are true
(B) both I and II are false
(C) I is true and II is false
(D) I is false and II is true

**Solution:**

**I:** Consider $f(x)=x+\sin x$ and $g(x)=-x$. Since

$$f'(x)=1+\cos x\geq 0$$

and $$g'(x)=1<0$$

we have that $f$ is increasing and $g$ is strictly decreasing. But $f(x)+g(x)=\sin x$ is neither increasing nor decreasing for $x>0$. Observe that the monotonicity nature of $\sin x$ can be decided on an interval only. Thus (I) is false.

**II:** Consider the function

$$f(x)=\frac{1}{x},\ x>0$$

so that

$$f(1)=1>0 \quad \text{and} \quad f'(x)=-\frac{1}{x^2}<0$$

Hence $f$ is continuous and decreasing for $x>0$. But $f(x)\neq 0$ for all $x\neq 0$. Thus (II) is false.

**Answer: (B)**

**106.** Consider the following two statements.

**P:** If $f$ is differentiable and strictly increasing on $\mathbb{R}$, then $f'(x)>0$ for all $x\in\mathbb{R}$.

**Q:** If $f(x)$ and $g(x)$ are not monotonic functions on $\mathbb{R}$, then $f(x)+g(x)$ is also not monotonic on $\mathbb{R}$.

Then

(A) both P and Q are false
(B) P is false and Q is true
(C) P is true whereas Q is false
(D) both P and Q are true

**Solution:** **P is false:** Consider $f(x)=x^3, x\in\mathbb{R}$. Then

$$f'(x)=3x^2>0 \quad \text{for} \quad x\neq 0$$

Thus $f$ is strictly increasing on $\mathbb{R}$ and $f'(0)=0$.

**Q is also false:** Let $f(x)=x^2+x$ and $g(x)=x-x^2$ so that $f$ and $g$ are not monotonic on $\mathbb{R}$. But the sum $f(x)+g(x)=2x$ is monotonic on $\mathbb{R}$.

***Note:*** $f'(x)=2x+1<0$ for $x<-1/2$ and greater than zero for $x>-1/2$. Thus, $f$ is decreasing in $(-\infty,-1/2)$ and increasing in $(-1/2,\infty)$. Similarly, $g'(x)=1-2x \Rightarrow g$ is decreasing for $x>1/2$ and increasing for $x<1/2$.

**Answer: (A)**

**107.** Consider the following two propositions $P_1$ and $P_2$.

**$P_1$:** If $f$ and $g$ are increasing functions on $[a,b]$, then $f+g$ is also increasing on $[a,b]$.

**$P_2$:** If $f(x)=x$ and $g(x)=1-x$ are defined on $[0,1]$, then the product $fg$ is increasing on $[0,1]$.

Then

(A) both $P_1$ and $P_2$ are true
(B) $P_1$ is true, but $P_2$ is false
(C) $P_1$ is false whereas $P_2$ is true
(D) both $P_1$ and $P_2$ are false

**Solution:** **$P_1$ is true:** Let $x_1, x_2\in[a,b]$ and $x_1<x_2$. Now

$$\begin{aligned}(f+g)(x_1)&=f(x_1)+g(x_1)\\ &\leq f(x_2)+g(x_2)\ (\because f \text{ and } g \text{ are increasing})\\ &=(f+g)(x_2)\end{aligned}$$

Therefore, $f+g$ is increasing.

**$P_2$ is false:** We have

$$(fg)(x)=f(x)g(x)=x-x^2$$

Differentiating we get

$$(fg)'(x)=\begin{cases}1-2x<0 & \text{for } x>1/2\\ 1-2x>0 & \text{for } x<1/2\end{cases}$$

Thus, $fg$ decreases for $x>1/2$ and increases for $x<1/2$. That is, in $[0,1/2]$, $fg$ is increasing and in $[1/2,1]$, $fg$ is decreasing. Thus $P_2$ is not true.

**Answer: (B)**

**108.** The function $f(x)=xe^{-3x}$

(A) increases on $\mathbb{R}$
(B) decreases on $\mathbb{R}$
(C) increases in $(-\infty, 1/3)$
(D) decreases in $(-\infty, 1/3)$

**Solution:** Differentiating the given function we have

$$f'(x)=e^{-3x}+x(-3)e^{-3x}=e^{-3x}(1-3x)$$

We know that $e^{-3x}>0$ for all real $x$. Therefore

$$\begin{aligned}f'(x)>0 &\Leftrightarrow 1-3x>0\\ &\Leftrightarrow x<\frac{1}{3}\end{aligned}$$

So $f'(x)>0$ for all $x<1/3$ and hence $f$ increases in $(-\infty,1/3)$ and decreases in $(1/3,\infty)$.

**Answer: (C)**

**109.** In $[0, 1]$, $f$ is twice differentiable and $f''(x) < 0$. Let

$$\phi(x) = f(x) + f(1-x)$$

for $x \in [0, 1]$. Then

(A) $\phi$ is increasing on $[0, 1]$
(B) $\phi$ is decreasing on $[0, 1]$
(C) $\phi$ increases on $[1/2, 1]$
(D) $\phi$ decreases on $[1/2, 1]$

**Solution:** Differentiating the given function we have

$$\phi'(x) = f'(x) - f'(1-x)$$

$f''(x) < 0$ for $x \in [0,1] \Rightarrow f'(x)$ is decreasing in $[0,1]$. Therefore

$$x < 1 - x \Leftrightarrow x < \frac{1}{2}$$

so that for $0 \le x \le 1/2$, we have

$$\phi'(x) = f'(x) - f'(1-x) > 0$$

Therefore $\phi$ is increasing in $[0, 1/2]$. Again

$$x > 1 - x \Leftrightarrow x > 1/2$$

Therefore ($\because f'$ is decreasing)

$$f'(x) - f'(1-x) < 0 \quad \text{for } \frac{1}{2} \le x \le 1$$

So $\phi'(x) < 0$ in $[1/2, 1]$. $\phi$ is decreasing in $[1/2, 1]$.

**Answer: (D)**

**110.** If $f(x) = (m+2)x^3 - 3mx^2 + 9mx - 1$ decreases on $\mathbb{R}$, then $m$ belongs to

(A) $(-\infty, -3) \cup (0, \infty)$
(B) $(-\infty, 3]$
(C) $(-\infty, 0] \cup (0, 3]$
(D) $[-1, 0] \cup (0, 1]$

**Solution:** Differentiating the given function we get

$$f'(x) = 3(m+2)x^2 - 6mx + 9m$$
$$= 3[(m+2)x^2 - 2mx + 3m]$$

Since $f$ is decreasing on $\mathbb{R}$, we have $f'(x) \le 0\ \forall\ x \in \mathbb{R}$. Therefore

$$(m+2)x^2 - 2mx + 3m \le 0\ \forall\ x \in \mathbb{R}$$

So,

$$\begin{aligned}
4m^2 - 12m(m+2) \le 0 &\quad \text{and} \quad m + 2 < 0\\
m^2 - 3m(m+2) \le 0 &\quad \text{and} \quad m < -2\\
-2m(m+3) \le 0 &\quad \text{and} \quad m < -2\\
m(m+3) \ge 0 &\quad \text{and} \quad m < -2\\
m \le -3 \text{ or } m > 0 &\quad \text{and} \quad m < -2\\
m \le -3 &\quad \text{or} \quad m \ge 0
\end{aligned}$$

But, when $m = -3$ or $m = 0$, $f(x)$ is not decreasing for all $x \in \mathbb{R}$ (check this point). Therefore

$$m < -3 \text{ or } m > 0$$

So, $m \in (-\infty, -3) \cup (0, \infty)$.

**Answer: (A)**

**111.** Let $f$ be differentiable on $\mathbb{R}$ and

$$h(x) = f(x) - [f(x)]^2 + [f(x)]^3$$

Then

(A) $h$ is increasing whenever $f$ is increasing
(B) $h$ is increasing whenever $f$ is decreasing
(C) $h$ is decreasing whenever $f$ is decreasing or increasing
(D) $h$ is decreasing whenever $f$ is increasing

**Solution:** Differentiating the given function we get

$$\begin{aligned}
h'(x) &= f'(x) - 2f(x)f'(x) + 3f^2(x)f'(x)\\
&= f'(x)[3f^2(x) - 2f(x) + 1]\\
&= \frac{f'(x)}{3}\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right]
\end{aligned}$$

Therefore

$$h'(x) \ge 0 \Leftrightarrow f'(x) \ge 0$$

So $h$ is increasing whenever $f$ is increasing.

**Answer: (A)**

**112.** The function $f(x) = \sin^4 x + \cos^4 x$ increases if

(A) $0 < x < \dfrac{\pi}{8}$
(B) $\dfrac{\pi}{4} < x < \dfrac{3\pi}{8}$
(C) $\dfrac{3\pi}{8} < x < \dfrac{5\pi}{8}$
(D) $\dfrac{5\pi}{8} < x < \dfrac{3\pi}{4}$

**Solution:** We have

$$\begin{aligned}
f(x) &= \sin^4 x + \cos^4 x\\
&= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\\
&= 1 - \frac{1}{2}\sin^2 2x
\end{aligned}$$

Differentiating we get

$$\begin{aligned}
f'(x) &= -\frac{1}{2}(2\sin 2x)(\cos 2x)(2)\\
&= -2\sin 2x \cos 2x\\
&= -\sin 4x
\end{aligned}$$

Now $-\sin 4x > 0$, if $\sin 4x < 0$. But $\sin 4x < 0$, if $\pi < 4x < 3\pi/2$ or $3\pi/2 < 4x < 2\pi$. Therefore

$$\sin 4x < 0, \quad \text{if } \frac{\pi}{4} < x < \frac{3\pi}{8} \text{ or } \frac{3\pi}{8} < x < \frac{\pi}{2}$$

So (B) is the correct answer.

**Answer: (B)**

**113.** If $f$ is a real-valued function defined on $\mathbb{R}$ such that

$$f'(x) = e^x (x-1)(x-2)$$

then $f$ decreases in the interval

(A) $(-\infty, -2)$ (B) $(-2, -1)$
(C) $(1, 2)$ (D) $(2, +\infty)$

**Solution:** We have to determine the values of $x$ for which $f'(x) < 0$. Observe that $e^x > 0\ \forall \in \mathbb{R}$. Therefore

$$f'(x) < 0 \Leftrightarrow (x-1)(x-2) < 0$$
$$\Leftrightarrow 1 < x < 2$$

Hence, (C) is the correct answer.

**Answer: (C)**

**114.** For all $x \in (0, 1)$

(A) $e^x < 1 + x$ (B) $\log_e(1+x) < x$
(C) $\sin x > x$ (D) $\log_e x > x$

**Solution:**

(A) Let $f_1(x) = e^x - 1 - x$. Then

$$f_1'(x) = e^x - 1 > 0 \quad \text{for } x > 0$$

So $f_1(x)$ increases in $(0, 1)$. Therefore

$$0 < x \Rightarrow f_1(0) < f_1(x)$$
$$\Rightarrow 0 < f_1(x) = e^x - (1+x)$$
$$\Rightarrow e^x > 1 + x$$

So (A) is not the correct answer.

(B) Let $f_2(x) = \log_e(1+x) - x$. Then

$$f_2'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x} < 0 \quad \text{for } x > 0$$

So $f_2(x)$ is decreasing for $x > 0$. Therefore

$$0 < x \Rightarrow f_2(0) > f_2(x)$$
$$\Rightarrow 0 > f_2(x) = \log_e(1+x) - x$$
$$\Rightarrow \log_e(1+x) < x \quad \text{for } x > 0$$

So (B) is the correct answer.

(C) Let $f_3(x) = \sin x - x$. Then

$$f_3'(x) = \cos x - 1 < 0 \quad \text{for } x > 0$$

So $f_3(x)$ is decreasing for $x > 0$. Therefore

$$x > 0 \Rightarrow f_3(x) < f_3(0) = 0$$
$$\Rightarrow \sin x < x$$

Hence (C) is not the correct answer

(D) Let $f_4(x) = \log_e x - x$. Then

$$f_4'(x) = \frac{1}{x} - 1 = \frac{1-x}{x} > 0 \quad (\because 0 < x < 1)$$

Therefore $f_4(x)$ is increasing in $(0, 1)$. So,

$$0 < x < 1 \Rightarrow \lim_{x \to 0+0} \log_e x = -\infty$$

so that

$$\lim_{x \to 0+0} f_4(x) = -\infty$$

So $f_4(x) < 0$ in $(0, 1)$. This implies $\log_e x < x$ in $(0, 1)$. Hence (D) is not the correct answer.

**Answer: (B)**

**115.** If $f(x) = x\, e^{x(1-x)}$, then $f(x)$ is

(A) increasing in $[-1/2, 1]$
(B) decreasing in $[-1/2, 1]$
(C) increasing in $\mathbb{R}$
(D) decreasing in $\mathbb{R}$

**Solution:** Differentiating the given function we get

$$f'(x) = e^{x(1-x)} + x\, e^{x(1-x)}(1-2x)$$
$$= e^{x(1-x)}[1 + x - 2x^2]$$
$$= -e^{x(1-x)}[2x^2 - x - 1]$$
$$= -e^{x(1-x)}(2x+1)(x-1)$$

Now

$$f'(x) \ge 0 \Leftrightarrow (2x+1)(x-1) \le 0$$
$$\Leftrightarrow x \in \left[-\frac{1}{2}, 1\right]$$

So $f$ increases in $[-1/2, 1]$ and decreases in $(-\infty, -1/2) \cup (1, +\infty)$. Thus (A) is correct.

***Note*:** If $f(x) = xe^{x(x-1)}$, then $f'(x) = e^{x(x-1)}(2x^2 - x + 1) > 0\ \forall\ x \in \mathbb{R}$ and hence $f(x)$ increases in $\mathbb{R}$.

**Answer: (A)**

**116.** If $f(x)$ is a function whose derivative is

$$e^{-(x^2+1)^2}(2x) - e^{-x^4}(2x)$$

then $f(x)$ increases in

(A) $(1, 2)$ (B) no value of $x$
(C) $(0, \infty)$ (D) $(-\infty, 0)$

**Solution:** Given that

$$f'(x) = (2x)e^{-x^4}(e^{-2x^2-1} - 1)$$
$$= (2x)e^{-(x^2+1)^2}(1 - e^{2x^2+1})$$

Since $e^x > 1$, for $x > 0$, we have $1 - e^{2x^2+1} < 0$ for all $x$. Therefore $f'(x) > 0$, only when $x < 0$. So $f(x)$ increases in $(-\infty, 0)$.

**Answer: (D)**

**117.** The function $f(x) = 2\log(x-2) - x^2 + 4x + 1$

(A) increases in $(2, 3)$
(B) decreases in $(2, 3)$
(C) increases in $(3, +\infty)$
(D) cannot be determined

**Solution:** Note that $f(x)$ is defined for $x > 2$. Now

$$\begin{aligned} f'(x) &= \frac{2}{x-2} - 2x + 4 \\ &= \frac{2}{x-2} - 2(x-2) \\ &= \frac{2[1-(x-2)^2]}{x-2} \\ &= \frac{2(1+x-2)(1-x+2)}{x-2} \\ &= -\frac{2}{(x-2)^2}(x-1)(x-2)(x-3) \end{aligned}$$

Therefore, $f'(x) > 0$ for $x < 1$ or $2 < x < 3$. But

$$x > 2 \Rightarrow f'(x) > 0 \quad \text{for } x \in (2, 3)$$

So $f$ increases in $(2, 3)$. Thus, (A) is the correct answer.

**Answer: (A)**

**118.** The set of all values of the parameter $a$ for each of which the function

$$f(x) = \sin 2x - 8(a+1)\sin x + (4a^2 + 8a - 14)x$$

increases for all $x \in \mathbb{R}$ and has no critical points is

(A) $(-\infty, -2-\sqrt{5}) \cup (\sqrt{5}, \infty)$

(B) $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, 5)$

(C) $(-\infty, 0) \cup (0, \sqrt{5})$

(D) $(0, 5) \cup (-\infty, 2-\sqrt{5})$

**Solution:** Differentiating the given function we get

$$\begin{aligned} f'(x) &= 2\cos 2x - 8(a+1)\cos x + 4a^2 + 8a - 14 \\ &= 4\cos^2 x - 8(a+1)\cos x + 4a^2 + 8a - 16 \\ &= 4\,[(\cos x - (a+1)^2 - 5] > 0 \\ &\Leftrightarrow \text{either } \cos x - (a+1) < -\sqrt{5} \\ &\qquad \text{or } \cos x - (a+1) > \sqrt{5} \end{aligned}$$

for all $x \in \mathbb{R}$. That is

$$\cos x < a + 1 - \sqrt{5} \quad \text{for all real } x$$

or $$\cos x > (a+1) + \sqrt{5} \quad \text{for all } x$$

But $-1 \le \cos x \le 1$ for all real $x$. Therefore,

$$\text{either } 1 < a + 1 - \sqrt{5} \quad \text{or} \quad -1 > a + 1 + \sqrt{5}$$

So

$$\sqrt{5} < a \quad \text{or} \quad -2 - \sqrt{5} > a$$

Hence $a \in (-\infty, -2-\sqrt{5}) \cup (\sqrt{5}, \infty)$.

**Answer: (A)**

**119.** Consider the function

$$f(x) = \begin{cases} x\,e^{ax} & \text{for } x \le 0 \\ x + ax^2 - x^3 & \text{for } x > 0 \end{cases}$$

where $a > 0$ is constant. Then $f'(x)$ is

(A) increasing in $\left(\frac{-2}{a}, \frac{a}{3}\right)$

(B) increasing in $\left(\frac{a}{3}, \infty\right)$

(C) decreasing in $\left(\frac{-2}{a}, \frac{a}{3}\right)$

(D) increasing in $\left(-\infty, \frac{-2}{a}\right)$

**Solution:** Differentiating the given function we have

$$f'(x) = \begin{cases} e^{ax}(ax+1) & \text{for } x \le 0 \\ 1 + 2ax - 3x^2 & \text{for } x > 0 \end{cases}$$

$$f''(x) = \begin{cases} 2ae^{ax} + a^2 x\,e^{ax} & \text{for } x \le 0 \\ 2a - 6x & \text{for } x > 0 \end{cases}$$

We determine the values of $x$ for which $f''(x) \ge 0$.

**(i)** $f''(x) = 0$ at $x = -\dfrac{2}{a}$ when $x \le 0$

**(ii)** $f''(x) = 0$ at $x = \dfrac{a}{3}$ when $x > 0$

Also, when $x \le 0$,

$$f''(x) = ae^{ax}(2 + ax) > 0 \quad \text{if } x > -\frac{2}{a}$$

When $x > 0$,

$$f''(x) = 2a - 6x > 0 \quad \text{if } x < \frac{a}{3}$$

Therefore $f''(x) > 0$ if $-2/a < x < a/3$. Hence $f'(x)$ increases in $(-2/a, a/3)$.

**Answer: (A)**

**120.** The function $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$, $0 \le x \le \pi$ is

(A) increasing in $\left(0, \frac{\pi}{2}\right)$

(B) increasing in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(C) decreasing in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(D) neither increasing nor decreasing in $[0, \pi]$

**Solution:** Differentiating the given function we get

$$\begin{aligned} f'(x) &= 12\cos^3 x(-\sin x) \\ &\quad + 30\cos^2 x(-\sin x) + 12\cos x(-\sin x) \\ &= -3\sin 2x\,(2\cos^2 x + 5\cos x + 2) \\ &= -3\sin 2x\,(2\cos x + 1)(\cos x + 2) \end{aligned}$$

Observe that $\cos x + 2 > 0\ \forall\ x \in [0, \pi]$. Therefore

$$f'(x) > 0 \Leftrightarrow \sin 2x(2\cos x + 1) < 0 \text{ for } 0 \le x \le \pi$$

**(i)** When $0 \le x \le \pi$,

$$\sin 2x < 0 \Leftrightarrow \frac{\pi}{2} < x < \pi$$

and $$2\cos x + 1 > 0 \Leftrightarrow \cos x > -\frac{1}{2}$$
$$\Leftrightarrow x > \frac{2\pi}{3}$$

**(ii)** We have

$$\sin 2x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$$

and $$\cos x < -\frac{1}{2} \Leftrightarrow x < \frac{2\pi}{3}$$

From (i) and (ii) we have $\sin 2x(2\cos x + 1) < 0$ for $x \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

Therefore $f'(x) > 0$ for $x \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

This implies $f$ is increasing in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

**Answer: (B)**

**121.** Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \le 2 \\ 1 & \text{for } x = 0 \end{cases}$

Then at $x = 0$, $f$ has

(A) a local maximum
(B) no local maximum
(C) a local minimum
(D) no extremum

**Solution:** We have

$$f(x) = \begin{cases} -x & \text{for } -2 \le x < 0 \\ 1 & \text{for } x = 0 \\ x & \text{for } 0 < x \le 2 \end{cases}$$

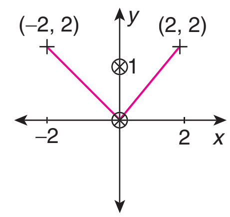



**FIGURE 3.21** Single correct choice type question 121.

See Fig. 3.21. Clearly $f$ is discontinuous at $x = 0$. But this cannot disturb the local extremity of the function. Clearly in a neighbourhood of zero,

$$f(x) = |x| < 1 = f(0)$$

Hence $f$ has local maximum at $x = 0$.

***Note:*** This question is purely based on the concept.

**Answer: (A)**

**122.** If $f(x) = a\log|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

(A) $a = 2, b = \frac{1}{2}$ (B) $a = \frac{1}{2}, b = 2$

(C) $a = 2, b = -\frac{1}{2}$ (D) $a = -2, b = -\frac{1}{2}$

**Solution:** $f(x)$ is defined for all $x \ne 0$. Now

$$f'(x) = \frac{a}{x} + 2bx + 1$$

Since $x = -1$ and $x = 2$ are critical points, we have

$$f'(-1) = 0 \Rightarrow -a - 2b + 1 = 0 \tag{3.38}$$

and $$f'(2) = 0 \Rightarrow \frac{a}{2} + 4b + 1 = 0 \tag{3.39}$$

Solving Eqs. (3.38) and (3.39), we get

$$a = 2, b = -\frac{1}{2}$$

**Answer: (C)**

**123.** Let $P(x) \equiv a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$ be a polynomial in real variable $x$ with $0 < a_0 < a_1 < a_2 < \cdots < a_n$. The function $P(x)$ has

(A) neither maximum nor minimum
(B) only one maximum
(C) only one minimum
(D) only one maximum and only one minimum

**Solution:** Differentiating the given function we get

$$P'(x) = 2a_1x + 4a_2x^3 + \cdots + (2n)a_nx^{2n-1}$$

Since all the powers of $x$ are odd and coefficients are positive, $P'(x) \ne 0$ for all real $x \ne 0$ and $P'(0) = 0$. Thus $x = 0$ is the only critical point. Further,

$$P''(x) = 0 = 2a_1 > 0$$

Hence $P(x)$ has only one minimum at $x = 0$.

**Answer: (C)**

**124.** On the interval $[0, 1]$, the function $x^{25}(1-x)^{75}$ takes its maximum value at the point

(A) 0 (B) 1/4 (C) 1/2 (D) 1/3

**Solution:** Let $f(x) = x^{25}(1-x)^{75}$. Differentiating we get

$$\begin{aligned} f'(x) &= 25x^{24}(1-x)^{75} - 75x^{25}(1-x)^{74} \\ &= x^{24}(1-x)^{74}[25(1-x) - 75x] \\ &= x^{24}(1-x)^{74}(25 - 100x) \end{aligned}$$

Also $0 < x < 1$ and $f'(x) = 0 \Rightarrow x = 1/4$.

Now

$$x < \frac{1}{4} \Rightarrow f'(x) > 0$$

and $$x > \frac{1}{4} \Rightarrow f'(x) < 0$$

That is $f'(x)$ changes sign from positive to negative (see the testing procedure given in Sec. 3.4.1). Therefore, $f(x)$ has maximum at $x = 1/4$.

Note that $f''(0) = f''(1) = 0$.

**Answer: (B)**

**125.** The minimum value of

$$f(x) = x^8 + x^6 - x^4 - 2x^3 - x^2 - 2x + 9$$

on the real number set $\mathbb{R}$ is

(A) 1 (B) 5 (C) 0 (D) 3

**Solution:** The given function can be written as

$$f(x) = (x^4 - 1)^2 + (x^3 - 1)^2 + (x^2 - 1)^2 + (x - 1)^2 + 5 \geq 5$$

for all real $x$ and equality holds if and only if $x = 1$. Therefore, the minimum value of $f$ is 5 and it attains its minimum value at $x = 1$.

**Answer: (B)**

**126.** The minimum value of

$$Z = 2x^2 + 2xy + y^2 - 2x + 2y + 2$$

is

(A) 2 (B) −2 (C) 3 (D) −3

**Solution:** The given function can be written as

$$Z = (x + y + 1)^2 + (x - 2)^2 - 3 \geq -3 \ \forall \ x, y$$

and equality holds when $x = 2, y = -3$. Therefore, the minimum value of $Z$ is $-3$.

**Answer: (D)**

**127.** The value of "$a$" such that the sum of the squares of the roots of the equation

$$x^2 - (a-2)x - a - 1 = 0$$

assumes least value is

(A) 0 (B) −1 (C) +1 (D) 2

**Solution:** Let $\alpha, \beta$ be the roots. Therefore

$$\alpha + \beta = a - 2, \ \alpha\beta = -(a+1)$$

Let

$$\begin{aligned} Z &= \alpha^2 + \beta^2 \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a-2)^2 + 2(a+1) \\ &= a^2 - 2a + 6 \end{aligned}$$

Then

$$\frac{dZ}{da} = 0 \Leftrightarrow 2a - 2 = 0 \Leftrightarrow a = 1$$

Also

$$\frac{d^2Z}{da^2} = 2 < 0$$

Thus $Z$ is minimum at $a = 1$.

**Answer: (C)**

**128.** $f(x) = 2 - (x-1)^{2/3}$ is maximum at

(A) $x = -1$ (B) $x = 1$

(C) $x = 0$ (D) $x = 2\sqrt{2}$

**Solution:** The given function is continuous $\forall$ real $x$. Now,

$$f'(x) = -\frac{2}{3}(x-1)^{-1/3}$$

so that $f'(1)$ does not exist. Also $f'(x) \neq 0$ for any value of $x \neq 1$. Hence $x = 1$ is the only critical point for $f(x)$. Also

$$x < 1 \Rightarrow f'(x) > 0$$

and $$x > 1 \Rightarrow f'(x) > 0$$

Thus, $f'(x)$ changes sign from positive to negative at $x = 1$. Hence, $f$ is maximum at $x = 1$.

**Answer: (B)**

**129.** $f(x)$ is a function satisfying the following conditions.

**(i)** $f(0)=2, f(1)=1$

**(ii)** $f$ has minimum value at $x=\frac{5}{2}$ and

**(iii)** $f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

for all $x$ where $a$ and $b$ are constants.

Then

(A) $a=\frac{1}{4}, b=-\frac{5}{4}$ (B) $a=-\frac{1}{4}, b=\frac{5}{4}$

(C) $a=-\frac{1}{4}, b=-\frac{5}{4}$ (D) $a=\frac{1}{4}, b=\frac{5}{4}$

**Solution:** On $f'(x)$ perform the following elementary row operation. That is, applying $R_3 - R_1 - 2R_2$ we get

$$f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

Therefore

$$\begin{aligned} f'(x) &= 2ax(b+1)-b(2ax-1) \\ &= 2ax+b \end{aligned}$$

$f$ has local minimum at $x = 5/2$. This implies

$$f'\left(\frac{5}{2}\right)=0$$

$$\Rightarrow 2a\left(\frac{5}{2}\right)+b=0$$

$$\Rightarrow 5a+b=0 \tag{3.40}$$

Now,

$$f'(x)=2ax+b$$

$$\Rightarrow f(x)=ax^2+bx+c$$

where $c$ is constant. Again

$$f(0)=2 \Rightarrow c=2$$

and
$$f(1)=1 \Rightarrow a+b+2=1$$

This gives

$$a+b=-1 \tag{3.41}$$

Solving Eqs. (3.40) and (3.41), we get

$$a=\frac{1}{4} \quad \text{and} \quad b=-\frac{5}{4}$$

**Answer: (A)**

**130.** If $p(x)$ is a polynomial of degree 3 satisfying $P(-1) = 10$, $P(1) = -6$, $P'(x)$ has local minima at $x = 1$ and $P(x)$ has maximum at $x = -1$, then the distance between local maxima and local minima of the curve is

(A) $2\sqrt{65}$ (B) $3\sqrt{65}$

(C) $4\sqrt{65}$ (D) $5\sqrt{65}$

**Solution:** Since $P(x)$ is a polynomial of degree 3, let

$$P(x)=ax^3+bx^2+cx+d$$

Therefore

$$P(-1)=10 \Rightarrow -a+b-c+d=10 \tag{3.42}$$

$$P(1)=-6 \Rightarrow a+b+c+d=-6 \tag{3.43}$$

New

$$P'(x)=3ax^2+2bx+c$$

$P'(x)$ has local minimum at $x=1 \Rightarrow P''(1)=0$. That is

$$6a+2b=0$$

$$\Rightarrow 3a+b=0 \tag{3.44}$$

Now $P(x)$ has maximum at $x = -1 \Rightarrow P'(-1) = 0$. This means

$$3a-2b+c=0 \tag{3.45}$$

Solving Eqs. (3.42)–(3.45), we obtain that $a = 1$, $b = -3$, $c = -9$, $d = 5$. Therefore

$$P(x)=x^3-3x^2-9x+5$$

Differentiating we get

$$\begin{aligned} P'(x) &= 3x^2-6x-9 \\ &= 3(x^2-2x-3) \\ &= 3(x-3)(x+1) \end{aligned}$$

So $P'(x)=0 \Rightarrow x=3, -1$. Also

$$P''(x)=6x-6$$

This implies

$$P''(3)=12>0$$

and
$$P''(-1)=-12<0$$

Therefore, $P(x)$ is maximum at $x = -1$ and minimum at $x = 3$.

Let $A$ and $B$ be the points on the curve $y = P(x)$ at which $P(x)$ is maximum and minimum, respectively. Therefore $A = (-1, 10)$ and $B = (3, -22)$. So

$$\begin{aligned} \text{Distance } AB &= \sqrt{(-1-3)^2+(10+22)^2} \\ &= \sqrt{16+1024} \\ &= \sqrt{1040} \\ &= \sqrt{16\times 65} \\ &= 4\sqrt{65} \end{aligned}$$

**Answer: (C)**

**131.** Let $f(x)=\begin{cases} 2-x+a^2-9a-9 & \text{for } x<2 \\ 2x-3 & \text{for } x\geq 2 \end{cases}$

where $a$ is a positive constant. If $f(x)$ has local minimum at $x = 2$, then $a$ lies in the interval

(A) $(0, 1]$ (B) $[10, \infty)$
(C) $[1, 10]$ (D) $[2, 10)$

**Solution:** Since $f$ has local minimum at $x = 2$, we have $f(x) > f(2)$ for all $x < 2$. Therefore

$$\lim_{x \to 2-0} f(x) \ge f(2) = 1$$

So

$$a^2 - 9a - 9 \ge 1$$

which implies that

$$(a+1)(a-10) \ge 0$$
$$\Rightarrow a \le -1 \text{ or } a \ge 10$$

But $a$ is positive. Hence $a \ge 10$. That is $a \in [10, \infty)$. Note that when $a = 10$,

$$f(x) = \begin{cases} 3 - x & \text{for } x < 2 \\ 2x - 3 & \text{for } x \ge 2 \end{cases}$$

so that $f$ is continuous at 2, $f'(2)$ does not exist and $f'(x) \ne 0 \ \forall \ x \in \mathbb{R}$. Therefore, $x = 2$ is a critical point and $f'(x)$ changes sign from negative to positive at $x = 2$.

**Answer: (B)**

**132.** The sum of an infinitely decreasing geometric progression is equal to the least value of the function

$$f(x) = 3x^2 - x + \frac{25}{12}$$

and the first term of the progression is equal to the square of its common ratio. Then, the common ratio is

(A) $\sqrt{2} - 1$ (B) $\sqrt{3} - 1$
(C) $2 - \sqrt{3}$ (D) $\dfrac{2\sqrt{2}}{3}$

**Solution:** Differentiating the given function we have

$$f'(x) = 6x - 1 = 0 \Rightarrow x = \frac{1}{6}$$

$f''(x) = 6 \Rightarrow f$ is minimum at $x = 1/6$ and the least value of $f$ is

$$f\left(\frac{1}{6}\right) = \frac{3}{36} - \frac{1}{6} + \frac{25}{12} = 2$$

Let $r$ be the common ratio $(|r| < 1)$ so that the progression is $r^2, r^3, r^4 \ldots$. By hypothesis,

$$\frac{r^2}{1-r} = 2$$
$$\Rightarrow r^2 + 2r - 2 = 0$$
$$\Rightarrow r = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$$

Now $|r| < 1 \Rightarrow r = -1 + \sqrt{3}$.

**Answer: (B)**

**133.** The values of the perameter "$a$" for which the values of the function

$$f(x) = x^3 - 6x^2 + 9x + a$$

at the point $x = 2$ and at the points of extremum, taken in a certain order, form a geometric progression are

(A) $\dfrac{8}{3}, \dfrac{4}{3}$ (B) $-\dfrac{8}{3}, \dfrac{4}{3}$
(C) $\dfrac{8}{3}, -\dfrac{4}{3}$ (D) $-\dfrac{8}{3}, -\dfrac{4}{3}$

**Solution:** We have

$$f(x) = x^3 - 6x^2 + 9x + a$$

Now

$$f(2) = a + 2 \tag{3.46}$$
$$f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0$$
$$\Rightarrow 3(x^2 - 4x + 3) = 0$$
$$\Rightarrow 3(x-1)(x-3) = 0$$
$$\Rightarrow x = 1, 3$$

Now,

$$f(1) = a + 4 \tag{3.47}$$

and

$$f(3) = a \tag{3.48}$$

By hypothesis $f(2) = a + 2, f(1) = a + 4$ and $f(3) = a$ form a geometric progression taken in a certain order. We can easily show that $a, a + 2, a + 4$ cannot be in GP. So

$$a, a+4, a+2 \text{ are in GP} \Leftrightarrow (a+4)^2 = a\,(a+2)$$
$$\Leftrightarrow 6a = -16$$
$$\Leftrightarrow a = -\frac{8}{3}$$

Again

$$a+2, a, a+4 \text{ are in GP} \Leftrightarrow (a+2)(a+4) = a^2$$
$$\Leftrightarrow 6a + 8 = 0$$
$$\Leftrightarrow a = -\frac{4}{3}$$

Therefore $a = -8/3, -4/3$.

**Answer: (D)**

**134.** The number of minima of the polynomial

$$f(x) = 10x^6 - 24x^5 + 15x^4 + 40x^2 + 108$$

is

(A) 5 (B) 1 (C) 2 (D) 3

**Solution:** Differentiating the given function we have

$$f'(x) = 60x^5 - 120x^4 + 60x^3 + 80x$$
$$= 20x[3x^4 - 6x^3 + 3x^2 + 4]$$
$$= 20x[3x^2 (x-1)^2 + 4]$$

Therefore

$$f'(x) = 0 \Leftrightarrow x = 0$$

So $x = 0$ is the only critical point. Also $f'(x)$ changes sign from negative to positive at $x = 0$. Hence, $f$ is minimum at $x = 0$ and the minimum value of $f$ is 108.

**Answer: (B)**

**135.** For $x \geq 0$, the least value of the expression $\dfrac{1+x^2}{1+x}$ is

(A) $\sqrt{2}$ (B) 1
(C) $2(\sqrt{2}+1)$ (D) $2(\sqrt{2}-1)$

**Solution:** Let

$$f(x) = \frac{1+x^2}{1+x}, \quad x \geq 0$$

Differentiating we get

$$f'(x) = \frac{2x(1+x) - 1(1+x^2)}{(1+x)^2}$$
$$= \frac{x^2 + 2x - 1}{(1+x)^2}$$
$$= 1 - \frac{2}{(1+x)^2}$$

Now

$$f'(x) = 0 \Leftrightarrow (x+1)^2 = 2$$
$$\Leftrightarrow x = (\pm\sqrt{2}) - 1$$

Now $x \geq 0 \Rightarrow x = \sqrt{2} - 1$ is the only critical point. Also

$$f''(x) = \frac{4}{(1+x)^3}$$

and
$$f''(\sqrt{2}-1) = \frac{4}{2\sqrt{2}} > 0$$

$f$ is least at $x = \sqrt{2} - 1$ and the least value is

$$f(\sqrt{2}-1) = \frac{1+(\sqrt{2}-1)^2}{1+\sqrt{2}-1}$$
$$= \frac{4-2\sqrt{2}}{\sqrt{2}}$$
$$= 2\sqrt{2} - 2$$
$$= 2(\sqrt{2}-1)$$

**Answer: (D)**

**136.** Let $f(x) = 5 - 4(x-2)^{2/3}$. Then at $x = 2$, $f(x)$

(A) attains minimum value
(B) attains maximum value
(C) is neither maximum nor minimum value
(D) is not defined

**Solution:** Differentiating we get

$$f'(x) = -\frac{8}{3}(x-2)^{-1/3}$$

$f'(x) \neq 0 \; \forall \; x \neq 2$ and $f'(2)$ is not defined. Hence, $x = 2$ is the only critical point. Also

$$f'(x) > 0 \quad \text{for } x < 2$$

and
$$f'(x) < 0 \quad \text{for } x > 2$$

Thus, $f$ is maximum at $x = 2$.

**Answer: (B)**

**137.** As $x$ varies over all real numbers, the range of the function

$$f(x) = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

is

(A) $[-7, 7]$ (B) $\left[-\dfrac{1}{7}, 7\right]$

(C) $\left[\dfrac{1}{7}, 7\right]$ (D) $\left(-\infty, \dfrac{1}{7}\right) \cup (7, \infty)$

**Solution:** The given function can be written as

$$f(x) = 1 - \frac{6x}{x^2 + 3x + 4}$$

Observe that

$$x^2 + 3x + 4 = \left(x + \frac{3}{2}\right)^2 + \frac{7}{4} > 0$$

for all real $x$. Now

$$f'(x) = -\frac{6[1(x^2+3x+4) - x(2x+3)]}{(x^2+3x+4)^2}$$
$$= -\frac{6(4-x^2)}{(x^2+3x+4)^2}$$
$$= \frac{6(x^2-4)}{(x^2+3x+4)^2}$$
$$= \frac{6(x+2)(x-2)}{(x^2+3x+4)^2}$$

Now $f'(x) = 0 \Leftrightarrow x = -2, 2$. That is $-2, 2$ are the only critical points. Also

$$f'(x) > 0 \quad \text{when } x < -2$$

and $$f'(x) < 0 \quad \text{when } x > -2$$

Hence $f$ is maximum at $x = -2$. Similary, we can see that $f$ is minimum at $x = 2$. Therefore

$$\text{Minimum value of } f = f(2) = \frac{4-6+4}{4+6+4} = \frac{1}{7}$$

$$\text{Maximum value of } f = f(-2) = \frac{4+6+4}{4-6+4} = 7$$

So

$$\text{Range of } f = \left[\frac{1}{7}, 7\right]$$

**Answer: (C)**

> **Try it out** Try the above problem by using quadratic equation.

**138.** Let $f(x) = (1+b^2)x^2 + 2bx + 1$. Let $m(b)$ be the minimum value of $f(x)$ (that is least value). As $b$ varies, the range of $m(b)$ is

(A) $[0, 1]$ (B) $(0, 1/2]$

(C) $[1/2, 1]$ (D) $(0, 1]$

**Solution:** Differentiating the given function we get

$$f'(x) = 2(1+b^2)x+2b$$

Now

$$f'(x) = 0 \Leftrightarrow x = \frac{-b}{1+b^2}$$

So

$$x < \frac{-b}{1+b^2} \Rightarrow f'(x) < 0$$

and $$x > \frac{-b}{1+b^2} \Rightarrow f'(x) > 0$$

So $f$ is minimum at $x = -b/(1+b^2)$ and the minimum value of $f(x)$ is

$$f\left(\frac{-b}{1+b^2}\right) = (1+b^2)\left(\frac{b^2}{(1+b^2)^2}\right) - \frac{2b^2}{1+b^2} + 1$$

$$= \frac{b^2}{1+b^2} - \frac{2b^2}{1+b^2} + 1 = 1 - \frac{b^2}{1+b^2}$$

Therefore

$$m(b) = 1 - \frac{b^2}{1+b^2} \tag{3.49}$$

Differentiating we get

$$m'(b) = -\frac{[2b(1+b^2) - b^2(2b)]}{(1+b^2)^2}$$

$$= -\frac{2b}{(1+b^2)^2}$$

So $m'(b) = 0 \Leftrightarrow b = 0$. Now,

$$b < 0 \Rightarrow m'(b) > 0$$

and $$b > 0 \Rightarrow m'(b) < 0$$

Hence $m(b)$ is maximum at $b = 0$ and the maximum value of $m(b) = m(0) = 1$ [by Eq. (3.49)]. Also

$$m(b) = 1 - \frac{b^2}{1+b^2} = \frac{1}{1+b^2} > 0$$

That is $m(b) > 0$ and $m(b) \le 1$ and $m(b) = 1$ when $b = 0$. So

$$\text{Range of } m(b) = (0, 1]$$

**Answer: (D)**

**139.** The maximum value of $f(x) = \cos x\sqrt{\sin x}$ on the interval $[0, \pi/2]$ is

(A) $2^{1/2} \cdot 3^{-3/4}$ (B) $2^{1/2} \cdot 3^{3/4}$

(C) $2^{3/4} \cdot 3^{1/2}$ (D) $2^{3/4} \cdot 3^{-1/2}$

**Solution:** We have $f(x) = \cos\sqrt{\sin x}$. So,

$$f(0) = 0, f\left(\frac{\pi}{2}\right) = 0$$

and $$f(x) > 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

Differentiating we get

$$f'(x) = -\sin x\sqrt{\sin x} + \cos x\left(\frac{1}{2\sqrt{\sin x}}\right)\cos x$$

$$= -(\sin x)^{3/2} + \frac{\cos^2 x}{2\sqrt{\sin x}}$$

$$= \frac{-2\sin^2 x + \cos^2 x}{2\sqrt{\sin x}}$$

$$= \frac{1 - 3\sin^2 x}{2\sqrt{\sin x}}$$

Now (since $\sin x > 0$) we have

$$f'(x) = 0 \Rightarrow \sin x = \frac{1}{\sqrt{3}}$$

Also $f'(x) > 0$ when $\sin x < 1/\sqrt{3}$ and $f'(x) < 0$ when $\sin x > 1/\sqrt{3}$. Therefore, $f$ is maximum when $\sin x = 1/\sqrt{3}$ and the maximum value of $f$ is

$$f\left(\text{Sin}^{-1}\frac{1}{\sqrt{3}}\right) = \sqrt{1-\frac{1}{3}}\cdot\sqrt{\frac{1}{\sqrt{3}}} = 2^{1/2}\cdot 3^{-3/4}$$

**Answer: (A)**

**140.** The greatest value of the function

$$f(x) = \frac{x}{ax^2+b}$$

is

(A) $\frac{2}{\sqrt{ab}}$ (B) $\frac{1}{2\sqrt{ab}}$

(C) $2\sqrt{ab}$ (d) $2ab$

**Solution:** Differentiating the given function we have

$$f'(x) = \frac{(ax^2+b)-x(2ax)}{(ax^2+b)^2} = \frac{b-ax^2}{(ax^2+b)^2}$$

So

$$f'(x) = 0 \Rightarrow x = \pm\sqrt{\frac{b}{a}}$$

Now

$$x < \sqrt{\frac{b}{a}} \Rightarrow b-ax^2 > 0$$

and $$x > \sqrt{\frac{b}{a}} \Rightarrow b-ax^2 < 0$$

Therefore, $f'(x)$ changes sign from positive to negative at $x = \sqrt{b/a}$. Hence, $f$ is maximum at $x = \sqrt{b/a}$ and the maximum value is

$$f\left(\sqrt{\frac{b}{a}}\right) = \frac{\sqrt{b/a}}{a(b/a)+b} = \frac{1}{2\sqrt{ab}}$$

**Answer: (B)**

**141.** The number of values of $x$ at which the function $f(x) = (x-1)\,x^{2/3}$ has extremum values is

(A) 4 (B) 3 (C) 2 (D) 1

**Solution:** Differentiating the given function we have

$$f'(x) = x^{2/3} + (x-1)\frac{2}{3}x^{-1/3} = \frac{3x+2(x-1)}{3x^{1/3}} = \frac{5x-2}{3x^{1/3}}$$

Now $f'(x) = 0 \Leftrightarrow x = 2/5$. Also $f(x)$ is defined and continuous at $x = 0$, $f$ is not differentiable at $x = 0$. Thus, zero is a critical point. Therefore, 0 and 2/5 are critical points of $f(x)$.

**(i)** $x < 0 \Rightarrow f'(x) > 0$ and $x > 0 \Rightarrow f'(x) < 0$. Therefore at $x = 0$, $f$ is maximum and the maximum value $= f(0) = 0$.

**(ii)** $x < 2/5 \Rightarrow f'(x) < 0$ and $x > 2/5 \Rightarrow f'(x) > 0$. Thus $f$ is minimum at $x = 2/5$ and the minimum value

$$f\left(\frac{2}{5}\right) = -\frac{3}{5}\left(\frac{4}{25}\right)^{1/3}$$

**Answer: (C)**

**142.** Consider the function $f(x) = x^3-3x+3$ on the interval $[-3, 3/2]$. Let $M = \text{Max}\, f(x)$ and $m = \text{Min}\, f(x)$ on $[-3, 3/2]$. Then

(A) $M = 15, m = 5$ (B) $M = 5, m = -15$

(C) $M = 15, m = -5$ (D) $M = -5, m = -15$

**Solution:** Differentiating the given function we have

$$f'(x) = 3x^2 - 3 = 3\,(x^2-1)$$

Therefore $f'(x) = 0 \Leftrightarrow x = \pm 1$. That is 1, −1 are the only critical points. Now

$$f''(x) = 6x \Rightarrow f''(1) > 0 \text{ and } f''(-1) < 0.$$

Therefore, $f$ is maximum at $x = -1$ and minimum at $x = 1$. Let

$$S = \left\{f(-3), f(-1), f(1), f\left(\frac{3}{2}\right)\right\} = \left\{-15, 5, 1, \frac{15}{8}\right\}$$

According to the procedure given in Sec. 3.4.2, the maximum of $f(x) = 5 = M$ and $m$ = minimum of $f(x) = -15$ on the interval $[-3, 3/2]$.

**Answer: (B)**

**143.** Let $f(x) = \sin x\cos^2 x, x \in [0, \pi]$. Let $m$ be the number of values of $x$ at which $f(x)$ is minimum and $n$ be the number of values of $x$ at which $f(x)$ is maximum. Then

(A) $m = 2, n = 2$ (B) $m = 2, n = 1$

(C) $m = 3, n = 2$ (D) $m = 1, n = 2$

**Solution:** Differentiating the given function we get

$$f'(x) = \cos x\cos^2 x - 2\sin^2 x\cos x = \cos x[\cos^2 x - 2\sin^2 x] = \cos x[3\cos^2 x - 2]$$

Now $f'(x)=0$ implies

$$\cos x = 0 \quad \text{or} \quad \cos x = \pm\sqrt{\frac{2}{3}}$$

$$\Rightarrow x = \frac{\pi}{2} \quad \text{or} \quad x = \text{Cos}^{-1}\left(\sqrt{\frac{2}{3}}\right), \text{Cos}^{-1}\left(-\sqrt{\frac{2}{3}}\right)$$

Again differentiating we get

$$f''(x) = -\sin x(3\cos^2 x - 2) + \cos x(-6\cos x \sin x)$$

Now

$$f''\left(\frac{\pi}{2}\right) = -1(-2) = 2 > 0$$

$$f''\left(\text{Cos}^{-1}\sqrt{\frac{2}{3}}\right) = 0 - 6\left(\frac{2}{3}\right)(+) < 0$$

Similary, $f''(\text{Cos}^{-1}(-\sqrt{2}/3))$ is also negative. Therefore $m=1, n=2$.

**Answer: (D)**

**144.** The range of the function $f(x) = \dfrac{x}{1+x^2}$, $x \in \mathbb{R}$ is

(A) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (B) $[-1, 0]$

(C) $[-2, -1]$ (D) $\left[-\frac{1}{2}, 0\right]$

**Solution:** Differentiating the given function we get

$$f'(x) = \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

Therefore

$$f'(x) = 0 \Rightarrow x = \pm 1$$

**(i)** $x < -1 \Rightarrow f'(x) < 0$ and $x > -1 \Rightarrow f'(x) > 0$. Therefore, $f$ is minimum at $x = -1$ and the minimum value $= f(-1) = -1/2$.

**(ii)** $x < 1 \Rightarrow f'(x) > 0$ and $x > 1 \Rightarrow f'(x) < 0$. Therefore, $f$ is maximum at $x = 1$ and the maximum value $= f(1) = 1/2$.

So

$$\text{Range of } f = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

**Answer: (A)**

**145.** The function $f(x) = x + \tan x$ has

(A) one maximum and one minimum

(B) only one maximum

(C) only one minimum

(D) neither maximum nor minimum

**Solution:** Differentiating the given function we get

$$f'(x) = 1 + \sec^2 x > 0$$

So $f$ is strictly increasing. Therefore, $f$ has neither maximum nor minimum.

**Answer: (D)**

**146.** The range of the function $f(x) = x(\log x)^2$ is

(A) $[1, e^2]$ (B) $[0, 4e^{-2}]$

(C) $[0, e^2]$ (D) $[0, e]$

**Solution:** $f(x) = x(\log x)^2$ is defined for all $x > 0$. Now

$$f'(x) = (\log x)^2 + \frac{2(\log x)x}{x} = \log x(\log x + 2)$$

Therefore

$$f'(x) = 0 \Rightarrow x = 1 \quad \text{or} \quad x = e^{-2}$$

Now,

$$f''(x) = \frac{2\log x}{x} + \frac{2}{x}$$

$$f''(1) = 2 > 0$$

and $$f''(e^{-2}) = -4e^2 + 2e^2 < 0$$

Hence, $f$ is minimum at $x = 1$ and maximum at $x = e^{-2}$. Now

$$f(1) = 0$$

and $$f(e^{-2}) = e^{-2}(\log e^{-2})^2 = 4e^{-2}$$

Therefore

$$\text{Range of } f = [0, 4e^{-2}]$$

**Answer: (B)**

**147.** The number of critical points for the function

$$f(x) = \frac{1}{3}\sin\theta \tan^3 x + (\sin\theta - 1)\tan x + \sqrt{\frac{\theta - 2}{8 - \theta}}$$

where $\pi < \theta < 2\pi$ is

(A) 4 (B) 2 (C) 1 (D) 0

**Solution:** Differentiating the given function we get

$$f'(x) = \frac{1}{3}\sin\theta(3\tan^2 x \sec^2 x) + (\sin\theta - 1)\sec^2 x$$

$$= \sin\theta \tan^2 x \sec^2 x + (\sin\theta - 1)\sec^2 x$$

Now

$$f'(x) = 0 \Rightarrow \sin\theta \tan^2 x + \sin\theta - 1 = 0$$

$$\Rightarrow \tan^2 x = \frac{1-\sin\theta}{\sin\theta} < 0 \quad (\because \pi < \theta < 2\pi)$$

which is absurd. Therefore, $f(x)$ has *no critical points*.

**Answer: (D)**

**148.** Let $f(x) = \frac{x^2-1}{x^2+1}, x \in \mathbb{R}$. Then the minimum value of $f$

(A) does not exist because $f$ is unbounded
(B) is not attained even though $f$ is bounded
(C) is equal to 1
(D) is equal to $-1$

**Solution:** We have for $x \in \mathbb{R}$

$$f(x) = \frac{x^2-1}{x^2+1} = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1}$$

Now

$$f'(x) = \frac{-2(-2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

Therefore $f'(x) = 0 \Leftrightarrow x = 0$. Also at $x = 0$, $f'(x)$ changes sign from negative to positive. Hence, $f$ is minimum at $x = 0$ and the minimum value $f(0) = -1$.

**Answer: (D)**

**149.** For $x \in (0, \pi)$, the least value of $f(x) = \left(\frac{2+\cos x}{\sin x}\right)^2$ is

(A) 2 (B) 3 (C) 4 (D) $\sqrt{3}$

**Solution:** We have

$$f(x) = \frac{(2+\cos x)^2}{\sin^2 x}$$

Put $t = \cos x$ so that $-1 < t < 1$. Let

$$\phi(t) = \frac{(2+t)^2}{1-t^2}, -1 < t < 1$$

Differentiating we get

$$\phi'(t) = \frac{2(2+t)(1-t^2) - (-2t)(2+t)^2}{(1-t^2)^2}$$

$$= \frac{2(2t+1)(t+2)}{(1-t^2)^2}$$

Since $-1 < t < 1$, $t = -1/2$ is the only critical point. Also

$$\phi'(t) < 0 \quad \text{for } t < -\frac{1}{2}$$

and

$$\phi'(t) > 0 \quad \text{for } t > -\frac{1}{2}$$

Hence $\phi$ is minimum at $t = -1/2$. Now
Least value of $f(x)$ on $(0, \pi)$ = Least value of $\phi$ on $(-1, 1)$

$$= \frac{\left(2-\frac{1}{2}\right)^2}{1-\frac{1}{4}} = 3$$

(See Note after Example 3.16.)

**Answer: (B)**

**150.** Let $f(x) = |x^2 - 5x + 6|$, $x \in [0, 2.4]$. Then, the sum of the maximum and minimum values of $f$ on [0, 2.4] is

(A) 6 (B) 5 (C) 6.5 (D) 5.5

**Solution:** The given function can be written as

$$f(x) = |(x-2)(x-3)| = \begin{cases} x^2 - 5x + 6, & 0 \le x \le 2 \\ -(x^2 - 5x + 6), & 2 < x \le 2.4 \end{cases}$$

Now differentiating we get

$$f'(x) = \begin{cases} 2x - 5, & 0 \le x \le 2 \\ -2x + 5, & 2 < x \le 2.4 \end{cases}$$

Now

$$f'(x) = 0 \Rightarrow x = \frac{2}{5} \notin [0, 2.4]$$

Also $f'(2)$ does not exist. Therefore $x = 2$ is the only critical point. $f'(x)$ changes sign from negative to positive. So $f$ is minimum at $x = 2$ and the minimum value = $f(2) = 0$.

Also $f(0) = 6$ and $f(2.4) = 0.24$. Therefore, on the interval [0, 2.4],

Minimum value of $f$ + Maximum value of $f = 0 + 6 = 6$

**Answer: (A)**

**151.** A closed circular cylinder of volume $k$ cubic units is to be formed with minimum amount of material. The ratio of its height to the radius of the base is

(A) 4 (B) 3 (C) 2 (D) 3/2

**Solution:** Let $h$ be the height and $r$ the radius of its base. By hypothesis,

$$\pi r^2 h = k \text{ (constant)} \qquad (3.50)$$

Let $S$ be the total surface area of the cylinder. Therefore

$$\begin{aligned} S &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2}\right) \quad \text{[from Eq. (3.50)]} \\ &= 2\pi r^2 + \frac{2k}{r} \end{aligned}$$

where $2\pi r$ = sum of the areas of the base and the top and $2\pi rh$ is the lateral surface area. Therefore differentiating we get

$$\frac{dS}{dr} = 4\pi r - \frac{2k}{r^2}$$

Now

$$\begin{aligned} \frac{dS}{dr} = 0 &\Rightarrow r^3 = \frac{k}{2\pi} \\ &\Rightarrow r = \left(\frac{k}{2\pi}\right)^{1/3} \end{aligned} \qquad (3.51)$$

Since $r > 0, k > 0$, we have

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4k}{r^3} > 0$$

So, $S$ is minimum when $r = (k/2\pi)^{1/3}$. Hence

$$\begin{aligned} \frac{h}{r} &= \frac{k/\pi r^2}{r} \\ &= \frac{k}{\pi}\cdot\frac{1}{r^3} \\ &= \frac{k}{\pi}\cdot\frac{2\pi}{k} \quad \text{[from Eq. (3.51)]} \\ &= 2 \end{aligned}$$

**Answer: (C)**

**152.** If $h$ is the height and $r$ is the radius of the base of a circular cylinder of greatest volume that can be inscribed in a given sphere, then $h$ is equal to (Fig. 3.22)

(A) $2r$ (B) $3r$
(C) $\sqrt{3}\,r$ (D) $\sqrt{2}\,r$

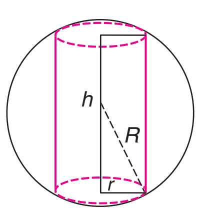



**FIGURE 3.22** Single correct choice type question 152.

**Solution:** Let $R$ be the radius of the sphere. Then

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2 \qquad (3.52)$$

Let $V$ is the volume of the cylinder, so that

$$\begin{aligned} V &= \pi r^2 \mathrm{h} \\ &= \pi h\left(R^2 - \frac{h^2}{4}\right) \quad \text{[from Eq. (3.52)]} \end{aligned}$$

Differentiating we get

$$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4}$$

Now

$$\begin{aligned} \frac{dV}{dh} = 0 &\Rightarrow h^2 = \frac{4R^2}{3} \\ &\Rightarrow h = \frac{2}{\sqrt{3}}R \end{aligned}$$

Also since $h > 0$

$$\frac{d^2V}{dh^2} = -\frac{3\pi}{2}h < 0$$

So, $V$ is greatest, when $h = 2R/\sqrt{3}$. In such a case

$$r^2 = R^2 - \frac{h^2}{4} = R^2 - \frac{4R^2}{3\times 4} = \frac{2R^2}{3}$$

Therefore

$$r = \sqrt{\frac{2}{3}}\cdot R$$

and

$$h = \frac{2R}{\sqrt{3}} = \sqrt{2}\left(\sqrt{\frac{2}{3}}R\right) = \sqrt{2}r$$

**Answer: (D)**

**153.** From a rectangular cardboard of size $3 \times 8$, equal square pieces are removed from the four corners, and an open rectangular box is formed from the remaining. The maximum volume of the box is

(A) $\frac{250}{6}$ cubic units (B) $\frac{250}{3}$ cubic units

(C) 125 cubic units (D) $\frac{200}{27}$ cubic units

**Solution:** See Fig. 3.23. Let $x$ be the side of the square piece removed. Therefore the dimensions of the box are $3 - 2x$, $8 - 2x$ and $x$.

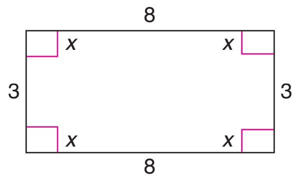



**FIGURE 3.23** Single correct choice type question 153.

Let $V$ be the volume of the box so that

$$V = x(3-2x)(8-2x)$$

Now,

$$\begin{aligned}\frac{dV}{dx} &= (3-2x)(8-2x) + x(8-2x)(-2) + x(3-2x)(-2)\\ &= (24-22x+4x^2) + (4x^2-16x) + (4x^2-6x)\\ &= 12x^2 - 44x + 24\\ &= 4(3x^2 - 11x + 6)\\ &= 4(3x-2)(x-3)\end{aligned}$$

Therefore the critical points are $x = 2/3$ and $x = 3$. Now $x = 3$ is impossible. Hence $x = 2/3$ is the only critical point. So

$$\frac{d^2V}{dx^2} = 24x - 44$$

$$\Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=\frac{2}{3}} = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$$

Hence, $V$ is maximum at $x = 2/3$ and the maximum volume is

$$\frac{2}{3}\left(3-\frac{4}{3}\right)\left(8-\frac{4}{3}\right) = \frac{2}{3}\left(\frac{5}{3}\right)\left(\frac{20}{3}\right) = \frac{200}{27}\text{ cubic units}$$

**Answer: (D)**

**154.** Let $x, y, z$ be positive numbers such that

$$x + y + z = 26 \quad \text{and} \quad y = 3x$$

If $x^2 + y^2 + z^2$ is to be least, then the value of $x$ is

(A) 3 (B) 4
(C) 3.5 (D) 4.5

**Solution:** Let

$$\begin{aligned}S &= x^2 + y^2 + z^2\\ &= x^2 + 9x^2 + (26-4x)^2\end{aligned}$$

($\because y = 3x, z = 26 - x - y$). Now

$$0 < z = 26 - 4x \Rightarrow x < \frac{13}{2}$$

Therefore, $S$ is a function of $x$ on the interval $(0, 13/2)$. Now

$$\frac{dS}{dx} = 0 \Rightarrow 2x + 18x - 8(26-4x) = 0$$

$$\Rightarrow 52x = 8 \times 26$$

$$\Rightarrow x = \frac{8\times 26}{52} = 4 \in \left(0, \frac{13}{2}\right)$$

Also

$$\frac{d^2S}{dx^2} = 20 + 32 > 0$$

Therefore $S$ is least at $x = 4$.

**Answer: (B)**

**155.** If $h$ is the height of a circular cone of greatest volume of given slant height $l$, then $h$ is equal to

(A) $\frac{l}{\sqrt{3}}$ (B) $\frac{l}{\sqrt{2}}$ (C) $\sqrt{3}\,l$ (D) $\sqrt{2}\,l$

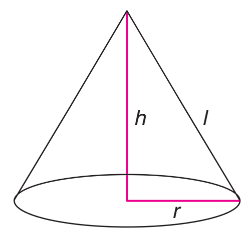



**FIGURE 3.24** Single correct choice type question 155.

**Solution:** See Fig. 3.24. Let $r$ be the radious of the base circle. Therefore

$$h^2 + r^2 = l^2 \qquad (l \text{ is constant})$$

The volume of the cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(l^2 - h^2)h$$

Differentiating we get

$$\frac{dV}{dh} = \frac{\pi}{3}(l^2 - 3h^2)$$

Now

$$\frac{dV}{dh} = 0 \Rightarrow h = \frac{l}{\sqrt{3}}$$

Also

$$\left(\frac{d^2V}{dh^2}\right) = -2\pi h < 0, \text{ if } \quad h = \frac{l}{\sqrt{3}}$$

So $V$ is maximum when $h = \frac{l}{\sqrt{3}}$.

**Answer: (A)**

**156.** The least distance of the point $Q$ (0, –2) from the point $P(x, y)$ where $y = \frac{16}{\sqrt{3}x^3} - 2$ and $x > 0$ is

(A) 4/3 (B) $4\sqrt{3}$ (C) $4/\sqrt{3}$ (D) $3\sqrt{3}$

**Solution:** We have

$$D = \text{distance } PQ = \sqrt{x^2+(y+2)^2}$$

Therefore

$$D^2 = x^2 + \left(\frac{16}{\sqrt{3}x^3}\right)^2 = x^2 + \frac{256}{3x^6}$$

Now

$$(2D)\frac{dD}{dx} = 2x - \frac{6\times 256}{3}x^{-7}$$

So

$$\frac{dD}{dx} = 0 \Rightarrow x^8 = 256$$

$$\Rightarrow x = 2 \quad (\because x > 0)$$

Also when $x = 2$

$$\frac{d^2D}{dx^2}(2D) = 2 + 7(2\times 256)x^{-8} > 0$$

So $D$ is minimum if $x = 2$ and the minimum value of $D$ is

$$\sqrt{4+\left(\frac{16}{\sqrt{3}}\times\frac{1}{8}\right)^2} = \sqrt{4+\frac{4}{3}} = \frac{4}{\sqrt{3}}$$

**Answer: (C)**

**157.** The greatest volume of a circular cylinder whose total surface area is $2\pi$, is

(A) $\dfrac{2\pi}{3\sqrt{3}}$ (B) $\dfrac{2\pi}{3}$ (C) $\dfrac{2\pi}{\sqrt{3}}$ (D) $\dfrac{4\pi}{3\sqrt{3}}$

**Solution:** Let $r$ be the radius of the base circle and $h$ is the height of the cylinder (Fig. 3.25). By hypothesis

$$2\pi = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow r^2 + rh = 1 \tag{3.53}$$

The volume of the cylinder is given by

$$V = \pi r^2 h$$

$$= \pi r^2 \frac{(1-r^2)}{r} \quad \text{[from Eq. (3.53)]}$$

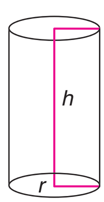



**FIGURE 3.25** Single correct choice type question 157.

$$= \pi r(1-r^2)$$

$$= \pi(r-r^3)$$

Now

$$\frac{dV}{dr} = 0 \Rightarrow \pi(1-3r^2) = 0$$

$$\Rightarrow r = \frac{1}{\sqrt{3}}$$

Also

$$\frac{d^2V}{dr^2} = -6\pi r < 0 \quad \text{when } r = \frac{1}{\sqrt{3}}$$

So $V$ is maximum if $r = 1/\sqrt{3}$ and the greatest volume is

$$\pi\left(\frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3\right) = \pi\left(\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right)$$

$$= \frac{\pi}{\sqrt{3}}\left(\frac{2}{3}\right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$

**Answer: (A)**

**158.** $ABCD$ is a trapezium in which $AB$ and $CD$ are non-parallel sides with $BC$ and $AD$ of lengths of 6 and 10 units, respectively. Further it is given that $AB$ is perpendicular to the parallel sides $BC$ and $AD$. Thus, the greatest area of the rectangle inscribed in the trapezium so that one of its sides lies on the larger side of the trapezium, given that $AB$ is of length 8 units is

(A) 46 square units (B) 48 square units
(C) 36 square units (D) 72 square units

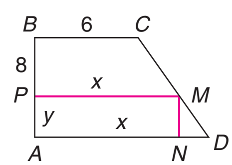



**FIGURE 3.26** Single correct choice type question 158.

**Solution:** Let $APMN$ (see Fig. 3.26) be a rectangle inscribed in the trapezium with $AN$ on the larger side $AD$. Let $AN = x$ and $AP = y$ so that $6 \le x < 10$ and $0 \le y \le 8$. Now, Area of trapezium $ABCD = \frac{1}{2}\cdot 8\cdot(6+10) = 64$
Also,

Area of $ABCD$ = Area of $PBCM$ + Area of $APMN$ + Area of $\Delta MND$

$$64 = \frac{1}{2}(8-y)(6+x) + xy + \frac{1}{2}(10-x)y$$
$$128 = (48 + 8x - 6y - xy) + 2xy + (10y - xy)$$
$$= 8x + 4y + 48$$

Therefore

$$2x + y = 20 \tag{3.54}$$

Let $S$ be the area of the rectangle $APMN$, so that, using Eq. (3.54),

$$S = xy = x(20 - 2x)$$

where $6 \le x < 10$. Now

$$\frac{dS}{dx} = 20 - 4x$$

so that

$$\frac{dS}{dx} = 0 \Leftrightarrow x = 5 \notin [6, 10)$$

Further $dS/dx < 0$ for $x \ge 6$. Therefore $S$ is decreasing for $x \ge 6$ and hence $S$ is maximum at $x = 6$. The greatest value of $S$ is

$$6 \times 8 = 48$$

because, from Eq. (3.54), $x = 6$ implies $y = 8$.

**Answer: (B)**

**159.** The area of the greatest rectangle inscribed in the segment of the curve $y^2 = 2px$ cut off by the line $x = 2a$ is

(A) $\dfrac{16a^{3/2}\sqrt{p}}{3\sqrt{3}}$ (B) $\dfrac{8a^{3/2}\sqrt{p}}{3\sqrt{3}}$

(C) $\dfrac{4a^{3/2}\sqrt{p}}{3}$ (D) $4a^{3/2}\sqrt{\dfrac{p}{3}}$

**Solution:** See. Fig. 3.27. First, observe that the curve $y^2 = 2px$ passes through $(0, 0)$ and is symmetric about the $x$-axis. That is, if $P(x, y)$ is a point on the curve, then $Q(x, -y)$ is also a point on the curve. Draw $PN$ and $QM$ perpendicular to the line $x = 2a$. Therefore, $PQMN$ is a rectangle inscribed in the segment. Let its area be $A$.

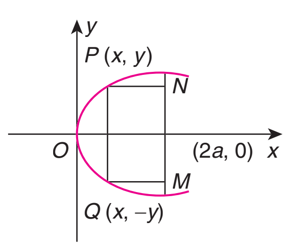



**FIGURE 3.27** Single correct choice type question 159.

Then

$$A = 2y(2a - x)$$
$$= 4ay - 2xy$$
$$= 4ay - 2\left(\frac{y^2}{2p}\right)y$$
$$= 4ay - \frac{y^3}{p} \tag{3.55}$$

Now

$$\frac{dA}{dy} = 0 \Rightarrow 4a - \frac{3y^2}{p} = 0$$
$$\Rightarrow y = \pm 2\sqrt{\frac{ap}{3}}$$

For this value of $y$

$$\frac{d^2A}{dy^2} = -\frac{6y}{p} < 0$$

Therefore, $A$ is maximum, when $y = \pm 2\sqrt{ap/3}$ and the greatest area is

$$\text{Greatest area} = 4ay - \frac{y^3}{p} \quad \text{[from Eq. (3.55)]}$$
$$= y\left(4a - \frac{y^2}{p}\right)$$

where $y = 2\sqrt{ap/3}$, so that the greatest area equals

$$2\left(\sqrt{\frac{ap}{3}}\right)\left(4a - \frac{4ap}{3p}\right) = \frac{16a^{3/2}\sqrt{p}}{3\sqrt{3}}$$

**Answer: (B)**

**160.** A person wishes to lay a straight fence across a triangular field $ABC$ with $\underline{|A} < \underline{|B} < \underline{|C}$ so as to divide into two equal areas. The length of the fence with minimum expense is

(A) $\sqrt{2\Delta \tan \dfrac{A}{2}}$ (B) $\sqrt{2\Delta \cot \dfrac{B}{2}}$

(C) $\sqrt{2\Delta \tan \dfrac{C}{3}}$ (D) $\sqrt{\tan \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}}$

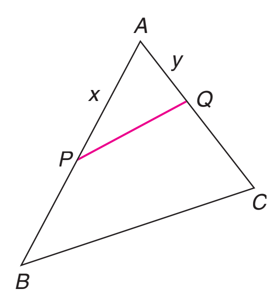



**FIGURE 3.28** Single correct choice type question 160.

**Solution:** Let $PQ$ be the fence opposite to the vertex $A$ (see Fig. 3.28). Let $AP = x$, $AQ = y$ and $PQ = Z_A$. By hypothesis

$$\begin{aligned}\frac{1}{2}xy\sin A &= \text{Area of } \Delta APQ \\ &= \frac{1}{2}(\text{Area of } \Delta ABC) \\ &= \frac{1}{2}\left(\frac{1}{2}bc\sin A\right)\end{aligned}$$

Therefore

$$xy = \frac{1}{2}bc \tag{3.56}$$

Now,

$$\begin{aligned}Z_A^2 &= (PQ)^2 \\ &= x^2 + y^2 - 2xy\cos A \\ &= x^2 + \frac{b^2c^2}{4x^2} - bc\cos A \quad \text{[from Eq. (3.56)]}\end{aligned}$$

Therefore

$$2Z_A\left(\frac{dZ_A}{dx}\right) = 2x - \frac{b^2c^2}{2x^3}$$

Now

$$\begin{aligned}\frac{dZ_A}{dx} = 0 &\Rightarrow x^4 = \frac{b^2c^2}{4} \\ &\Rightarrow x = \sqrt{\frac{bc}{2}}\end{aligned}$$

It can be easily seen that

$$\frac{d^2Z_A}{dx^2} > 0$$

Therefore, $Z_A$ is minimum if $x = \sqrt{bc/2}$ and minimum value of $Z_A$ is

$$\begin{aligned}\sqrt{\frac{bc}{2} + \frac{bc}{2} - bc\cos A} &= \sqrt{bc(1-\cos A)} \\ &= \sqrt{2bc\sin^2\frac{A}{2}} \\ &= \sqrt{\frac{2\Delta}{\sin A}\cdot 2\sin^2\frac{A}{2}} \\ &= \sqrt{2\Delta\tan\frac{A}{2}}\end{aligned}$$

Similarly, if $Z_B$ and $Z_C$ are the lengths of fences opposite to the vertices $B$ and $C$, respectively, then

$$Z_B = \sqrt{2\Delta\tan\frac{B}{2}} \quad \text{and} \quad Z_C = \sqrt{2\Delta\tan\frac{C}{2}}.$$

Since $\underline{|A} < \underline{|B} < \underline{|C}$, $Z_A = \sqrt{2\Delta\tan(A/2)}$ is the length of the fence with minimum expense.

**Answer: (A)**

## L'Hospital's Rule

**161.** The integer $n > 0$ for which

$$\lim_{x\to 0}\frac{(\cos x - 1)(\cos x - e^x)}{x^n}$$

is a finite non-zero number is

(A) 1 (B) 2 (C) 3 (D) 4

**Solution:** We have

$$\lim_{x\to 0}\frac{(\cos x - 1)(\cos x - e^x)}{x^n} = \lim_{x\to 0}\frac{\left(-2\sin^2\frac{x}{2}\right)(\cos x - e^x)}{x^n}$$

If $n = 1$ or 2, the limit is zero and hence $n \geq 3$. Therefore, the above limit is equal to

$$\begin{aligned}&\lim_{x\to 0}\left(-2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \times \frac{1}{4}\right)\frac{(\cos x - e^x)}{x^{n-2}} \\ &= -\frac{1}{2}\lim_{x\to 0}\left(\frac{\cos x - e^x}{x^{n-2}}\right)\left(\frac{0}{0}\right) \\ &= -\frac{1}{2}\lim_{x\to 0}\frac{(-\sin x - e^x)}{(n-2)x^{n-3}}\end{aligned}$$

If $n > 3$, then this limit does not exist and hence $n = 3$.

**Answer: (C)**

***Note:*** Here afterwards, if a certain limit is of $\left(\frac{0}{0}\right)$ form, it means that L'Hospital's rule is going to be applied.

**162.** If $f'(2) = 6$ and $f'(1) = 4$, then

$$\lim_{h\to 0}\frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)}$$

(A) does not exist (B) is equal to $-\frac{3}{2}$

(C) is equal to $\frac{3}{2}$ (D) is equal to 3

**Solution:** Since $f$ is differentiable at $x = 1$ and 2, $f$ is continuous at 1 and 2. Therefore

$$\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)} \text{ is } \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{(2+2h)f'(2h+2+h^2)}{(1-2h)f'(h-h^2+1)}$$

$$= \frac{2f'(2)}{1f'(1)} = \frac{2\times 6}{1\times 4} = 3$$

**Answer: (D)**

**163.** $\lim_{x\to 1}\frac{\log x}{\tan \pi x}$ is equal to

(A) 1 (B) $\frac{1}{\pi}$ (C) $\pi$ (D) $\infty$

**Solution:** The given limit is $\frac{0}{0}$ form. Therefore

$$\lim_{x\to 1}\frac{\log x}{\tan \pi x} = \lim_{x\to 1}\frac{1/x}{\pi \sec^2 \pi x} = \frac{1}{\pi}$$

**Answer: (B)**

**164.** $\lim_{x\to 0}\frac{e^{3x}-1}{\tan x}$ is

(A) 1 (B) $\frac{1}{3}$

(C) 3 (D) does not exist

**Solution:** The given limit is $\frac{0}{0}$ form. Therefore

$$\lim_{x\to 0}\frac{e^{3x}-1}{\tan x} = \lim_{x\to 0}\frac{3e^{3x}}{\sec^2 x} = \frac{3\cdot 1}{3} = 3$$

**Answer: (C)**

***Note:*** Without using L'Hospital's rule,

$$\lim_{x\to 0}\frac{e^{3x}-1}{\tan x} = \lim_{x\to 0}\frac{\left(\frac{e^{3x}-1}{3x}\right)\cdot 3}{\left(\frac{\tan x}{x}\right)} = \frac{1\cdot 3}{1}$$

$$\left(\because \frac{e^x-1}{x} \to 1 \text{ as } x \to 0 \text{ and } \lim_{x\to 0}\frac{\tan x}{x} = 1.\right)$$

**165.** $\lim_{x\to 0}\frac{x+\sin 2x}{x-\sin 2x}$ is equal to

(A) 3 (B) –3 (C) 1 (D) –1

**Solution:** $\lim_{x\to 0}\frac{x+\sin 2x}{x-\sin 2x}$ is $\left(\frac{0}{0}\right)$ form. It equals

$$\lim_{x\to 0}\frac{1+2\cos 2x}{1-2\cos 2x} = \frac{1+2(1)}{1-2(1)} = -3$$

**Answer: (B)**

**166.** $\lim_{x\to 0}\frac{\cos 2x-\cos x}{\sin^2 x}$ is

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $-\frac{3}{2}$

**Solution:** The given limit is $\frac{0}{0}$ form. Hence

$$\lim_{x\to 0}\frac{\cos 2x-\cos x}{\sin^2 x} = \lim_{x\to 0}\frac{-2\sin 2x+\sin x}{2\sin x\cos x}$$

$$= \lim_{x\to 0}\frac{-2\sin 2x+\sin x}{\sin 2x}$$

which is again $\frac{0}{0}$ form. So the limit becomes

$$\lim_{x\to 0}\frac{-4\cos 2x+\cos x}{2\cos 2x} = \frac{-4+1}{2} = -\frac{3}{2}$$

**Answer: (D)**

**167.** $\lim_{x\to 0}(\operatorname{cosec} x-\cot x)$ is

(A) 0 (B) 1

(C) –1 (D) does not exist

**Solution:** $\lim_{x\to 0}(\operatorname{cosec} x-\cot x) = \lim_{x\to 0}\frac{1-\cos x}{\sin x}$ is $\frac{0}{0}$ form. Therefore, it equals

$$\lim_{x\to 0}\left(\frac{\sin x}{\cos x}\right) = \frac{0}{1} = 0$$

**Answer: (A)**

**168.** $\lim_{x\to 0}\frac{1+x-e^x}{x^2}$ is

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 1 (D) –1

**Solution:** The given limit is $\frac{0}{0}$ form. So

$$\lim_{x\to 0}\frac{1+x-e^x}{x^2} = \lim_{x\to 0}\frac{1-e^x}{2x}$$

which is again $\left(\frac{0}{0}\right)$ form. Hence we have

$$\lim_{x\to 0}\left(\frac{-e^x}{2}\right)=\frac{-e^0}{2}=-\frac{1}{2}$$

**Answer: (B)**

**169.** $\lim_{x\to 0+0}(\sin x)^{\tan x}$ is

(A) 1 (B) $\infty$ (C) 0 (D) $-1$

**Solution:** The given limit is of the form $0^0$. As $x\to 0+0$, both $\sin x$ and $\tan x$ are positive and hence $(\sin x)^{\tan x}$ is defined. Let $y=(\sin x)^{\tan x}$. Then

$$\log_e y=\tan x\log_e\sin x=\frac{\log_e(\sin x)}{\cot x}$$

Now, let

$$l=\lim_{x\to 0+0}\log_e y=\lim_{x\to 0+0}\frac{\log_e(\sin x)}{\cot x}$$

which is of the $\frac{-\infty}{\infty}$ form. Therefore the limit is

$$\lim_{x\to 0+0}\frac{\cot x}{-\operatorname{cosec}^2 x}=-\lim_{x\to 0+0}(\cos x\sin x)=0$$

Therefore $l=e^0=1$.

**Answer: (A)**

**170.** $\lim_{x\to+\infty}\left(\frac{2^x}{x^{10}}\right)$ is equal to

(A) $\frac{(\log_e 2)^{10}}{10!}$ (B) $(\log_e 2)^{10}$

(C) $\frac{\log 2}{10!}$ (D) $+\infty$

**Solution:** The given limit is $\frac{+\infty}{+\infty}$ form. Therefore

$$\lim_{x\to+\infty}\left(\frac{2^x}{x^{10}}\right)=\lim_{x\to+\infty}\frac{(\log_e 2)\,2^x}{10x^9}\left(\frac{+\infty}{+\infty}\right)$$

Applying L'Hospital's rule 9 more times the given limit becomes

$$\lim_{x\to+\infty}\frac{(\log_e 2)^{10}\,2^x}{10!}=+\infty$$

**Answer: (D)**

**171.** $\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$ is equal to

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) 1 (D) 2

**Solution:** The given limit is of the form $\left(\frac{0}{0}\right)$. Therefore

$$\lim_{x\to 0}\left(\frac{e^{x^2}-\cos x}{x^2}\right)=\lim_{x\to 0}\frac{2xe^{x^2}+\sin x}{2x}$$

$$=\lim_{x\to 0}\left(e^{x^2}+\frac{\sin x}{2x}\right)$$

$$=1+\frac{1}{2}=\frac{3}{2}$$

**Answer: (A)**

**172.** $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ is

(A) $\frac{e}{2}$ (B) $\frac{-e}{2}$ (C) $\frac{e^2}{2}$ (D) $\frac{-e^2}{2}$

**Solution:** We know that $\lim_{x\to 0}(1+x)^{1/x}=e$ so that the given limit is of the form $\frac{0}{0}$. First, we calculate the derivative of $(1+x)^{1/x}$ w.r.t. $x$. Let

$$y=(1+x)^{1/x}$$

$$\Rightarrow \log_e y=\frac{1}{x}\log_e(1+x)$$

Differentiating both sides w.r.t. $x$ we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{\frac{x}{1+x}-\log_e(1+x)}{x^2}$$

$$=\frac{x-(1+x)\log_e(1+x)}{x^2+x^3}$$

So

$$\frac{dy}{dx}=\frac{y[x-(1+x)\log_e(1+x)]}{x^2+x^3}$$

Now

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}\frac{y[x-(1+x)\log_e(1+x)]}{x^2+x^3}$$

$$=\lim_{x\to 0}\frac{e[x-(1+x)\log_e(1+x)]}{x^2+x^3}\quad\left(\because \lim_{x\to 0}y=e\right)$$

$$=\lim_{x\to 0}\frac{e[1-\log(1+x)-1]}{2x+3x^2}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$= e\lim_{x\to 0}\frac{\left(\dfrac{-1}{1+x}\right)}{2+6x}$$
$$= e\left(\frac{-1}{2}\right)$$
$$= \frac{-e}{2}$$

**Answer: (B)**

**173.** Let $f$ be differentiable on some interval $(a, +\infty)$ and suppose

$$\lim_{x\to\infty}[f(x)+f'(x)] = l$$

Then $\lim_{x\to\infty} f(x)$ is equal to

(A) 0 (B) $-l$ (C) $l$ (D) $\dfrac{l}{2}$

**Solution:** We have

$$f(x) = \frac{f(x)e^x}{e^x}$$
$$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{f(x)e^x}{e^x}\left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x\to\infty}\frac{e^x[f(x)+f'(x)]}{e^x}$$
$$= \lim_{x\to\infty}[f(x)+f'(x)] = l$$

**Answer: (C)**

**174.** $\lim_{x\to a}\dfrac{x\sin a - a\sin x}{x-a}$ is

(A) $\sin a$ (B) $\sin a - a$
(C) $a\sin a$ (D) $\sin a - a\cos a$

**Solution:** The given limit is $\dfrac{0}{0}$ form. So

$$\lim_{x\to a}\frac{x\sin a - a\sin x}{x-a} = \lim_{x\to a}\left(\frac{\sin a - a\cos x}{1}\right)$$
$$= \sin a - a\cos a$$

**Answer: (D)**

**175.** $\lim_{x\to a}\dfrac{x^a - a^x}{x^x - a^a}$ equals

(A) $\dfrac{1-\log a}{1+\log a}$ (B) $\dfrac{1+\log a}{1-\log a}$

(C) 1 (D) does not exist

**Solution:** The given limit is $\dfrac{0}{0}$ form. So

$$\lim_{x\to a}\frac{x^a - a^x}{x^x - a^a} = \lim_{x\to a}\frac{ax^{a-1} - a^x\log a}{x^x(1+\log x)}$$
$$= \frac{a^a - a^a\log a}{a^a(1+\log a)}$$
$$= \frac{1-\log a}{1+\log a}$$

**Answer: (A)**

**176.** $\lim_{x\to 0+0}(x^n\log_e x)$ is

(A) $\infty$
(B) $-\infty$
(C) 0
(D) does not exists either finitely or infinitely

**Solution:** $\lim_{x\to 0+0}(x^n\log_e x)$ is of the form of $0\times(-\infty)$. Therefore it equals

$$\lim_{x\to 0+0}\left(\frac{\log_e x}{x^{-n}}\right)$$

which is of the form $\dfrac{-\infty}{\infty}$. So it can be written as

$$\lim_{x\to 0+0}\frac{1/x}{-nx^{-(n+1)}} = \lim_{x\to 0+0}\left(-\frac{x^n}{n}\right) = \frac{0}{n} = 0$$

**Answer: (C)**

**177.** $\lim_{x\to x+0}(x^x)$ is

(A) $\infty$ (B) 1
(C) 0 (D) does not exist finitely

**Solution:** $\lim_{x\to 0+0}(x^x)$ is of the form $0^0$. Let $y = x^x$. Then

$$\log_e y = x\log_e x$$
$$\Rightarrow \lim_{x\to 0+0}\log_e y = \lim_{x\to 0+0}\left(\frac{\log_e x}{1/x}\right)$$

which is of the form $\dfrac{-\infty}{\infty}$. It equals

$$\lim_{x\to 0+0}\frac{1/x}{-1/x^2} = -\lim_{x\to 0+0}(x) = 0$$

Therefore

$$\lim_{x\to 0} y = e^0 = 1$$

**Answer: (B)**

**178.** $\lim_{x\to\infty} x(e^{1/x}-1)$ is equal to

(A) $\infty$ (B) 0 (C) 1 (D) $-1$

**Solution:** The given limit is of the form $\infty \times 0$. Therefore

$$\lim_{x\to\infty} x(e^{1/x}-1) = \lim_{x\to\infty}\left(\frac{e^{1/x}-1}{1/x}\right)\left(\frac{0}{0}\right)$$
$$= \lim_{x\to\infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2}$$
$$= \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

**Answer: (C)**

**179.** $\lim_{x\to 0}(\cos x + \sin x)^{1/x}$ equals

(A) $e$ (B) $-e$ (C) 1 (D) $-1$

**Solution:** $\lim_{x\to 0}(\cos x+\sin x)^{1/x}$ is of the form $1^\infty$. Let

$$y = (\cos x + \sin x)^{1/x}$$

Therefore

$$\log_e y = \frac{1}{x}\log_e(\cos x + \sin x)$$
$$= \frac{\log_e(\cos x+\sin x)}{x}$$

Now

$$\lim_{x\to 0}(\log_e y) = \lim_{x\to 0}\frac{\log_e(\cos x+\sin x)}{x}\left(\frac{0}{0}\right)$$
$$= \lim_{x\to 0}\frac{\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)}{1}$$
$$= \frac{1-0}{1+0} = 1$$

So $\lim_{x\to 0}(y) = e^1 = e$.

**Answer: (A)**

**180.** If $f(x)$ is differentiable and $f'(0)\neq 0$, then the value of

$$\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)} \text{ is}$$

(A) 1 (B) 0 (C) $-1$ (D) 2

**Solution:** The given limit is $\frac{0}{0}$ form. So

$$\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)} = \lim_{x\to 0}\frac{2xf'(x^2)-f'(x)}{f'(x)}$$
$$= \frac{0[f'(0)]-f'(0)}{f'(0)} = -1$$

**Answer: (C)**

**181.** $\lim_{x\to 0}\frac{a^x-1-x\log a}{x^2}$ is

(A) $\frac{1}{2}\log a$ (B) $\frac{1}{2}(\log a)^2$

(C) $(\log a)^2$ (D) 1

**Solution:** The given limit is of the form $\frac{0}{0}$. So

$$\lim_{x\to 0}\frac{a^x-1-x\log a}{x^2} = \lim_{x\to 0}\left(\frac{a^x\log a - \log a}{2x}\right)$$
$$= \lim_{x\to 0}\log a\frac{(a^x-1)}{x}\times\frac{1}{2}$$
$$= \log a(\log a)\frac{1}{2}$$
$$= \frac{1}{2}(\log a)^2$$

**Answer: (B)**

**182.** $\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x}$ is

(A) 4 (B) 3 (C) 0 (D) 1

**Solution:** The given limit

$$\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x} = \lim_{x\to 0}\frac{4\log(1+x^3)}{3\sin x - \sin 3x}$$

is $\frac{0}{0}$ form. Therefore, it equals

$$4\lim_{x\to 0}\frac{\left(\dfrac{3x^2}{1+x^3}\right)}{3\cos x - 3\cos 3x} = 4\lim_{x\to 0}\frac{x^2}{(1+x^3)(2\sin 2x\sin x)}$$

$$= \lim_{x\to 0}\left[\frac{1}{(1+x^3)\left(\frac{\sin 2x}{2x}\cdot\frac{\sin x}{x}\right)}\right]$$

$$= \frac{1}{(1+0)(1\cdot 1)} = 1$$

**Answer: (D)**

**183.** $\lim_{x\to 0}\left(\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}\right)$ equals

(A) $\frac{\pi}{8}$ (B) $\frac{\pi^2}{4}$ (C) $\pi^2$ (D) $\frac{\pi^2}{8}$

**Solution:** The limit is of the form $(\infty - \infty)$. The given limit equals

$$\frac{\pi}{4}\lim_{x\to 0}\frac{e^{\pi x}-1}{x(e^{\pi x}+1)}$$

which is $\left(\frac{0}{0}\right)$ form. Hence

$$\frac{\pi}{4}\lim_{x\to 0}\frac{e^{\pi x}-1}{x(e^{\pi x}+1)} = \frac{\pi}{4}\lim_{x\to 0}\frac{\pi e^{\pi x}}{e^{\pi x}+1+\pi x(e^{\pi x}+1)}$$

$$= \frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{8}$$

**ALITER**

$$\text{Given limit } = \frac{\pi}{4}\lim \frac{e^{\pi x}-1}{x(e^{\pi x}+1)}$$

$$= \frac{\pi}{4}\lim_{x\to 0}\left(\frac{e^{\pi x}-1}{\pi x}\right)\frac{\pi}{e^{\pi x}+1}$$

$$= \frac{\pi}{4}(1)\frac{\pi}{2} = \frac{\pi^2}{8}$$

**Answer: (D)**

**184.** $\lim_{x\to 0}(\cos x)^{\cot^2 x}$ is

(A) $e$ (B) $-e$ (C) $-\sqrt{e}$ (D) $\frac{1}{\sqrt{e}}$

**Solution:** $\lim_{x\to 0}(\cos x)^{\cot^2 x}$ is of the form $1^\infty$. Let $y = (\cos x)^{\cot^2 x}$. Then

$$\log_e y = \cot^2 x \log_e \cos x$$

$$= \frac{\log_e \cos x}{\tan^2 x}$$

So

$$\lim_{x\to 0}(\log_e y) = \lim_{x\to 0}\left(\frac{\log_e \cos x}{\tan^2 x}\right) \text{ is } \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0}\frac{-\tan x}{2\tan x \sec^2 x}$$

$$= -\frac{1}{2}$$

Therefore

$$\lim_{x\to 0} y = e^{-1/2} = \frac{1}{\sqrt{e}}$$

**Answer: (D)**

**185.** $\lim_{x\to 0}\left(\frac{1}{x^2} - \operatorname{cosec}^2 x\right)$ is equal to

(A) $\frac{1}{8}$ (B) $\frac{-1}{8}$ (C) $\frac{-1}{3}$ (D) $\frac{1}{3}$

**Solution:** The given limit is

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \operatorname{cosec}^2 x)\right) = \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$$

$$[(\infty - \infty) \text{ form}]$$

$$= \lim_{x\to 0}\frac{2(\sin^2 x - x^2)}{x^2(1-\cos 2x)} \quad \left(\frac{0}{0}\right) \text{form}$$

$$= 2\lim_{x\to 0}\frac{(\sin 2x - 2x)}{2x(1-\cos 2x)+2x^2\sin 2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{(2\cos 2x - 2)}{x(2\sin 2x)+(1-\cos 2x)+2x\sin 2x+2x^2\cos 2x} \quad \left(\frac{0}{0}\right)$$

$$= 2\lim_{x\to 0}\frac{\cos 2x - 1}{4x\sin 2x + 2x^2\cos 2x + 1 - \cos 2x} \quad \left(\frac{0}{0}\right)$$

$$= 2\lim_{x\to 0}\frac{-2\sin 2x}{4\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin 2x + 2\sin 2x}$$

$$= -4\lim_{x\to 0}\frac{\sin 2x}{6\sin 2x + 12x\cos 2x - 4x^2\sin 2x} \left(\frac{0}{0}\right)$$

$$= -4\lim_{x\to 0}\frac{2\cos 2x}{12\cos 2x + 12\cos 2x - 24x\sin 2x - 8x\sin 2x - 8x^2\cos 2x}$$

$$= \frac{-4(2)}{12+12-0-0-0} = \frac{-8}{24} = \frac{-1}{3}$$

**Answer: (C)**

**186.** Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f(1)=3$ and $f'(1)=6$. Then

$$\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x} =$$

(A) $e^2$ (B) $e^3$ (C) $e^4$ (D) $e$

**Solution:** The given limit is $(1^\infty)$ form. Let

$$y=\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$$

Therefore

$$\log_e y=\frac{\log_e f(1+x)-\log_e f(1)}{x}$$

So

$$\lim_{x\to 0}(\log_e y)=\lim_{x\to 0}\frac{\log_e f(1+x)-\log_e f(1)}{x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{f'(1+x)}{f(1+x)}$$

$$=\frac{f'(1)}{f(1)}=\frac{6}{3}=2$$

So $\lim_{x\to 0} y=e^2$.

**Answer: (A)**

**187.** If

$$\lim_{x\to 0}\frac{[(a-n)nx-\tan x]\sin nx}{x^2}=0$$

then $a$ is equal to

(A) 0 (B) $\dfrac{n+1}{n}$ (C) $n$ (D) $n+\dfrac{1}{n}$

**Solution:** Given that

$$\lim_{x\to 0}\frac{[(a-n)nx-\tan x]\sin nx}{x^2}=0$$

Left-hand side limit is of the form $\frac{0}{0}$. Therefore

$$\lim_{x\to 0}\frac{[(a-n)n-\sec^2 x]\sin nx+n[(a-n)nx-\tan x]\cos nx}{2x}=0$$

$$\lim_{x\to 0}\left[[(a-n)n-\sec x]\frac{\sin nx}{nx}(n)+n\left((a-n)n-\frac{\tan x}{x}\right)\cos nx\right]=0$$

$$[(a-n)n-1]n+n[(a-n)n-1]=0$$

$$(a-n)n-1=0$$

$$an=n^2+1$$

So

$$a=\frac{n^2+1}{n}=n+\frac{1}{n}$$

**Answer: (D)**

**188.** If $f(9)=9, f'(9)=4,$ then

$$\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}=$$

(A) 3 (B) 4 (C) 9 (D) 2

**Solution:** The given limit is $\left(\frac{0}{0}\right)$ form. Hence

$$\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}=\lim_{x\to 9}\left(\frac{\dfrac{f'(x)}{2\sqrt{f(x)}}}{\left(\dfrac{1}{2\sqrt{x}}\right)}\right)$$

$$=\lim_{x\to 9}\frac{\sqrt{x}f'(x)}{\sqrt{f(x)}}$$

$$=\frac{\sqrt{9}(4)}{\sqrt{9}}=4$$

**Answer: (B)**

**189.** $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$ is equal to

(A) 0 (B) 1 (C) –1 (D) $-\infty$

**Solution:** The given limit is of the form $(\infty-\infty)$. So

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)=\lim_{x\to 0}\frac{\sin x-x\cos x}{x\sin x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}$$

$$=\lim_{x\to 0}\frac{x\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}$$

$$=\frac{0+0}{2-0}=0$$

**Answer: (A)**

**190.** $\lim_{x\to 1}\sec\frac{\pi}{2x}\log x$ equals

(A) $\dfrac{1}{\pi}$ (B) $\pi$ (C) $\dfrac{2}{\pi}$ (D) $\dfrac{\pi}{2}$

**Solution:** We have

$$\lim_{x\to 1}\sec\frac{\pi}{2x}\log x = \lim_{x\to 1}\frac{\log x}{\cos\dfrac{\pi}{2x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\frac{\left(\dfrac{1}{x}\right)}{-\sin\dfrac{\pi}{2x}\left(-\dfrac{\pi}{2x^2}\right)}$$

$$= \frac{1}{(-1)\left(-\dfrac{\pi}{2}\right)} = \frac{2}{\pi}$$

**Answer: (C)**

**191.** $\lim_{x\to 0}\dfrac{2\sin x - \sin 2x}{\tan^3 x}$ is equal to

(A) 0 (B) 1 (C) 2 (D) –1

**Solution:** The given limit is $\left(\dfrac{0}{0}\right)$ form. So

$$\lim_{x\to 0}\frac{2\sin x - \sin 2x}{\tan^3 x} = \lim_{x\to 0}\frac{2\sin x(1-\cos x)\cos^3 x}{\sin^3 x}$$

$$= 2\lim_{x\to 0}\frac{(1-\cos x)\cos^3 x}{\sin^2 x}$$

$$= 2\lim_{x\to 0}\frac{\cos^3 x}{1+\cos x}$$

$$= 2\times\frac{1}{2} = 1$$

**Answer: (B)**

**Try it out** In the above problem, use of L'Hospital's rule is a lengthy process. Direct method is easy. Try it.

**192.** $\lim_{h\to 0}\dfrac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$ is

(A) 0 (B) 1 (C) –1 (D) $\dfrac{1}{2}$

**Solution:** Given limit is of the form $\left(\dfrac{0}{0}\right)$. Therefore

$$\lim_{h\to 0}\frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2} = \lim_{h\to 0}\left(\frac{\dfrac{2}{1+2h} - \dfrac{2}{1+h}}{2h}\right)$$

$$= \lim_{h\to 0}\frac{-h}{h(1+h)(1+2h)} = -1$$

**Answer: (C)**

**193.** $\lim_{h\to 0}\dfrac{e^x + e^{-x} - 2\cos x}{2x\sin x}$ is equal to

(A) 0 (B) 1 (C) $\dfrac{1}{2}$ (D) –1

**Solution:** Given limit is of the form $\left(\dfrac{0}{0}\right)$. Therefore

$$\lim_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{2x\sin x} = \lim_{x\to 0}\frac{e^x - e^{-x} + 2\sin x}{2\sin x + 2x\cos x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x} + 2\cos x}{2\cos x + 2\cos x - 2x\sin x}$$

$$= \frac{4}{4+0} = 1$$

**Answer: (B)**

**194.** $\lim_{x\to\frac{\pi}{4}}\dfrac{\sec^2 x - 2\tan x}{1+\cos 4x}$ is

(A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{1}{4}$ (D) 2

**Solution:** Given limit is of the form $\left(\dfrac{0}{0}\right)$. Therefore

$$\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x - 2\tan x}{1+\cos 4x} = \lim_{x\to\frac{\pi}{4}}\frac{2\sec^2 x\tan x - 2\sec^2 x}{-4\sin 4x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x\tan x - \sec^2 x}{-2\sin 4x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{2\sec^2 x\tan^2 x + \sec^4 x - 2\sec^2 x\tan x}{-8\cos 4x}$$

$$= \frac{4+4-4}{-8(-1)} = \frac{1}{2}$$

**Answer: (A)**

**195.** $\lim_{x\to 0+0}(\operatorname{cosec} x)^x$ is

(A) 1 (B) 0 (C) $\dfrac{1}{2}$ (D) 2

**Solution:** We have $y = (\operatorname{cosec} x)^x$, $x > 0$. Now

$$\log_e y = x\log_e(\operatorname{cosec} x)$$
$$= -x\log_e(\sin x)$$

$$= -\frac{\log_e(\sin x)}{1/x}$$

Therefore

$$\lim_{x\to 0+0} \log_e y = \lim_{x\to 0+0} -\frac{\log_e(\sin x)}{1/x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0+0} \frac{-\cot x}{-1/x^2}$$

$$= \lim_{x\to 0+0} \frac{x^2 \cos x}{\sin x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0+0} \left(\frac{2x\cos x - x^2 \sin x}{\cos x}\right)$$

$$= \frac{0}{1} = 0$$

Therefore $\lim_{x\to 0+0} y = e^0 = 1$.

**Answer: (A)**

**196.** $\lim_{x\to 0+0} \log_{\sin x}(\sin 2x)$ is equal to

(A) 2 (B) 1 (C) 0 (D) $\infty$

**Solution:** We have

$$\lim_{x\to 0+0} \log_{\sin x}(\sin 2x) = \lim_{x\to 0+0} \frac{\log_e \sin 2x}{\log_e \sin x}\left(\frac{-\infty}{-\infty}\right)$$

$$= \lim_{x\to 0+0} \frac{\left(\dfrac{2\cos 2x}{\sin 2x}\right)}{\left(\dfrac{\cos x}{\sin x}\right)}$$

$$= 2 \lim_{x\to 0+0} \left(\frac{\sin x \cos 2x}{\cos x \sin 2x}\right)$$

$$= \lim_{x\to 0+0} \left(\frac{\sin x}{x}\right)\left(\frac{2x}{\sin 2x}\right)\left(\frac{\cos 2x}{\cos x}\right)$$

$$= 1 \times 1 \times 1 = 1$$

**Answer: (B)**

**197.** $f$ is a real-valued differentiable and "$a$" is a real constant. Further $f'$ is continuous. If

$$F(x) = \begin{vmatrix} f(x+a) & f(x+2a) & f(x+3a) \\ f(a) & f(2a) & f(3a) \\ f'(a) & f'(2a) & f'(3a) \end{vmatrix}$$

then $\lim_{x\to 0} F(x)/x$ is

(A) 1 (B) –1 (C) 2 (D) 0

**Solution:** Since $f(x)$ is differentiable, $F(x)$ is also differentiable. Hence

$$F'(x) = \begin{vmatrix} f'(x+a) & f'(x+2a) & f'(x+3a) \\ f(a) & f(2a) & f(3a) \\ f'(a) & f'(2a) & f'(3a) \end{vmatrix}$$

and $F'(0) = 0$ ($\because$ Two rows are identical)

Now $\lim_{x\to 0} \dfrac{F(x)}{x}$ is $\left(\dfrac{0}{0}\right)$ form. Therefore

$$\lim_{x\to 0} \frac{F(x)}{x} = \lim_{x\to 0} \frac{F'(x)}{1} = F'(0) = 0$$

**Answer: (D)**

**198.** $\lim_{x\to\infty} \dfrac{\log_e\left(\dfrac{x+1}{x}\right)}{\log_e\left(\dfrac{x-1}{x}\right)}$ is

(A) 0 (B) 1 (C) –1 (D) $\infty$

**Solution:** We have

$$\lim_{x\to\infty} \frac{\log_e\left(\dfrac{x+1}{x}\right)}{\log_e\left(\dfrac{x-1}{x}\right)} = \lim_{x\to\infty} \frac{\log_e\left(1+\dfrac{1}{x}\right)}{\log_e\left(1-\dfrac{1}{x}\right)}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \left(\frac{\dfrac{1}{1+(1/x)}\left(-\dfrac{1}{x^2}\right)}{\dfrac{1}{1-(1/x)}\left(\dfrac{1}{x^2}\right)}\right)$$

$$= \lim_{x\to\infty} \left(\frac{-1}{1+(1/x)}\right) \div \left(\frac{1}{1-(1/x)}\right)$$

$$= -1$$

**Answer: (C)**

**199.** $\lim_{x\to\infty} \dfrac{e^x}{x^3}$ is

(A) 6 (B) 1 (C) 3 (D) $\infty$

**Solution:** The given limit is of $\left(\dfrac{\infty}{\infty}\right)$ form. Therefore

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{e^x}{6x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

**Answer: (D)**

## Multiple Correct Choice Type Questions

**1.** If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

(A) $a > 0, b > 0$  (B) $a > 0, b < 0$
(C) $a < 0, b > 0$  (D) $a < 0, b > 0$

**Solution:** Suppose $ax + by + c = 0$ is normal to the curve $xy = 1$ at $(x_1, y_1)$. Differentiating $xy = 1$ with respect $x$, we have

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-1}{x_1^2}$$

$(\because x_1y_1 = 1 \Rightarrow x_1 \neq 0$ and $y_1 \neq 0)$. Therefore equation of the normal at $(x_1, y_1)$ is

$$y - y_1 = x_1^2(x - x_1)$$
$$\Rightarrow x_1^2 x - y + y_1 - x_1^3 = 0$$

But $ax + by + c = 0$ is the normal at $(x_1, y_1)$. Therefore

$$\frac{a}{x_1^2} = \frac{b}{-1} = \frac{c}{y_1 - x_1^3}$$
$$\Rightarrow \frac{a}{b} = -x_1^2 < 0$$

So $a$ and $b$ must have opposite signs. This means

$$a > 0, b < 0 \quad \text{or} \quad a < 0, b > 0$$

Therefore, both (B) and (C) are correct.

**Answers: (B), (C)**

**2.** The tangent at a point $P_1$ (other than origin) on the curve $y = x^3$ meets the curve again at $P_2$. The tangent at $P_2$ meets the curve again at $P_3$ and so on. Then

(A) The abscissae of $P_1, P_2, \ldots, P_n$ are in AP
(B) The absscissae of $P_1, P_2, \ldots, P_n$ are in GP
(C) (Area of $\Delta P_1P_2P_3$):(Area of $P_2P_3P_4$) = 1:8
(D) (Area of $\Delta P_1P_2P_3$):(Area of $P_2P_3P_4$) = 1:16

**Solution:** Let $P_1$ be $(x_1, x_1^3)$. Now

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2$$

So the equation of the tangent at $P_1(x_1, x_1^3)$ is

$$y - x_1^3 = 3x_1^2(x - x_1)$$

Suppose this meets the curve at $P_2(x_2, x_2^3)$.

So

$$x_2^3 - x_1^3 = 3x_1^2(x_2 - x_1)$$

As $x_2 \neq x_1$, we have

$$x_2^2 + x_2x_1 + x_1^2 = 3x_1^2$$
$$\Rightarrow x_2^2 + x_2x_1 - 2x_1^2 = 0$$
$$\Rightarrow (x_2 + 2x_1)(x_2 - x_1) = 0$$

So

$$x_1 \neq x_2 \Rightarrow x_2 = -2x_1$$

Similarly, if the tangent $P_2(x_2, x_2^3)$ meets the curve in $P_3(x_3, x_3^3)$, then

$$x_3 = -2x_2 = -2(-2x_1) = 4x_1$$

Continuing this process, the abscissae of $P_1, P_2, \ldots, P_n$ are respectively

$$x_1, -2x_1, 4x_1, -8x, 16x_1, \ldots$$

which are in GP with the common ratio –2. Hence (B) is correct.

Area of $\Delta P_1P_2P_3$ is the absolute value of the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ x_2 & x_2^3 & 1 \\ x_3 & x_3^3 & 1 \end{vmatrix}$$

Now

$$\begin{vmatrix} x_1 & x_1^3 & 1 \\ x_2 & x_2^3 & 1 \\ x_3 & x_3^3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix}$$
$$= x_1^4 \begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix} \tag{3.57}$$

Again area of $\Delta P_2P_3P_4$ is the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_2 & x_2^3 & 1 \\ x_3 & x_3^3 & 1 \\ x_4 & x_4^3 & 1 \end{vmatrix}$$

Now

$$\begin{vmatrix} x_2 & x_2^3 & 1 \\ x_3 & x_3^3 & 1 \\ x_4 & x_4^3 & 1 \end{vmatrix} = \begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & 512x_1^3 & 1 \end{vmatrix}$$
$$= (-2)(-8)x_1^4 \begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix} \tag{3.58}$$

From Eqs. (3.57) and (3.58), we get

$$\frac{\text{Area of }\Delta P_1P_2P_3}{\text{Area of }\Delta P_2P_3P_4} = \frac{1}{16}$$

Thus (D) is correct.

**Answers: (B), (D)**

**3.** The abscissae of the points on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical are

(A) $\frac{4}{\sqrt{3}}$ (B) $\frac{11}{\sqrt{3}}$

(C) $\frac{-11}{\sqrt{3}}$ (D) $\frac{-4}{\sqrt{3}}$

**Solution:** Differentiating the curve equation w.r.t. $x$ we get

$$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\frac{dy}{dx}(y^2-4) = -2x$$

Therefore

$$\frac{dy}{dx} = \frac{-2x}{y^2-4}$$

Now

$$\text{Tangent is vertical} \Leftrightarrow y^2 = 4$$
$$\Leftrightarrow y = \pm 2$$

So

$$y = \pm 2 \Rightarrow (\pm 8) + 3x^2 = 12(\pm 2)$$

Therefore

$$x^2 = \frac{24-8}{3} = \frac{16}{3} \quad (\because y=-2 \Rightarrow x^2 < 0)$$

$$\Rightarrow x = \pm\frac{4}{\sqrt{3}}$$

Hence (A) and (D) are correct.

**Answers: (A), (D)**

**4.** The number of points on the curve $y = x^4 - 6x^3 + 13x^2 - 10x$ at which the tangents are parallel to the line $y = 2x$ and the number of points having the same tangent that is parallel to the line $y = 2x$ is

(A) 3 (B) 2

(C) 4 (D) 4

**Solution:** Differentiating $y = x^4 - 6x^3 + 13x^2 - 10x$ we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

Since, the slope of the line $y = 2x$ is 2, we have

$$4x^3 - 18x^2 + 26x - 10 = 2$$
$$\Rightarrow 2x^3 - 9x^2 + 13x - 6 = 0$$

Clearly $x = 1$ is a root. So

$$(x-1)(2x^2 - 7x + 6) = 0$$
$$\Rightarrow (x-1)(x-2)(2x-3) = 0$$

Therefore $x = 1$, $x = 2$, $x = 3/2$. So the points are (1, 3), (2, 5), (3/2, 15/6) at which the tangents are parallel to the line $y = 2x$ and the tangents at (1, 3) and (2, 5) are same and the common tangent is $y = 2x + 1$. Therefore, (A) and (B) are correct.

**Answers: (A), (B)**

**5.** The slopes of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2) are

(A) $2\sqrt{3}$ (B) $4\sqrt{3}$

(C) $-2\sqrt{3}$ (D) $-4\sqrt{3}$

**Solution:** Differentiating the given equation with respect to $x$ we get

$$2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y-2}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(h,k)} = \frac{3h^2}{k-2}$$

Now equation of the tangent at $(h, k)$ is

$$y - k = \frac{3h^2}{k-2}(x-h)$$

This tangent passes through the point (1, 2). This implies

$$2 - k = \frac{3h^2}{k-2}(1-h)$$

$$3h^3 - 3h^2 + 2k - 4 - k^2 + 2k = 0 \quad (3.59)$$

Also $(h, k)$ lies on the curve. This implies

$$k^2 - 2h^3 - 4k + 8 = 0 \quad (3.60)$$

Adding Eqs. (3.59) and (3.60), we have

$$h^3 - 3h^2 + 4 = 0$$
$$\Rightarrow (h+1)(h-2)^2 = 0$$
$$\Rightarrow h = -1, 2$$

Now $h = -1 \Rightarrow k$ is imaginary. So

$$h = 2 \quad \text{and} \quad k = 2 \pm 2\sqrt{3}$$

Hence the slopes are $\pm 2\sqrt{3}$.

**Answers: (A), (C)**

**6.** The line $y = x$ is a tangent to the curve $y = ax^2 + bx + c$ at the point $x = 1$. If the curve passes through the point $(-1, 0)$, then

(A) $a = -1/2$ (B) $b = 1/2$

(C) $c = 1/4$ (D) $a + c - b = 0$

**Solution:** Differentiating the given curve we have

$$\frac{dy}{dx} = 2ax + b$$

Since the line $y = x$ touches the curve, at the point $x = 1$, we have

$$1 = \frac{dy}{dx} = 2a + b$$

$$\Rightarrow 2a + b = 1 \quad (3.61)$$

Also $(1, 1)$ lies on the curve implies

$$a + b + c = 1 \quad (3.62)$$

Again, the curve passes through the point $(-1, 0)$ implies

$$a - b + c = 1 \quad (3.63)$$

Solving Eqs. (3.61)–(3.63), we get

$$a = \frac{1}{4}, b = \frac{1}{2}, c = \frac{1}{4}$$

Hence (B), (C) and (D) are correct.

**Answers: (B), (C), (D)**

**7.** The points on the curve $4x^2 + 9y^2 = 1$, at which the tangents are parallel to the line $8x = 9y$ are

(A) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (B) $\left(-\frac{2}{5}, \frac{1}{5}\right)$

(C) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (D) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

**Solution:** Differentiating the curve equation w.r.t. $x$, we have

$$8x + 18y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y} = \frac{8}{9}$$

so that $x = -2y$. Since the point lies on the curve, we have

$$4(-2y)^2 + 9y^2 = 1$$

$$\Rightarrow y = \pm\frac{1}{5}$$

Now

$$y = \pm\frac{1}{5} \Rightarrow x = \mp\frac{2}{5}$$

Therefore the points are $\left(-\frac{2}{5}, \frac{1}{5}\right)$ and $\left(\frac{2}{5}, -\frac{1}{5}\right)$.

**Answers: (B), (D)**

**8.** For the curve $x = 2\cos t + \cos 2t$, $y = 2\sin t - \sin 2t$ at any point $t$

(A) the tangent equation is $x\sin\frac{t}{2} + y\cos\frac{t}{2} = \sin\frac{3t}{2}$

(B) normal equation is $x\cos\frac{t}{2} - y\sin\frac{t}{2} = 3\cos\frac{3t}{2}$

(C) sub-tangent length $= \left|y\cot\frac{t}{2}\right|$

(D) sub-normal length $= \left|y\tan\frac{t}{2}\right|$

**Solution:** Differentiating both the equations we get

$$\frac{dx}{dt} = -2\sin t - 2\sin 2t$$

$$\frac{dy}{dt} = 2\cos t - 2\cos 2t$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt}\\ &= \frac{\cos t - \cos 2t}{-(\sin t + \sin 2t)}\\ &= \frac{\cos 2t - \cos t}{\sin 2t + \sin t}\\ &= \frac{-2\sin\frac{3t}{2}\sin\frac{t}{2}}{2\sin\frac{3t}{2}\cos\frac{t}{2}}\\ &= -\tan\frac{t}{2}\end{aligned}$$

The tangent equation is

$$y - (2\sin t - \sin 2t) = \frac{-\sin\frac{t}{2}}{\cos\frac{t}{2}}(x - 2\cos t - \cos 2t)$$

That is

$$\begin{aligned}x\sin\frac{t}{2} + y\cos\frac{t}{2} &= 2\sin\left(t + \frac{t}{2}\right) - \sin\left(2t - \frac{t}{2}\right)\\ &= \sin\frac{3t}{2}\end{aligned}$$

Thus, (A) is correct. Now the equation of the normal is

$$y-(2\sin t-\sin 2t)=\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}(x-2\cos t-\cos 2t)$$

That is

$$x\cos\frac{t}{2}-y\sin\frac{t}{2}=\left[+2\cos\left(t+\frac{t}{2}\right)+\cos\left(2t-\frac{t}{2}\right)\right]$$
$$=3\cos\frac{3t}{2}$$

Hence (B) is true. By definition,

$$\text{Sub-tangent}=\left|y\frac{dx}{dy}\right|$$
$$=\left|y\left(-\cot\frac{t}{2}\right)\right|$$
$$=\left|y\left(\cot\frac{t}{2}\right)\right|$$

So (C) is correct. Finally, the

$$\text{Sub-normal}=\left|y\frac{dy}{dx}\right|=\left|y\tan\frac{t}{2}\right|$$

Hence (D) is correct.

**Answers: (A), (B), (C), (D)**

**9.** The equations of the normals to the curve $y=\dfrac{x}{1-x^2}$ at the points where the tangents make angles of $\pi/4$ with the positive direction of $x$-axis are

(A) $x+y=0$ (B) $x+y=\dfrac{\sqrt{3}}{2}$

(C) $x+y=2\sqrt{2}$ (D) $x+y=\dfrac{-\sqrt{3}}{2}$

**Solution:** Since the slopes of the tangents is 1, we have

$$\frac{1+x^2}{(1-x^2)^2}=\frac{dy}{dx}=1$$
$$\Rightarrow x^4-3x^2=0$$
$$\Rightarrow x=0,\pm\sqrt{3}$$

Now,

$$x=0\Rightarrow y=0$$
$$x=\sqrt{3}\Rightarrow y=-\frac{\sqrt{3}}{2}$$
$$x=-\sqrt{3}\Rightarrow y=\frac{\sqrt{3}}{2}$$

Thus, the points on the curve at which the tangents make angles of $\pi/4$ with the positive direction of $x$-axis are $(0, 0)$, $(\sqrt{3},-\sqrt{3}/2)$ and $(-\sqrt{3},\sqrt{3}/2)$.

**(i)** Normal at $(0, 0)$: $y-0=-1(x-0)$. That is $x+y=0$. Hence (A) is true.

**(ii)** Normal at $(\sqrt{3},-\sqrt{3}/2)$ is

$$y+\frac{\sqrt{3}}{2}=-1(x-\sqrt{3})$$

That is

$$x+y=\sqrt{3}-\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}$$

So (B) is true

**(iii)** Normal at $(-\sqrt{3},\sqrt{3}/2)$ is

$$y-\frac{\sqrt{3}}{2}=-1(x+\sqrt{3})$$

That is

$$x+y=\frac{\sqrt{3}}{2}-\sqrt{3}=-\frac{\sqrt{3}}{2}$$

So (D) is true.

**Answers: (A), (B), (D)**

**10.** For the curve $y=\dfrac{x}{1-x^2}$, let $m$ be the number of points on the curve at which the tangent is parallel to $x$-axis and $n$ be the number of points at which the tangent is vertical. Then

(A) $m=0$ (B) $n=0$

(C) $m=1, n=1$ (D) $m=0, n=2$

**Solution:** The given curve is defined for $x\neq\pm1$. Now

$$\frac{dy}{dx}=\frac{1+x^2}{(1-x^2)^2}$$
$$\Rightarrow m=0\quad\text{and}\quad n=0$$

**Answers: (A), (B)**

**11.** Consider the following two statements:

**I:** For the curve $x^m y^n=a^{m+n}$, the sub-tangent at any point varies at the abscissa of point.

**II:** The equation of the common tangent to the curves

$$y=x^3-3x^2-8x-4$$
and
$$y=3x^2+7x+4$$

at their point of contact is $x-y+1=0$.

Then

(A) I is true (B) II is true

(C) I is false (D) II is false

**Solution:**

**I:** We have $x^m y^n = a^{m+n}$. Therefore

$$mx^{m-1}y^n + x^m(ny^{n-1})\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{my}{nx}$$

The sub-tangent is

$$\left|y\frac{dx}{dy}\right| = \left|y\left(-\frac{nx}{my}\right)\right| = \left|\frac{n}{m}x\right|$$

Therefore the sub-tangent varies as the abscissa of the point. Thus I is true.

**II:** At a common point,

$$x^3 - 3x^2 - 8x - 4 = 3x^2 + 7x + 4$$
$$\Rightarrow x^3 - 6x^2 - 15x - 8 = 0$$
$$\Rightarrow (x+1)(x^2 - 7x - 8) = 0$$
$$\Rightarrow (x+1)(x+1)(x-8) = 0$$

So $x = -1$ is a repeated root. Therefore at the point $(-1, 0)$, the two curves must touch each other. Also the value of $dy/dx$ at $(-1, 0)$ for the two curves is same (is equal to 1). Therefore the common tangent at $(-1, 0)$ is

$$y - 0 = 1(x+1)$$
$$\Rightarrow x - y + 1 = 0$$

Thus (B) is also true.

**Answers: (A), (B)**

**12.** The angles of intersection of the curves $x^2 = 4ay$ and $2y^2 = ax$

(A) $\frac{\pi}{2}$ (B) $\text{Tan}^{-1}\left(\frac{2}{5}\right)$

(C) $\frac{\pi}{4}$ (D) $\text{Tan}^{-1}\left(\frac{3}{5}\right)$

**Solution:** Solving the equations $x^2 = 4ay$ and $2y^2 = ax$, we have $x = 0, 2a$. Therefore the points of intersection are $(0, 0)$ and $(2a, a)$. Now

$$x^2 = 4ay \Rightarrow \frac{dy}{dx} = \frac{x}{2a} \quad (3.64)$$

$$2y^2 = ax \Rightarrow \frac{dy}{dx} = \frac{a}{4y} \quad (3.65)$$

From Eqs. (3.64) and (3.65), it is easy to see that the angle of intersection of the two curves at $(0, 0)$ is $\pi/2$. Thus (A) is true.

Again, from Eq. (3.64),

$$\left(\frac{dy}{dx}\right)_{(2a,a)} = 1$$

and from Eq. (3.65),

$$\left(\frac{dy}{dx}\right)_{(2a,a)} = \frac{a}{4a} = \frac{1}{4}$$

Therefore if $\theta$ is the acute angle of intersection of the curves at $(2a, a)$, then

$$\tan\theta = \left|\frac{1-(1/4)}{1+1(1/4)}\right| = \frac{3}{5}$$

Therefore

$$\theta = \text{Tan}^{-1}\left(\frac{3}{5}\right)$$

Hence (D) is true.

**Answers: (A), (D)**

**13.** Tangent to the curve $x^2 + y^2 - 2x - 3 = 0$ is parallel to the $x$-axis at the points:

(A) $(\pm 3, 0)$ (B) $(\pm 1, 2)$

(C) $(1, 2)$ (D) $(1, -2)$

**Solution:** Differentiating the given equation with respect to $x$ we get

$$x + y\frac{dy}{dx} - 1 = 0$$

Now

$$\frac{dy}{dx} = 0 \Rightarrow \frac{1-x}{y} = 0$$
$$\Rightarrow x = 1$$

So

$$x = 1 \Rightarrow y^2 = 4$$

Therefore the points are $(1, 2)$ and $(1, -2)$.

**Answers: (C), (D)**

**14.** If the normals to the curve $y^2 = 4ax$ at the points $P(at_2^2, 2at_2)$ and $Q(at_3^2, 2at_3)$ intersect on the curve at $R(at_1^2, 2at_1)$, then

(A) $t_1 = -t_2 - \frac{2}{t_2}$

(B) $t_1 + t_3 = -\frac{2}{t_3}$

(C) $t_2 t_3 = 2$

(D) product of the ordinates of $P$ and $Q$ is $8a^2$

**Solution:** We have

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

So

$$\left(\frac{dy}{dx}\right)_P = \frac{2a}{2at_2} = \frac{1}{t_2}$$

The normal at $P$ is

$$y - 2at_2 = -t_2(x - at_2^2)$$
$$\Rightarrow t_2 x + y = 2at_2 + at_2^3$$

This passes through the point $R(at_1^2, 2at_1)$. Therefore,

$$t_2(at_1^2) + 2at_1 = 2at_2 + at_2^3$$
$$t_2(t_1^2 - t_2^2) = 2(t_2 - t_1)$$

Since $t_2 \neq t_1$, we have

$$t_2(t_1 + t_2) = -2$$
$$\Rightarrow t_1 = -t_2 - \frac{2}{t_2}$$

Thus (A) is true. Similarly

$$t_1 = -t_3 - \frac{2}{t_3} \Rightarrow t_1 + t_3 = -\frac{2}{t_3}$$

Thus (B) is also true. Now

$$-t_2 - \frac{2}{t_2} = t_1 = -t_3 - \frac{2}{t_3}$$
$$\Rightarrow t_3 - t_2 = \frac{2}{t_2} - \frac{2}{t_3} = \frac{2(t_3 - t_2)}{t_2 t_3}$$
$$\Rightarrow t_2 t_3 = 2$$

Hence (C) is also true. Now, the product of the ordinates of $P$ and $Q$ is

$$(2at_2)(2at_3) = 4a^2 t_2 t_3 = 4a^2(2) = 8a^2$$

Thus (D) is true.

**Answers: (A), (B), (C), (D)**

**15.** Consider the curve $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$.

(A) The distance of the normal at $\theta$ from the origin is 1

(B) Equation of the tangent at $\theta$ is $x\sin\theta - y\cos\theta = 1$

(C) Length of the sub-normal at $\theta = \frac{\pi}{4}$ is $\frac{4-\pi}{4\sqrt{2}}$

(D) Length of the sub-tangent at $\theta = \frac{\pi}{4}$ is $\frac{4-\pi}{4\sqrt{2}}$

**Solution:** From the given equations, we have

$$\frac{dx}{d\theta} = \theta\cos\theta$$

and $$\frac{dy}{d\theta} = \theta\sin\theta$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\sin\theta}{\cos\theta}$$

Equation of the normal at $\theta$ is

$$y - (\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - \cos\theta - \theta\sin\theta)$$

That is,

$$x\cos\theta + y\sin\theta = \sin^2\theta - \theta\sin\theta\cos\theta + \cos^2\theta + \theta\sin\theta\cos\theta$$
$$\Rightarrow x\cos\theta + y\sin\theta = 1$$

Therefore the distance of the normal from the origin 1. Thus (A) is correct. Equation of the tangent at $\theta$ is

$$y - (\sin\theta - \theta\cos\theta) = \frac{\sin\theta}{\cos\theta}(x - \cos\theta - \theta\sin\theta)$$

That is,

$$x\sin\theta - y\cos\theta = \sin\theta\cos\theta + \theta\sin^2\theta - \sin\theta\cos\theta + \theta\cos^2\theta$$
$$\Rightarrow x\sin\theta - y\cos\theta = \theta$$

Hence (B) is not correct. Now

$$\theta = \frac{\pi}{4} \Rightarrow y = \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}$$

and $$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{4}} = 1$$

Therefore sub-normal at $\theta = \pi/4$ is

$$\left|\frac{4-\pi}{4\sqrt{2}} \times 1\right| = \frac{4-\pi}{4\sqrt{2}}$$

and sub-tangent at $\theta = \pi/4$ is

$$\left|\frac{4-\pi}{4\sqrt{2}}(+1)\right| = \frac{4-\pi}{4\sqrt{2}}$$

So (C) and (D) are correct.

**Answers: (A), (C), (D)**

**16.** $f(x)$ is a cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minimum at $x = 0$. Then

(A) the distance between $(-1,2)$ and $(a, f(a))$, where $x = a$ is the point of local minima, is $2\sqrt{5}$

(B) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$

(C) $f(x)$ has local minima at $x = 1$

(D) the value of $f(0) = 15$

**Solution:** Let

$$f(x) = ax^3 + bx^2 + cx + d$$

Then

$$f(1) = -1 \Rightarrow a + b + c + d = -1 \quad (3.66)$$

$$f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18 \quad (3.67)$$

$f(x)$ has local minima at $x = -1$ implies

$$f'(-1) = 0$$

$$\Rightarrow 3a - 2b + c = 0 \quad (3.68)$$

$f'(x)$ has local minima at $x = 0$ implies

$$f''(0) = 0 \Rightarrow b = 0 \quad (3.69)$$

From Eqs. (3.66), (3.67) and (3.69), we get

$$a + c + d = -1 \quad (3.70)$$

and

$$8a + 2c + d = -18 \quad (3.71)$$

From Eqs. (3.70) and (3.71),

$$7a + c = 19 \quad (3.72)$$

Again from Eqs. (3.68) and (3.69),

$$3a + c = 0 \quad (3.73)$$

From Eqs. (3.72) and (3.73), we get

$$a = \frac{19}{4}, c = -\frac{57}{4}$$

Substituting the values of $a$ and $c$ in Eq. (3.70) we get

$$d = -1 - a - c = -1 - \frac{19}{4} + \frac{57}{4} = \frac{37}{4}$$

Therefore

$$a = \frac{19}{4}, b = 0, c = -\frac{57}{4}, d = \frac{34}{4}$$

or

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34)$$

Now $a = 1$ is a point of local minima so that

$$(a, f(a)) = (1, -1) \quad [\because f(1) = -1]$$

Therefore distance between $(-1, 2)$ and $(1, -1)$ is

$$\sqrt{2^2 + 3^2} = \sqrt{13}$$

So (A) is not correct. Again

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34)$$

$$\Rightarrow f'(x) = \frac{1}{4}(57x^2 - 57)$$

Now

$$f'(x) = 0 \Rightarrow x = \pm 1.$$

Therefore, the critical points of $f$ are $\pm 1$. Now

$$f'(x) = \frac{57}{4}(x^2 - 1) = \frac{57}{4}(x+1)(x-1)$$

so that $f'(x) > 0$ for $x < -1$, $f'(x) < 0$ for $-1 < x < 1$ and $f'(x) > 0$ for $x > 1$. That is $f$ increases in $(-\infty, -1)$, decreases in $(-1, 1)$ and again increases in $(1, \infty)$. So $f(x)$ has local maxima at $x = -1$ and local minima at $x = 1$. Also, $f(x)$ increases in $(1, \infty) \Rightarrow f$ is increasing in $[1, 2\sqrt{5}]$. Thus (B) is correct. $f(x)$ has local minima at $x = 1$ so that (C) is true. Now

$$f(0) = \frac{34}{4} \neq 15$$

Hence (D) is not true.

**Answers: (B), (C)**

**17.** For the function $f(x) = x\cos\frac{1}{x}, x \geq 1$,

(A) for at least one $x$ in $[1, \infty)$, $f(x+2) - f(x) < 2$

(B) $\lim_{x \to \infty} f'(x) = 1$

(C) for all $x$ in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

**Solution:** For $x \neq 0$,

$$f'(x) = \cos\frac{1}{x} + x\sin\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right)$$

$$= \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}$$

Since $\lim_{x \to \infty} (1/x) = 0$, we have

$$\lim_{x \to \infty} f'(x) = \cos 0 + 0 = 1$$

because as $x \to \infty$, $1/x \to 0$ and $\sin(1/x)$ is a bounded function. So,

$$\lim_{x \to \infty} \frac{1}{x}\sin\frac{1}{x} = 0$$

Thus, (B) is correct. Now, for $x \geq 1$,

$$f''(x) = \frac{1}{x^2}\sin\frac{1}{x} - \frac{1}{x^2}\sin\frac{1}{x} - \frac{1}{x^3}\cos\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x^3}\cos\frac{1}{x} < 0$$

Therefore, $f'(x)$ is strictly decreasing in $[1, \infty)$. So (D) is correct.

Using Lagrange's mean value theorem for $f(x)$ on the interval $[x, x+2]$ $(x \geq 1)$, there exists $c \in (x, x+2)$ such that

$$\frac{f(x+2)-f(x)}{(x+2)-x} = f'(c)$$

$$f(x+2)-f(x) = 2f'(c) \qquad (3.74)$$

Now $\lim_{x\to\infty} f'(x) = 1$ [(B) is correct] and $f'(x)$ is strictly decreasing for $x \geq 1$ [(D) is correct]. We have that $f'(c) > 1$. Hence from Eq. (3.74),

$$f(x+2)-f(x) > 2$$

for all $x \geq 1$. Therefore (C) is true.

**Answers: (B), (C), (D)**

**18.** Let $f(x)$ be a differentiable function and for every real $x$,

$$h(x) = f(x) - (f(x))^2 + (f(x))^3$$

Then

(A) $h$ is increasing whenever $f$ is increasing
(B) $h$ is increasing whenever $f$ is decreasing
(C) $h$ is decreasing whenever $f$ is decreasing
(D) $h$ is decreasing whenever $f$ is increasing

**Solution:** Since $f$ is differentiable, $h$ is also differentiable. Also

$$h'(x) = f'(x)[1-2f(x)+3(f(x))^2]$$

$$= 3f'(x)\left[\left(f(x)-\frac{1}{3}\right)^2 + \frac{1}{9}\right]$$

Now

$$h'(x) \geq 0, \quad \text{if } f'(x) \geq 0$$

and $$h'(x) \leq 0, \quad \text{if } f'(x) \leq 0$$

So $h$ is increasing or decreasing according as $f$ is increasing or decreasing.

**Answers: (A), (C)**

**19.** Which of the following curves cut the curve $y^2 = 4ax$ orthogonally?

(A) $x^2 + y^2 = a^2$ (B) $y = e^{-x/2a}$
(C) $y = ax$ (D) $x^2 = 4ay$

**Solution:**

(A) Put $y^2 = 4ax$ in $x^2 + y^2 = a^2$ so that, we have

$$x^2 + 4ax - a^2 = 0$$
$$\Rightarrow (x+2a)^2 - 5a^2 = 0$$
$$\Rightarrow x = -2a \pm \sqrt{5}a = a(-2 \pm \sqrt{5})$$

As $y^2 \geq 0$, $x$ cannot be $a(-2-\sqrt{5})$. Therefore $x = a(-2+\sqrt{5})$ which implies

$$y = \pm 2a(-2+\sqrt{5})$$

At the points $(a(-2+\sqrt{5}), \pm 2a - 2 + \sqrt{5})$, we can see the two curves cannot intersect orthogonally. Therefore (A) is not true.

(B) We have

$$y = e^{-x/2a} \Rightarrow \frac{dy}{dx} = -\frac{1}{2a}e^{-x/2a} = -\frac{y}{2a}$$

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Therefore

$$\left(-\frac{y}{2a}\right)\left(\frac{2a}{y}\right) = -1$$

implies that the two curves $y = e^{-x/2a}$ and $y^2 = 4ax$ intersect orthogonally. So (B) is correct.

(C) We have

$$y = ax \Rightarrow \frac{dy}{dx} = a$$

and $$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Further $a(2a/y) \neq -1$, because the points of intersection are $(0, 0)$ and $(4/a, 4)$. Hence (C) is not true.

(D) We have

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

and $$x^2 = 4ay \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$$

Further, $y = 0$ (i.e., $x$-axis) is a tangent to $x^2 = 4ay$ at $(0, 0)$ and $x = 0$ (i.e., $y$-axis) is tangent to $y^2 = 4ax$ at, $(0, 0)$. Hence, at $(0, 0)$, the two curves $y^2 = 4ax$ and $x^2 = 4ay$ intersect orthogonally. Therefore (D) is correct.

**Answers: (B), (D)**

**20.** If $f$ is a function, whose derivative is

$$f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$$

then $f$ has minimum value at $x$ is equal to

(A) 0 (B) 1
(C) 2 (D) 3

**Solution:** We have

$$f'(x)=0 \Rightarrow x=0, 1, 2 \text{ and } 3$$

First, observe $e^x \geq 1$ for $x \geq 0$ and $e^x$ is strictly increasing on $\mathbb{R}$.

**(i)** At $x=0$: $x<0 \Rightarrow$ the sign of $f'(x)$ is $(-)(-)(-)(-)(-) < 0$.
$x>0 \Rightarrow$ the sign of $f'(x)$ is $(+)(+)(-)(-)(-) < 0$.
Therefore, $f$ has no extremum value at $x = 0$. Thus, (A) is not correct.

**(ii)** At $x = 1$: $x < 1 \Rightarrow$ sign of $f'(x)$ is $(+)(+)(-)(-)(-) < 0$.
$x>1 \Rightarrow$ sign of $f'(x)$ is $(+)(+)(+)(-)(-) > 0$.
Therefore, $f$ has minimum value at $x = 1$. Thus, (B) is true.
For a continuous function, minimum and maximum values occur alternately. So $f$ has maximum at $x = 2$ and minimum at $x = 3$. Therefore, (C) is not true whereas (D) is correct.

**Answers: (B), (D)**

**21.** Let $f(x)=2+(x-1)^{2/3}$ and $g(x)=x^{2/3}$. Then

(A) Rolle's theorem is applicable for $f(x)$ on $[0, 2]$ and Lagrange's mean value theorem is applicable for $g(x)$ on $[-1, 1]$

(B) Rolle's theorem is not applicable for $f(x)$ on $[0,2]$

(C) Lagrange's mean value theorem is applicable to $g(x)$ on $[-1, 1]$

(D) Rolle's theorem is not applicable to $f(x)$ on $[0,2]$ and Lagrange's mean value theorem is not applicable to $g(x)$ on $[-1, 1]$

**Solution:** The function $f(x)$ is continuous $\forall\, x \in [0,2]$ and

$$f'(x)=\frac{2}{3}(x-1)^{-1/3}$$

is not defined at $x = 1$. Thus $f$ is not differentiable at $x=1 \in (0,2)$. Therefore (B) is correct. Now

$$g'(x)=\frac{2}{3}x^{-1/3}$$

is not defined at $x=0 \in (-1,1)$. Thus $g$ is not differentiable at $x=0$. Hence (D) is true.

**Answers: (B), (D)**

**22.** Let $f(x)=\begin{cases} \dfrac{3-x^2}{2} & \text{for } 0 \leq x \leq 1 \\ \dfrac{1}{x} & \text{for } 1 \leq x \leq 2 \end{cases}$

Then, the value of $c$ in the Lagrange's mean value theorem over $[0, 2]$ is

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$

(C) $\dfrac{3}{2}$ (D) $\sqrt{2}$

**Solution:** We have

$$\lim_{x \to 1-0} f(x)=\frac{3-1}{2}=1$$

and
$$\lim_{x \to 1+0} f(x)=\frac{1}{1}=1$$

Therefore $f$ is continuous at $x = 1$. Using Lagrange's mean value theorem for $f(x)$ on $[0, 1]$, there exists $c \in (0, 1)$ such that

$$\frac{f(1)-f(0)}{1-0}=-\frac{2c}{2}$$
$$\Rightarrow 1-\frac{3}{2}=-c$$
$$\Rightarrow c=\frac{1}{2}$$

Therefore (A) is correct.

Again, using Lagrange's mean value theorem for $f(x)$ on $[1, 2]$, there exists $c \in (1, 2)$ such that

$$\frac{f(2)-f(1)}{2-1}=-\frac{1}{c^2}$$
$$\Rightarrow \frac{1}{2}-1=-\frac{1}{c^2}$$
$$\Rightarrow c^2=2$$
$$\Rightarrow c=\sqrt{2}$$

Therefore (D) is correct.

**Answers: (A), (D)**

**22.** Suppose $f$ is differentiable for all real values of $x$. Then by the Lagrange's mean value theorem, there exists $\theta \in (0, 1)$ such that

$$f(x+h)=f(x)+hf'(x+\theta h)$$

**This statement can be considered as Lagrange's Mean Value Theorem**. Then

(A) the value of $\theta$ in Lagrange's mean value theorem for $f(x)=x^2$ is 1/2

(B) the value of $\theta$ for $f(x)=x^2+ax+b$ is 1/2

(C) the value of $\theta$ for $g(x)=x^3$ is 1/3

(D) the value of $\theta$ for $g(x)=x^3+bx^2+cx+d$ is 1/3

**Solution:** We consider $f(x)=ax^2+bx+c$. Therefore

$$f(x+h)=f(x)+f'(x+\theta h) \quad (0<\theta<1)$$

We have

$$a(x+h)^2+b(x+h)+c=(ax^2+bx+c)+[2a(x+\theta h)+b]h$$
$$\Rightarrow 2axh+ah^2+bh=2axh+2a\theta h^2+bh$$
$$\Rightarrow ah^2=2a\theta h^2$$
$$\Rightarrow 2\theta=1 \quad \text{or} \quad \theta=\frac{1}{2}$$

Thus (A) and (B) are correct. Consider $g(x) = x^3$. Therefore

$$g(x+h)=g(x)+hg'(x+\theta h)$$
$$\Rightarrow (x+h)^3=x^3+h[3(x+\theta h)^2]$$
$$\Rightarrow 3x^2h+3xh^2+h^3=3x^2h+6x\theta h^2+\theta^2h^3$$
$$\Rightarrow 3x+h=6x\theta+\theta^2 h$$

so that $\theta \neq 1/3$. Hence (C) and (D) are not correct.

**Answers: (A), (B)**

**Try it out** If

$$f(x+h)=f(x)+\frac{h}{\lfloor 1}f'(x)+\frac{h^2}{\lfloor 2}f''(x+\theta h)$$

where $0<\theta<1$, then for

$$f(x)=ax^3+bx^2+cx+d$$

and $a\neq 0$, the value of $\theta$ is 1/3.

**23.** Consider the function $f(x)=4x^3-12x,\ x\in[-1,3]$. Then

(A) $f$ has local maximum at $x=-1$

(B) $f$ has local minimum at $x=1$

(C) the image of the interval $[-1,3]$ under the function is $[-8,72]$

(D) $f$ has no extremum value in $[-1,3]$

**Solution:** Differentiating the given function we get

$$f'(x)=12x^2-12$$

Now

$$f'(x)=0 \Leftrightarrow x=-1,1$$

So the critical points of $f$ are $\pm1$. Since $f''(x)=24x$, we have $f''(-1)<0$ and $f''(1)>0$. Hence $f$ has local maximum at $x=-1$ and local minimum at $x=1$. Thus (A) and (B) are correct. Also,

$$f(-1)=-4+12=8$$
$$f(1)=4-12=-8$$

and $$f(3)=108-36=72$$

The least and greatest value of $f$ on $[-1,3]$ are $-8$ and 72, respectively. Therefore, the image of $[-1, 3]$ under $f$ is $[-8, 72]$. So(C) is correct.

**Answers: (A), (B), (C)**

**24.** On the interval $[0,3]$, the function $f(x)=4x^3-x|x-2|$

(A) has local minimum at $x=1/3$

(B) has greatest value at $x=3$

(C) $f$ is increasing in $[1,2]$

(D) $f$ is decreasing in $[0,1/4]$

**Solution:** We have

$$f(x)=\begin{cases}4x^3+x(x-2) & \text{for } 0\le x\le 2\\ 4x^3-x(x-2) & \text{for } 2<x\le 3\end{cases}$$

Now,

$$f'(x)=\begin{cases}12x^2+2x-2 & \text{for } 0\le x\le 2\\ 12x^2-2x+2 & \text{for } 2<x\le 3\end{cases}$$

So

$$f'(2-0)=48+4-2=50$$

and $$f'(2+0)=48-4+2=46$$

Therefore, $f$ is not differentiable at $x = 2$. Hence 2 is a critical point. Now $0\le x\le 2$ and $f'(x)=0$ implies

$$6x^2+x-1=0$$
$$\Rightarrow (3x-1)(2x+1)=0$$
$$\Rightarrow x=1/3,-1$$

So $x=1/3$ is a critical point. Now $2\le x\le 3$ and $f'(x)=0$ implies

$$6x^2-x+1=0$$

But $6x^2-x+1=0$ has no real roots. Thus, the only critical points are $1/3, 2$.

**(i)** Also, $x < 1/3 \Rightarrow$ the sign of $f'(x)$ is $(-)(+)<0$ and $x>1/3 \Rightarrow f'(x)>0$. Hence $f$ has local minimum at $x = 1/3$.

**(ii)** $f$ is decreasing for $x < 1/3$ and increasing for $x > 1/3$. Further,

$$f(0)=0,\ f\left(\frac{1}{3}\right)=\frac{4}{27}-\frac{1}{3}\times\frac{5}{3}=\frac{4-15}{27}=-\frac{11}{27}$$
$$f(3)=108-3=105$$

On $[0, 3]$, the least value of $f$ is $-11/27$ and greatest value is 105. From (ii), we have $f$ is decreasing in $[0, 1/4]$ and increasing in $[1, 2]$. Hence all options are correct.

**Answers: (A), (B), (C), (D)**

**25.** $P(a\cos\theta, b\sin\theta)$ is a point on the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $a > 0, b > 0$ and $b^2 < a^2$. Then

(A) equation of the tangent $P$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

(B) minimum area of the triangle formed by the tangent at $P$ and the two coordinate axes is $ab$

(C) minimum area of the triangle in part (B) is $2ab$

(D) equation of the normal at $P$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

**Solution:** Differentiating the curve equation with respect to $x$ we get

$$\frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_P=-\frac{b^2(a\cos\theta)}{a^2(b\sin\theta)}=-\frac{b}{a}\cot\theta$$

So equation of the tangent at $P$ is

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$\Rightarrow bx\cos\theta+ay\sin\theta=ab(\sin^2\theta+\cos^2\theta)=ab$$

Dividing by $ab$ we get

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Thus (A) is true. Again, equation of the normal at $P$ is

$$y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$$

$$\Rightarrow \frac{x}{b\cos\theta}-\frac{y}{a\sin\theta}=\frac{a}{b}-\frac{b}{a}=\frac{a^2-b^2}{ab}$$

$$\Rightarrow \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

Thus (D) is correct.

If the intercepts made by the tangent $(x/a)\cos\theta + (y/b)\sin\theta = 1$ on the coordinate axes are $a\sec\theta$ and $b\operatorname{cosec}\theta$, respectively, then the area of the triangle thus formed is

$$\frac{1}{2}|(a\sec\theta)(b\operatorname{cosec}\theta)|=ab|\operatorname{cosec}2\theta|\ge ab$$

because $|\operatorname{cosec}2\theta|\ge 1$. Thus the minimum area of the triangle thus formed is $ab$. Hence (B) is correct.

**Answers: (A), (B), (D)**

**26.** Consider the function

$$f(x)=\left|\frac{1+x}{1-x}\right|$$

on the interval $[-2, 0]$. Then

(A) $f$ is decreasing in the interval $[-2, -1]$

(B) $f$ is increasing in $[-1, 0]$

(C) the minimum value of $f$ is 0

(D) the maximum value of $f$ is 1

**Solution:** We have

$$f(x)=\begin{cases}\dfrac{x+1}{x-1} & \text{if } -2\le x\le -1\\[2mm] \dfrac{x+1}{1-x} & \text{if } -1<x\le 0\end{cases}$$

$f$ is continuous at $x = -1$. Now

$$f'(x)=\begin{cases}-\dfrac{2}{(x-1)^2} & \text{for } -2\le x\le -1\\[2mm] \dfrac{2}{(x-1)^2} & \text{for } -1<x\le 0\end{cases}$$

Also,

$$f'(-1-0)=-\frac{1}{2}$$

and $$f'(-1+0)=\frac{1}{2}$$

so that $f'$ is **not** differentiable at $x = -1$ and hence $x = -1$ is the only critical point and $f'(x)\ne 0$ for $x\in[-2, 0]$.

Since $f'(x) < 0$ for $-2 \le x \le -1$ and $f'(x) > 0$ for $-1 \le x \le 0$, follows that $f$ is decreasing in $[-2, -1]$ and increasing in $[-1, 0]$, hence $f$ is minimum at $x = -1$ and the minimum value is $f(-1)$ which is 0. Now

$$f(-2)=\frac{1}{3},\ f(-1)=0 \text{ and } f(0)=1$$

This implies that the minimum value is 0 and maximum value is 1. Hence all options are correct.

**Answers: (A), (B), (C), (D)**

**27.** The sum of the third and ninth terms of an AP is equal to the least value of the quadratic expression $2x^2 - 4x + 10$. Then

(A) the common difference of the AP is $\dfrac{4-a}{5}$ where "$a$" is the first term of the AP.

(B) if the first term $a$ is $-1$, then the common difference is 1

(C) the common difference $d$ cannot be determined unless the first term is given

(D) sum of the first eleven terms is 44

**Answers: (A), (B), (C), (D)**

**Solution:** Let $a$ be the first term and $d$ be the common difference of the A.P. Let

$$f(x) = 2x^2 - 4x + 10$$

Therefore

$$f'(x) = 4x - 4 = 0 \Leftrightarrow x = 1$$

and $$f''(x) = 4 > 0$$

Therefore $f$ has least value at $x = 1$ and the least value is

$$f(1) = 2(1) - 4 + 10 = 8$$

By the hypothesis,

$$(a + 2b) + (a + 8d) = 8$$
$$a + 5d = 4 \tag{3.75}$$

From Eq. (3.75),

$$d = \frac{4-a}{5}$$

Thus (A) is correct and also (B) and (C) are correct.

Now, the sum of the first eleven terms is [by Eq. (3.75)]

$$\frac{11}{2}[a + (a + 10d)] = 11(a + 5d)$$
$$= 11 \times 4 = 44$$

Thus (D) is correct.

**28.** Let $f(x) = 2x^2 - \log_e |x|$ for $x \neq 0$. Then

(A) $f$ is decreasing in the set $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

(B) $f$ is increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(C) minimum value of $f = \frac{1}{2} + \log_e 2$

(D) maximum value for $f$ does not exist

**Solution:** Differentiating the given function we get

$$f'(x) = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x} = \frac{(2x+1)(2x-1)}{x} \tag{3.76}$$

Now

$$f'(x) = 0 \Rightarrow x = \pm\frac{1}{2}$$

$$f''(x) = 4 + \frac{1}{x^2} > 0 \;\; \forall \; x \neq 0$$

Therefore, $f$ is minimum at $x = \pm 1/2$. Also, from Eq. (3.76),

$$f'(x) < 0 \quad \text{for } x < -\frac{1}{2}$$

$$f'(x) > 0 \quad \text{for } -\frac{1}{2} < x < 0$$

$$f'(x) < 0 \quad \text{for } 0 < x < \frac{1}{2}$$

and finally

$$f'(x) > 0 \quad \text{for } x > \frac{1}{2}$$

Therefore $f$ is decreasing in $(-\infty, -1/2) \cup (0, 1/2)$ and increasing in $(-1/2, 0) \cup (1/2, \infty)$. Thus (A) and (B) are correct.

Now $f''(\pm 1/2) > 0 \Rightarrow f$ is minimum at $x = \pm 1/2$ and the minimum value is

$$f\left(\pm\frac{1}{2}\right) = \frac{2}{4} - \log_e \frac{1}{2} = \frac{1}{2} + \log_e 2$$

Therefore (C) is correct. Because at both $x = \pm 1/2$, $f$ is minimum, there is no maximum value for $f$. Hence (D) is correct.

***Note:*** In the above problem, $f$ has two consecutive minima which, in general, is not true. But, here $f$ is not defined at $x = 0 \in (-1/2, 1/2)$. Thus, $f$ is minimum at two consecutive points.

**Answers: (A), (B), (C), (D)**

**29.** The points on the curve $y = x^2 - 5x + 6$ at which the tangents drawn intersect in $(1, 1)$ are

(A) $(2, 0)$ (B) $(1, 2)$
(C) $(0, 6)$ (D) $(-1, 12)$

**Solution:** Differentiating the given function we get

$$\frac{dy}{dx} = 2x - 5$$

Suppose $(x_1, y_1)$ is a point on the curve. Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2x_1 - 5$$

Equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = (2x_1 - 5)(x - x_1)$$

This tangent passes through the point $(1, 1)$. Therefore

$$1 - y_1 = (2x_1 - 5)(1 - x_1)$$
$$= -2x_1^2 + 7x_1 - 5 \tag{3.77}$$

Since $(x_1, y_1)$ lies on the curve, we have

$$y_1 = x_1^2 - 5x_1 + 6 \tag{3.78}$$

From Eqs. (3.77) and (3.78),

$$1 - x_1^2 + 5x_1 - 6 = -2x_1^2 + 7x_1 - 5$$
$$x_1^2 - 2x_1 = 0$$
$$x_1 = 0, 2$$

Now

$$x_1 = 0 \Rightarrow y_1 = 6$$

and $$x_1 = 2 \Rightarrow y_1 = 0$$

Therefore the points on the curve are (0, 6) and (2, 0). Hence (A) and (C) are correct.

**Answers: (A), (C)**

**30.** $P$ is a point on the curve $y = x^3 - 3x^2 - 7x + 6$ at which the intercept made by the tangent on the positive $x$-axis is half of the intercept made by it on the negative $y$-axis. Then

(A) $P = (0, 6)$

(B) $P = (3, -15)$

(C) $P = (1, -3)$

(D) length of the intercept on positive $x$-axis is 21/2

**Solution:** Let $P$ be $(x_0, y_0)$. Therefore

$$y_0 = x_0^3 - 3x_0^2 - 7x_0 + 6 \tag{3.79}$$

Differentiating the given equation with respect to $x$, we get

$$\frac{dy}{dx} = 3x^2 - 6x - 7$$

The equation of the tangent at $(x_0, y_0)$ is

$$y - y_0 = (3x_0^2 - 6x_0 - 7)(x - x_0)$$
$$\Rightarrow y - x_0^3 + 3x_0^2 + 7x_0 - 6 = (3x_0^2 - 6x_0 - 7)(x - x_0) \tag{3.80}$$

Putting $y = 0$ in Eq. (3.80) we get

$$x = -\frac{x_0^3 + 3x_0^2 + 7x_0 - 6}{3x_0^2 - 6x_0 - 7} + x_0$$
$$= \frac{2x_0^3 - 3x_0^2 - 6}{3x_0^2 - 6x_0 - 7}$$

So

$$x\text{-intercept} = \frac{2x_0^2 - 3x_0^2 - 6}{3x_0^2 - 6x_0 - 7} > 0 \text{ (By hypothesis)}$$

Put $x = 0$ in Eq. (3.80). Therefore (by hypothesis)

$$y\text{-intercept} = -2x_0^2 + 3x_0^2 + 6 < 0 \tag{3.81}$$

By hypothesis, $|y\text{-intercept}| = 2|x\text{-intercept}|$. Therefore

$$2x_0^3 - 3x_0^2 - 6 = 2\left(\frac{2x_0^3 - 3x_0^2 - 6}{3x_0^2 - 6x_0 - 7}\right)$$
$$(2x_0^3 - 3x_0^2 - 6)\left[1 - \frac{2}{3x_0^2 - 6x_0 - 7}\right] = 0$$
$$(2x_0^3 - 3x_0^2 - 6)[3x_0^2 - 6x_0 - 9] = 0$$
$$(2x_0^2 - 3x_0 - 6)(x_0 + 1)(3x_0 - 9) = 0$$

Since $2x_0^3 - 3x_0 - 6$ is positive [see Eq. (3.81)], $x_0 = -1$ or $x_0 = 3$. Now

$$x_0 = -1 \Rightarrow x\text{-intercept} < 0$$

Therefore, $x_0 = 3$ which implies $y_0 = -15$. Thus (B) is correct and length of the intercept on positive $x$-axis is

$$\frac{2x_0^3 - 3x_0^2 - 6}{3x_0^2 - 6x_0 - 7} = \frac{54 - 27 - 6}{27 - 18 - 7} = \frac{21}{2}$$

Hence (D) is correct.

**Answers: (B), (D)**

## *Matrix-Match Type Questions*

**1.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) The equations to the curve $y = x^3 + 2x + 6$ which are perpendicular to the line $x + 14y + 4 = 0$ is (are) | (p) $14x - y = 10$ |
| | (q) $14x + y = 14$ |
| (B) The equation of the tangent to the curve $y = 3e^{-x/2}$ at the point where it crosses the $y$-axis is | (r) $3x + 2y = 6$ |

(*Continued*)

| *Column I* | *Column II* |
|---|---|
| (C) The tangent of the curve $y = 14e^{-x}$ at the point (0, 14) is | (s) $x - y = 0$ |
| (D) Equation of the normal to the curve $x^3 + y^3 = 6xy$ at (3, 3) is | (t) $14x - y + 22 = 0$ |

**Solution:**

(A) Differentiating the given function we have

$$\frac{dy}{dx} = 3x^2 + 2$$

Since the tangent is perpendicular to the line $x + 14y + 4 = 0$, we have

$$\frac{dy}{dx} = +14$$
$$\Rightarrow 3x^2 + 2 = 14$$
$$\Rightarrow x = \pm 2$$

Hence the required points on the curve are (2, 18) and (–2, –6).
Equation of the tangent at (2, 18) is

$$y - 18 = 14(x - 2)$$
$$\Rightarrow 14x - y = 10$$

Equation of the tangent at (–2, –6) is

$$y + 6 = 14(x + 2)$$
$$\Rightarrow 14x - y + 22 = 0$$

**Answer: (A) → (p), (t)**

(B) Consider the curve $y = be^{-x/a}$. It meets $y$-axis in $(0, b)$. Differentiating we get

$$\frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)} = -\frac{b}{a}$$

So the equation of the tangent to $y = be^{-x/a}$ at $(0, b)$ is

$$y - b = -\frac{b}{a}(x - 0)$$
$$\Rightarrow bx + ay = ab$$

If $b = 3$, $a = 2$, then the equation of the tangent at (0, 3) is

$$3x + 2y = 6$$

**Answer: (B) → (r)**

(C) If $b = 14$ and $a = 1$, then the equation of the tangent at (0, 14) is

$$14x + y = 14$$

**Answer: (C) → (q)**

(D) Differentiating the given equation with respect to $x$ we get

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) = 6\left(y + x\frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dy}{dx}(y^2 - 2x) = 2y - x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(3,3)} = \frac{6-9}{9-6} = -1$$

Hence normal at (3, 3) is

$$y - 3 = 1(x - 3)$$
$$\Rightarrow x - y = 0$$

**Answer: (D) → (s)**

**2.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) Equation of the tangent to the curve $y^2 = 8x$, that is parallel to the line $4x - y + 3 = 0$ is | (p) $x + 3y - 8 = 0$ |
| | (q) $x - y = \sqrt{2}$ |
| (B) A normal line to the curve $3x^2 - y^2 = 8$ having slope $-1/3$ is | (r) $8x - 2y + 1 = 0$ |
| (C) Equation of the tangent to the curve $x = \sin 3t$ and $y = \cos 3t$ at $t = \pi/4$ is | (s) $y = 1$ |
| (D) Equation of the tangent of the curve $y = \cot^2 x - 2\cot x + 2$ at $x = \pi/4$ is | (t) $x + 3y + 8 = 0$ |

**Solution:**

(A) Differentiating $y^2 = 8x$ we get

$$\frac{dy}{dx} = \frac{4}{y}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{4}{y_1}$$

Since the tangent at $(x_1, y_1)$ is parallel to the line $4x - y + 3 = 0$, we have

$$\frac{4}{y_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 4$$
$$\Rightarrow y_1 = 1$$

Since $y_1^2 = 8x_1$ and $y_1 = 1$, we have $x_1 = 1/8$. Therefore

$$(x_1, y_1) = \left(\frac{1}{8}, 1\right)$$

Equation of the tangent at this point is

$$y-1=4\left(x-\frac{1}{8}\right)$$
$$\Rightarrow 2y-2=8x-1$$
$$\Rightarrow 8x-2y+1=0$$

**Answer: (A) → (r)**

(B) Differentiating $3x^2 - y^2 = 8$ we get

$$6x-2y\frac{dy}{dx}=0$$
$$\Rightarrow \frac{dy}{dx}=\frac{3x}{y}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{3x_1}{y_1}$$

Hence slope of the normal at $(x_1, y_1)$ is

$$-\frac{y_1}{3x_1}=-\frac{1}{3} \quad \text{(by hypothesis)}$$
$$\Rightarrow y_1 = x_1 \tag{3.82}$$

But

$$3x_1^2 - y_1^2 = 8 \tag{3.83}$$

From Eqs. (3.82) and (3.83) we get $x_1 = \pm 2$ and $y_1 = \pm 2$, either both are "+" signs or both are "–" signs. This implies that the points are (2, 2) and (–2, –2). Now the equation of the normal at (2, 2) is

$$y-2=-\frac{1}{3}(x-2)$$
$$\Rightarrow x+3y-8=0$$

Similarly, equation of the normal at (–2, –2) is

$$y+2=-\frac{1}{3}(x+2)$$
$$x+3y+8=0$$

**Answer: (B) → (p), (t)**

(C) The point given on the curve $=\left(1/\sqrt{2}, -1/\sqrt{2}\right)$. Now

$$x=\sin 3t \Rightarrow \frac{dx}{dt}=3\cos 3t$$
$$y=\cos 3t \Rightarrow \frac{dy}{dt}=-3\sin 3t$$

Therefore

$$\frac{dy}{dx}=-\tan 3t \Rightarrow \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}}=-\tan\frac{3\pi}{4}=-(-1)=1$$

So equation of the tangent at $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$ is

$$y+\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right)$$
$$\Rightarrow x-y=\sqrt{2}$$

**Answer: (C) → (q)**

(D) The given point is $(\pi/4, 1)$. Differentiating the given curve we get

$$\frac{dy}{dx}=-2\cot x \operatorname{cosec}^2 x+2\operatorname{cosec}^2 x$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{4},1\right)}=-2(2)+2(2)=0$$

Therefore equation of the tangent at $(\pi/4, 1)$ is

$$y-1=0\left(x-\frac{\pi}{4}\right)$$
$$\Rightarrow y=1$$

**Answer: (D) → (s)**

**3.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) Tangent of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in $P$ and $Q$. Then, the mid-point of $PQ$ lies on the circle $$x^2+y^2=\frac{a^2}{k^2}$$ where $k^2$ is equal to | (p) 1<br>(q) 4 |
| (B) Tangent at $(a, b)$ to the curve $x^3 + y^3 = c^3$ meets the curve again in $(a_1, b_1)$. Then $$\frac{a_1}{a}+\frac{b_1}{b}=$$ | (r) 2 |
| (C) If the sum of the squares of the intercepts on the axes cut off by a tangent to the curve $x^{1/3}+y^{1/3}=a^{1/3}$ $(a > 0)$ at the point $(a/8, a/8)$ is 2, then the value of $a$ is | (s) –1 |
| (D) Tangents are drawn to the curve $y = \sin x$ from the origin. Then the points of contact lie on the curve $$\frac{1}{y^2}-\frac{1}{x^2}=p$$ where the value of $p$ is | (t) –2 |

**Solution:**

(A) Differentiating $x = a\cos^3\theta$, $y = a\sin^3\theta$ we get

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

and $$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = -\tan\theta$$

So, tangent at $\theta$,

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$\Rightarrow x\sin\theta + y\cos\theta = a\sin\theta\cos\theta$$

$$\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$$

Hence $P = (a\cos\theta, 0)$ and $Q = (0, a\sin\theta)$. If $(x_1, y_1)$ is the mid-point of $PQ$, then

$$2x_1 = a\cos\theta \quad \text{and} \quad 2y_1 = a\sin\theta$$

Therefore

$$4(x_1^2 + y_1^2) = a^2$$

or $$x_1^2 + y_1^2 = \frac{a^2}{4}$$

So, $(x_1, y_1)$ lies on

$$x^2 + y^2 = \frac{a^2}{4}$$

**Answer: (A) → (q)**

(B) Differentiating the given curve we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,b)} = -\frac{a^2}{b^2}$$

So, equation of the tangent at $(a, b)$ is

$$y - b = -\frac{a^2}{b^2}(x - a)$$

The tangent at $(a, b)$ meets the curve again in $(a_1, b_1)$. This implies that

$$b_1 - b = -\frac{a^2}{b^2}(a_1 - a)$$

$$\Rightarrow \frac{b_1 - b}{a_1 - a} = -\frac{a^2}{b^2} \tag{3.84}$$

Also $a^3 + b^3 = c^3$ and $a_1^3 + b_1^3 = c^3$ implies

$$a^3 - a_1^3 = b_1^3 - b^3$$

$$\Rightarrow (a - a_1)(a^2 + aa_1 + a_1^2) = (b_1 - b)(b_1^2 + b_1 b + b^2)$$

$$\Rightarrow \frac{b_1 - b}{a_1 - a} = -\frac{(a^2 + aa_1 + a_1^2)}{b^2 + bb_1 + b_1^2} \tag{3.85}$$

From Eqs. (3.84) and (3.85), we have

$$\frac{a^2}{b^2} = \frac{a^2 + aa_1 + a_1^2}{b^2 + bb_1 + b_1^2}$$

On cross-multiplication and simplification we get

$$(ab_1 - a_1 b)(ab + a_1 b + ab_1) = 0$$

Now $(a, b) \neq (a_1, b_1)$ implies

$$ab + a_1 b + ab_1 = 0$$

$$\Rightarrow 1 + \frac{a_1}{a} + \frac{b_1}{b} = 0$$

$$\Rightarrow \frac{a_1}{a} + \frac{b_1}{b} = -1$$

**Answer: (B) → (s)**

(C) Differentiating the given curve we get

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{2/3}$$

At $(a/8, a/8)$, the value of $dy/dx = -1$. Therefore equation of the tangent at $(a/8, a/8)$ is

$$y - \frac{a}{8} = -1\left(x - \frac{a}{8}\right)$$

$$\Rightarrow x + y = \frac{a}{8} + \frac{a}{8} = \frac{a}{4}$$

Now the sum of the squares of the intercepts is

$$\left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^2 = \frac{2a^2}{16} = 2 \quad \text{(by hypothesis)}$$

Therefore $a^2 = 16$ or $a = 4$ ($\because a > 0$).

**Answer: (C) → (q)**

(D) Differentiating $y = \sin x$ we get

$$\frac{dy}{dx} = \cos x$$

At the point $(x_1, y_1)$, the value of $dy/dx = \cos x_1$.

Therefore equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = (\cos x_1)(x - x_1)$$

This tangent passes through (0,0) implies

$$-y_1 = -x_1 \cos x_1$$

$$\Rightarrow \frac{y_1}{x_1} = \cos x_1 \tag{3.86}$$

But $(x_1, y_1)$ lies on the curve implies

$$y_1 = \sin x_1 \tag{3.87}$$

From Eqs. (3.86) and (3.87), we get

$$\frac{y_1^2}{x_1^2} + y_1^2 = 1$$

$$\Rightarrow y_1^2 + x_1^2 y_1^2 = x_1^2$$

$$\Rightarrow x_1^2 - y_1^2 = x_1^2 y_1^2$$

$$\Rightarrow \frac{1}{y_1^2} - \frac{1}{x_1^2} = 1$$

So$(x_1, y_1)$ lies on the curve

$$\frac{1}{y^2} - \frac{1}{x^2} = 1$$

Hence $p = 1$.

**Answer: (D) → (p)**

**4.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) If the curves $xy = a (a > 0)$ and $y^2 = 4x$ cut orthogonally, then the value of $a/2\sqrt{2}$ is | (p) $\frac{1}{2}$ |
| (B) $\lim\limits_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ is | (q) $2\sqrt{2}$ |
| (C) $\lim\limits_{x \to 0} \frac{\tan x - x}{x - \sin x}$ is | (r) $-1$ |
| (D) If the curves $y^2 = 2x$ and $xy = a/2$ $(a > 0)$ cut at right angles, then $a/\sqrt{2}$ equals | (s) 2 |
| | (t) 1 |

**Solution:**

(A) We have

$$xy = a \Rightarrow \frac{dy}{dx} = -\frac{a}{x^2} \tag{3.88}$$

$$y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{2}{y} \tag{3.89}$$

The curves cut each other orthogonally means [using Eqs. (3.88) and (3.89)]

$$\left(-\frac{a}{x^2}\right)\left(\frac{2}{y}\right) = -1$$

$$\Rightarrow x^2 y = 2a$$

$$\Rightarrow x^2\left(\frac{a}{x}\right) = 2a \quad (\because xy = a)$$

$$\Rightarrow x = 2 \tag{3.90}$$

Since $(x, y)$ lies on $y^2 = 4x$ and $x = 2$ we have

$$y = \pm 2\sqrt{2}$$

Substituting the values of $x$ and $y$ in Eq. (3.90), we get

$$(4)(\pm 2\sqrt{2}) = 2a$$

$$\Rightarrow a = \pm 4\sqrt{2}$$

$$\Rightarrow a = 4\sqrt{2} \quad (\because a > 0)$$

$$\Rightarrow \frac{a}{2\sqrt{2}} = 2$$

**Answer: (A) → (s)**

(B) We have

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1+1}{1} = 2$$

**Answer: (B) → (s)**

(C) We have

$$\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\sec^2 x - 1}{1 - \cos x}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{\cos^2 x(1 - \cos x)}$$

$$= \lim_{x \to 0} \frac{1 + \cos x}{\cos^2 x} = 2$$

**Answer: (C) → (s)**

(D) We have

$$y^2 = 2x \Rightarrow \frac{dy}{dx} = \frac{1}{y}$$

and

$$xy = \frac{a}{2} \Rightarrow \frac{dy}{dx} = -\frac{a}{2x^2}$$

By hypothesis, product of the slopes of the curves at a common point = −1. Therefore

$$\frac{1}{y}\left(-\frac{a}{2x^2}\right) = -1$$

$$\Rightarrow x^2 y = \frac{a}{2}$$

$$\Rightarrow x^2\left(\frac{a}{2x}\right) = \frac{a}{2}\left(\because y = \frac{a}{2x}\right)$$

$$\Rightarrow x = 1 \quad (3.91)$$

Now

$$y^2 = 2x \text{ and } x = 1 \Rightarrow y = \pm\sqrt{2}$$

Substituting the values of $x = 1$ and $y = \pm\sqrt{2}$ in Eq. (3.91), we get

$$\frac{a}{2} = (1)(\pm\sqrt{2})$$

$$\frac{a}{2} = \sqrt{2} \qquad (\because a > 0)$$

$$\frac{a}{\sqrt{2}} = 2$$

**Answer: (D) → (s)**

**5.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) Value of $c$ in the Lagrange's mean value theorem for $f(x) = \sqrt{x^2 - 4}$ on the interval [2, 4] is | (p) 4 |
| (B) Value of $c$ in the Lagrange's mean value theorem for $f(x) = x^2 - 2x + 4$ on [1, 5] is | (q) 3 |
| (C) $f$ is continuous on [2, 4] and is differentiable in (2, 4). It is given that $f(2) = 5$ and $f(4) = 13$. Then there exists $c \in (2, 4)$ such that $f'(c)$ is equal to | (r) 1 |
| (D) If $c$ is a value in Rolle's theorem for $f(x) = e^{-x} \sin x$ on the interval $[0, \pi]$, then tan $c$ is equal to | (s) 2 |
| | (t) $\sqrt{6}$ |

**Solution:**

(A) Clearly $f$ is continuous on [2, 4] and differentiable in (2, 4). Hence by Lagrange's mean value theorem, there exists $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2}$$

$$\Rightarrow \frac{2c}{2\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2}$$

$$\Rightarrow c^2 = 3(c^2 - 4)$$

$$\Rightarrow 2c^2 = 12$$

$$\Rightarrow c = \sqrt{6} \in (2, 4)$$

**Answer: (A) → (t)**

(B) We have $f(x) = x^2 - 2x + 4$, $x \in [1, 5]$. Therefore $f(1) = 3, f(5) = 19$.
By Lagrange's mean value theorem, for some $c \in (1, 5)$

$$\frac{f(5) - f(1)}{5 - 1} = f'(c) = 2c - 2$$

$$\Rightarrow 4 = 2c - 2$$

$$\Rightarrow c = 3$$

**Answer: (B) → (q)**

(C) We have

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} = \frac{13 - 5}{2} = 4$$

where $c \in (2, 4)$.

**Answer: (C) → (p)**

(D) We have $f(x) = e^{-x} \sin x$. $f$ is continuous on $[0, \pi]$ and differentiable in $(0, \pi)$. Further

$$f(0) = 0 = f(\pi)$$

Therefore by Rolle's theorem, there exists $c \in (0, \pi)$ such that $f'(c) = 0$. So

$$e^{-c}(\cos c - \sin c) = 0,\ 0 < c < \pi$$

$$\Rightarrow \cos c - \sin c = 0\ (\because e^{-c} \neq 0)$$

$$\Rightarrow \tan c = 1$$

**Answer: (D) → (r)**

**6.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\lim_{x \to \infty} \frac{x + \sin x}{x}$ is | (p) 3 |
| (B) $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$ is | (q) −2 |
| (C) $\lim_{x \to 0+} (x^x)$ is | (r) 1 |
| (D) $\lim_{x \to 0} \frac{x - \sin x}{e^x - 1 - x - (x^2/2)}$ is | (s) 1/2 |

**Solution:**

(A) We have

$$\lim_{x\to\infty}\frac{x+\sin x}{x}=\lim_{x\to\infty}\left(1+\frac{\sin x}{x}\right)=1+0$$

because as $x\to\infty$, $1/x\to 0$ and $\sin x$ is bounded.

**Answer: (A) → (r)**

(B) We have

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x}\left(\frac{\infty}{-\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{3\sec^2 3x}$$

$$=\frac{1}{3}\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos^2 3x}{\cos^2 x}\right)\left(\frac{0}{0}\right)$$

$$=\frac{1}{3}\lim_{x\to\frac{\pi}{2}}\left(\frac{2\cos 3x(-\sin 3x)(3)}{2\cos x(-\sin x)}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{\sin 6x}{\sin 2x}\right)\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{6\cos 6x}{2\cos 2x}\right)$$

$$=\frac{3\cos 3\pi}{\cos\pi}=\frac{3(-1)}{(-1)}=3$$

**Answer: (B) → (p)**

(C) Let $l=\lim_{x\to 0+}(x^x)$. Therefore

$$\log_e l=\lim_{x\to 0+}x\log_e x$$

$$=\lim_{x\to 0+}\left(\frac{\log_e x}{1/x}\right)\left(\frac{-\infty}{\infty}\right)$$

$$=\lim_{x\to 0+}\frac{1/x}{-1/x^2}=\lim_{x\to 0+}(-x)=0$$

So, $l=e^0=1$.

**Answer: (C) → (r)**

(D) We have

$$\lim_{x\to 0}\frac{x-\sin x}{e^x-1-x-(x^2/2)}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{1-\cos x}{e^x-1-x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\sin x}{e^x-1}\left(\frac{0}{0}\right)$$

$$0=\lim_{x\to 0}\frac{\cos x}{e^x}=\frac{\cos 0}{e^0}=1$$

**Answer: (D) → (r)**

> **Try it out** In the above problem L'Hospital's rule cannot be applied. Why?

## *Comprehension-Type Questions*

**1. Passage:** If $f$ is continuous on closed $[a, b]$, differentiable on $(a, b)$ and $f(a)=f(b)$, then there exists $c\in(a, b)$ such that $f'(c)=0$. Answer the following three questions.

**(i)** If in the passage, $f(a)=f(b)=0$, then the equation $f'(x)+\lambda f(x)=0$ has

(A) solutions for all real $\lambda$

(B) no solution for any real $\lambda$

(C) exactly one solution for all real $\lambda$

(D) solution only for $\lambda=1$

**(ii)** If

$$\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+a_n=0$$

where $a_0, a_1, a_2,\ldots, a_n$ are reals, then the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$$

has a root in

(A) $(-\infty,-1)$ (B) $(-1,0)$

(C) $(0, 1)$ (D) $(1,\infty)$

**(iii)** The number of values of $c$ in Rolle's theorem for $f(x)=2x^3+x^2-4x-2$ in the interval $[-\sqrt{2},\sqrt{2}]$ is

(A) 0 (B) 1

(C) 2 (D) 3

**Solution:**

**(i)** Let $\phi(x)=e^{\lambda x}f(x)$ so that

(a) $\phi$ is continuous on $[a, b]$,

(b) differentiable in $(a, b)$,

(c) $\phi(a)=\phi(b)$ ($\because f(a)=f(b)=0$)

Therefore by Rolle's theorem $\phi'(c)=0$ for some $c\in(a, b)$. Then

$$e^{\lambda c}[\lambda f(c)+f'(c)]=0$$

$$\Rightarrow f'(c)+\lambda f(c)=0$$

That is, for each $\lambda$, there corresponds $c \in (a, b)$ such that

$$f'(c) + \lambda f(c) = 0$$

That is, $f'(x) + \lambda f(x) = 0$ is solvable whatever real $\lambda$ may be. Hence (A) is correct.

**Answer: (A)**

**(ii)** Let

$$F(x) = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \frac{a_2 x^{n-1}}{n-1} + \cdots + \frac{a_{n-1} x^2}{2} + a_n x$$

Obviously $F(x)$ continuous and differentiable for all real $x$. Also (by hypothesis) $F(0) = 0$ and $F(1) = 0$. Therefore by Rolle's theorem (using on [0, 1]), $F'(c) = 0$ for some $c \in (0, 1)$. That is

$$a_0 c^n + a_1 c^{n-1} + a_2 c^{n-2} + \cdots + a_n = 0, \quad 0 < c < 1$$

That is $a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$ has a root in $(0, 1)$. So (C) is correct.

**Answer: (C)**

**(iii)** We have $f(x) = 2x^3 + x^2 - 4x - 2$. Clearly

$$f(-\sqrt{2}) = 0 = f(\sqrt{2})$$

Using Rolle's theorem on $[-\sqrt{2}, \sqrt{2}]$, we get $f'(c) = 0$ for some $c \in (-\sqrt{2}, \sqrt{2})$. Therefore

$$6c^2 + 2c - 4 = 0$$
$$\Rightarrow 3c^2 + c - 2 = 0$$
$$\Rightarrow (3c - 2)(c + 1) = 0$$

So $c = -1, 2/3$ and both values belong to $(-\sqrt{2}, \sqrt{2})$. Hence (C) is correct.

**Answer: (C)**

**2. Passage:** Let $f$ be a function continuous in a neighbourhood of a critical point $x_0$ and differentiable at all points in that neighbourhood (except possibly at $x_0$). Then

**(a)** $f$ has local minima at $x_0$, if $f'$ changes sign from minus to plus as $x$ takes values from left of $x_0$ to right of $x_0$.

**(b)** $f$ has local maximum to $x_0$, if $f'$ changes sign from plus to minus.

Answer the following three questions:

**(i)** The total number of local maxima and local minima of the function

$$f(x) = \begin{cases} (x+2)^3 & -3 < x \le -1 \\ x^{2/3} & -1 < x < 2 \end{cases}$$

is

(A) 0 (B) 1

(B) 2 (D) 3

**(ii)** The minimum area of the triangle formed by any tangent to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with the coordinate axes is

(A) $\frac{1}{2}(a-b)^2$ (B) $\frac{1}{2}(a^2+b^2)$

(C) $\frac{1}{2}(a+b)^2$ (D) $ab$

**(iii)** The tangent at (1, 7) to the curve $y = x^2 - 6$ touches the curve $x^2 + y^2 + 16x + 12y + c = 0$ at

(A) $(6, 7)$ (B) $(-6, 7)$

(C) $(6, -7)$ (D) $(-6, -7)$

**Solution:**

**(i)** Clearly $f$ is continuous at $x = -1$, because

$$\lim_{x \to (-1)-0} f(x) = (-1+2)^3 = 1$$

and $$\lim_{x \to (-1)+0} f(x) = (-1)^{2/3} = 1$$

Now,

$$f'(x) = \begin{cases} 3(x+2)^2 & \text{for } -3 < x \le -1 \\ \frac{2}{3} x^{-1/3} & \text{for } -1 < x < 2 \end{cases}$$

So,

$$f'(-1-0) = 3(-1+2)^2 = 3$$

and $$f'(-1+0) = \frac{2}{3}(-1) = -\frac{2}{3}$$

imply that $f$ is not differentiable at $x = -1$ and $f'(0)$ does not exists. Hence $-1$ and 0 are critical points. Further $f'(-2) = 0$ gives that $-2$ is also a critical point. Hence all the critical points of $f$ are $-2, -1, 0$.

At $x = -2$: $f'(x)$ keeps the same sign so that $f$ has no local extremum value at $x = -2$.

At $x = -1$:

$$x < -1 \Rightarrow f'(x) > 0$$

and $$x > -1 \Rightarrow f'(x) = \frac{2}{3} x^{-1/3} < 0$$

Therefore $f$ has local maximum at $x = -1$.

At $x = 0$:

$$x < 0 \Rightarrow f'(x) = \frac{2}{3} x^{-1/3} < 0$$

and $\qquad x > 0 \Rightarrow f'(x) > 0$

Hence $f$ has local minimum at $x = 0$
Therefore, $f$ has one local maximum and one local minimum. Hence (C) is correct.

**Answer: (C)**

**(ii)** Now

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$$

Differentiating with respect to $x$ we get

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

We can see that $x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Therefore

$$\left(\frac{dy}{dx}\right)_\theta=-\frac{b^2(a\cos\theta)}{a^2(b\sin\theta)}=-\frac{b}{a}\cot\theta$$

The equation of the tangent at $(a\cos\theta, b\sin\theta)$ is

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$\Rightarrow (b\cos\theta)x+(a\sin\theta)y=ab$$

$$\Rightarrow \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Therefore area of the triangle formed by the tangent and the axes is

$$\frac{1}{2}\left|\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)\right|=ab|\operatorname{cosec}2\theta|\geq ab$$

It equals to $ab$ when $2\theta$ is an odd multiple of $\pi/2$. Hence (D) is correct.

**Answer: (D)**

**(iii)** Differentiating $y = x^2 - 6$ we get

$$\frac{dy}{dx}=2x$$

At $(1, 7)$, we have

$$\frac{dy}{dx}=2(1)=2$$

Therefore the equation of the tangent at $(1, 7)$ is

$$y-7=2(x-1)$$

$$\Rightarrow 2x-y+5=0 \qquad (3.92)$$

From Eq. (3.92), we have $y = 2x + 5$. Substituting this value of $y$ in the second curve equation we get

$$x^2+(2x+5)^2+16x+12(2x+5)+c=0$$

$$\Rightarrow 5x^2+60x+85+c=0$$

$$\Rightarrow x^2+12x+17+\frac{c}{5}=0 \qquad (3.93)$$

The line given by Eq. (3.92) touches the second curve if and only if the quadratic equation [Eq. (3.93)] has equal roots. This means

Quadratic equation has equal roots

$$\Leftrightarrow 144-4\left(17+\frac{c}{5}\right)=0$$

$$\Leftrightarrow c=95$$

Therefore the second curve equation is

$$x^2+y^2+16x+12y+95=0$$

Now, it can be verified that $(-6, -7)$ is the only common point for the line given by Eq. (3.92) and the second curve. Hence (D) is correct.

**Answer: (D)**

## *Assertion–Reasoning Type Questions*

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation for Statement I

(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I

(C) Statement I is true and Statement II is false

(D) Statement I is false and Statement II is true

**1. Statement I:** If $a_1, a_2, \ldots, a_n$ are positive reals, then

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq(a_1a_2\cdots a_n)^{1/n}$$

where the equality holds if and only if $a_1, a_2, \ldots, a_n$ are equal.

**Statement II:** $e^x \geq 1 + x$ for all real $x$.

**Solution:** Let $f(x) = e^x - 1 - x$. Differentiating this we get

$$f'(x) = e^x - 1$$

This implies that

$$f'(x) < 0 \ \forall \ x < 0$$
$$f'(0) = 0$$

and
$$f'(x) > 0 \ \forall \ x > 0$$

Therefore $f$ has absolute minimum (that is only one minimum) at $x = 0$. Hence $f(x) \geq 0$ for all real $x$. That is

$$e^x \geq 1 + x \quad \text{for all real } x \tag{3.94}$$

In Eq. (3.94) replace $x$ with $x - 1$ so that we have

$$e^{x-1} \geq x \quad \text{for all real } x \tag{3.95}$$

Let

$$\bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

In Eq. (3.95), replace $x$ with $a_1/\bar{a}, a_2/\bar{a}, \ldots, a_n/\bar{a}$ and multiply all the inequalities so obtained. Then we have

$$e^{\left(\frac{a_1}{\bar{a}}+\frac{a_2}{\bar{a}}+\cdots+\frac{a_n}{\bar{a}}-n\right)} \geq \frac{(a_1 \, a_2 \cdots a_n)}{(\bar{a})^n}$$
$$\Rightarrow e^{(n-n)} \geq \frac{(a_1 \, a_2 \cdots a_n)}{(\bar{a})^n}$$
$$\Rightarrow (\bar{a})^n \geq a_1 a_2 \cdots a_n$$
$$\Rightarrow \bar{a} \geq (a_1 \, a_2 \cdots a_n)^{1/n}$$

Hence both statements are true and Statement II is a correct explanation of Statement I.

**Answer: (A)**

**2. Statement I:** If

$$P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$$

then $P(x) = 0$ has a root in the open interval $(45^{1/100}, 46)$.

**Statement II:** If $f:[a,b] \to \mathbb{R}$ is continuous and $f$ is differentiable in $(a, b)$ such that $f(a) = f(b)$, then $f'(c) = 0$ for at least one $c \in (a, b)$.

**Solution:** Statement II is the Rolle's theorem. Now let

$$f(x) = \frac{51}{102}x^{102} - \frac{2323}{101}x^{101} - \frac{45}{2}x^2 + 1035x$$
$$= \frac{1}{2}x^{102} - 23x^{101} - \frac{45}{2}x^2 + 1035x$$

Therefore

$$f(45^{\frac{1}{100}}) = \frac{1}{2}(45)^{\frac{102}{100}} - 23(45)^{\frac{101}{100}} - \frac{45}{2}(45)^{\frac{2}{100}} + 1035(45)^{\frac{1}{100}}$$
$$= \frac{1}{2}(45)^{\frac{102}{100}} - 23(45)^{\frac{101}{100}} - \frac{1}{2}(45)^{\frac{102}{100}} + 23(45)^{\frac{101}{100}}$$
$$= 0$$

Hence $(45)^{1/100}$ is a root of $f(x) = 0$. Again

$$f(46) = \frac{1}{2}(46)^{102} - 23(46)^{101} - \frac{45}{2}(46)^2 + 1035(46)$$
$$= \frac{1}{2}(46)(46)^{101} - 23(46)^{101} - \frac{45 \times 46 \times 46}{2} + 23 \times 45 \times 46$$
$$= 23(46)^{101} - 23(46)^{101} - 23 \times 45 \times 46 + 23 \times 45 \times 46$$
$$= 0$$

Hence 46 is a root of $f(x) = 0$. Now using Rolle's theorem for $f(x)$ on the interval $[45^{1/100}, 46]$, we have $P(c) = f'(c) = 0$ for at least one $c \in (45^{1/100}, 46)$.

**Answer: (A)**

**3. Statement I:** The function $f(x) = x^3 + bx^2 + cx + d$ where $0 < b^2 < c$ is strictly increasing in $(-\infty, \infty)$.

**Statement II:** If $f$ is continuous on $[a, b]$, differentiable in $(a, b)$ and $f'(x) > 0 \ \forall \ x \in (a, b)$, then $f$ is strictly increasing in $(a, b)$.

**Solution:** Statement II is a consequence of Lagrange's mean value theorem (see Theorem 3.6 and the Note under it). Now

$$f(x) = x^3 + bx^2 + cx + d$$

Differentiating this, we get

$$f'(x) = 3x^2 + 2bx + c$$
$$= 3\left[x^2 + \frac{2b}{3}x + \frac{c}{3}\right]$$
$$= 3\left[\left(x + \frac{b}{3}\right)^2 + \frac{c}{3} - \frac{b^2}{9}\right]$$
$$= 3\left[\left(x + \frac{b}{3}\right)^2 + \frac{3c - b^2}{9}\right]$$
$$= 3\left[\left(x + \frac{b}{3}\right)^2 + \frac{2c}{9} + \frac{c - b^2}{9}\right] > 0 \quad (\because \ 0 < b^2 < c)$$

Hence $f$ is strictly increasing for all real $x$.

**Answer: (A)**

**4. Statement I:** The function

$$f(x) = \frac{\log_e(\pi + x)}{\log_e(e + x)}$$

is decreasing on $(0, \infty)$.

**Statement II:** If $f'(x) < 0$ for all $x \in (a, b)$, then $f$ is strictly decreasing in $(a, b)$.

**Solution:** Statement II is true according to Theorem 3.6 and the note under it. Now, let

$$f(x) = \frac{\log_e(\pi + x)}{\log_e(e + x)}$$

Therefore

$$f'(x) = \frac{(e+x)\log_e(e+x) - (\pi+x)\log_e(\pi+x)}{(e+x)(\pi+x)[\log_e(e+x)]^2} \quad (3.96)$$

It is known that, if $g(x) = x\log x$, then $g'(x) = \log_e x + 1 > 0$ for $x > 1/e$. That is $x\log_e x$ increases for $x > 1/e$.

Therefore if $x > 0$, then

$$\frac{1}{e} < e + x < \pi + x$$

so that

$$(e+x)\log_e(e+x) < (\pi+x)\log_e(\pi+x)$$

Hence from Eq. (3.95), $f'(x) < 0$. Hence $f(x)$ decreases on $(0, \infty)$.

**Answer: (A)**

**5. Statement I:** Tangent is drawn to the curve

$$\frac{x^2}{27} + y^2 = 1$$

at $(3\sqrt{3}\cos\theta, \sin\theta)$ where $0 < \theta < \pi/2$. Then the value of $\theta$ for which the sum of the intercepts on the coordinate axes made by this tangent is minimum is $\pi/3$.

**Statement II:** At a critical point $x_0$ a function is minimum if $f'(x)$ changes sign from negative to positive or if $f'(x_0) = 0$ and $f''(x_0) > 0$. Then $f$ is minimum at $x_0$.

**Solution:** Statement II is true according Sec. 3.4.1. It is known that the equation of the tangent at $(a\cos\theta, b\sin\theta)$ to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

(see, Problem 25 of the section "Multiple Correct Choice Type Questions"). Therefore the equation of the tangent to the curve

$$\frac{x^2}{27} + \frac{y^2}{1} = 1$$

at $(3\sqrt{3}\cos\theta, \sin\theta)$ is

$$\frac{x}{3\sqrt{3}}\cos\theta + y\sin\theta = 1$$

Hence the intercepts of the tangent on the axes are $3\sqrt{3}\sec\theta$ and $\operatorname{cosec}\theta$. Let

$$f(\theta) = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$$

Differentiating we get

$$\begin{aligned} f'(\theta) &= 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta \\ &= \frac{3\sqrt{3}\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta} \\ &= \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\cos^2\theta \cdot \sin^2\theta} \end{aligned}$$

Therefore

$$f'(\theta) = 0 \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Hence, Statement I is not correct.

**Answer: (D)**

## *Integer Answer Type Questions*

**1.** The number of values of $x$, where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is ________.

**Solution:** Clearly $f(x) \le 2$ and $f(x) = 2$, if $\cos x = 1$ and $\cos\sqrt{2}x = 1$. This happens when $x = 0$. Also

$$\cos x = 1 \Rightarrow x = 2m\pi$$

and $$\cos\sqrt{2}x = 1 \Rightarrow \sqrt{2}x = 2n\pi$$

$$\Rightarrow 2m\pi = x = \frac{n\pi}{\sqrt{2}}$$

$$\Leftrightarrow m = n = 0$$

Therefore at $x = 0$ only, $f(x)$ is maximum.

**Answer: 1**

**2.** Let $f(x) = \begin{cases} |x| & \text{if } -2 \le x \le 2 \text{ and } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

Then, the number of local maxima of $f$ is ______.

**Solution:** If $0 < \delta < 1$, then

$$x \in (-\delta, \delta) \Rightarrow f(x) \le f(0) = 1$$

Hence $f$ is locally maximum at $x = 0$.

**Answer: 1**

**3.** Let

$$f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$$

for all $x$ in $\mathbb{R}$. If $g:\mathbb{R}\to(0,\infty)$ is a function such that $f(x)=\log_e(g(x))$, the number of values of $x$ in $\mathbb{R}$ at which $g$ has local maxima is ________.

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**Solution:** We have

$$f(x) = \log_e(g(x))$$
$$\Rightarrow f'(x) = \frac{g'(x)}{g(x)}$$

and $$g(x) > 0 \;\forall\, x.$$

**(i)** Now

$$g'(x) = 0 \Leftrightarrow f'(x) = 0$$
$$\Leftrightarrow x = 2009, 2010, 2011 \text{ and } 2012$$

Now, $g'(x)>0$ for $x<2009$ and $g'(x)<0$ for $x>2009$. That is, $g'(x)$ changes sign from positive to negative. Hence $g$ has *local maximum* at $x=2009$.

**(ii)** $g'(x)<0$ for $x<2010$ and $x>2010$. Hence $g(x)$ has no local extremum at $x=2010$.

**(iii)** $g'(x)<0$ for $x<2011$ and $g'(x)>0$ for $x>2011$. Hence $g$ has local minimum at $x=2011$.

**(iv)** $g'(x)>0$ for $x>2011$. Therefore $g$ has only one local maximum.

**Answer: 1**

**4.** Let $P(x)$ be a polynomial of degree 4 having extremum at $x=1,2$ and

$$\lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right) = 2$$

Then the value of $P(2)$ is ___________.

**Solution:** Let

$$P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

Therefore

$$P'(x) = 4a_0x^3 + 3a_1x^2 + 2a_2x + a_3$$

Now $P'(1)=0$ and $P'(2)=0$ implies

$$4a_0 + 3a_1 + 2a_2 + a_3 = 0 \qquad (3.96)$$

and $$32a_0 + 12a_1 + 4a_2 + a_3 = 0 \qquad (3.97)$$

Now

$$\lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right) = 2$$
$$\Rightarrow \lim_{x\to 0}\frac{P(x)}{x^2} = 1$$
$$\Rightarrow \lim_{x\to 0}\frac{a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4}{x^2} = 1 \qquad (3.98)$$

By Corollary 1.3, it follows that

$$a_4 = 0 \qquad (3.99)$$

Therefore [by Eqs. (3.98) and (3.99)]

$$\lim_{x\to 0}\frac{a_0x^3 + a_1x^2 + a_2x + a_3}{x} = 1$$

Again, by the same Corollary 1.3, we have

$$a_3 = 0 \qquad (3.100)$$

So from Eqs. (3.98)–(3.100),

$$\lim_{x\to 0}(a_0x^3 + a_1x + a_2) = 1$$

This implies

$$a_2 = 1 \qquad (3.101)$$

Substituting the values $a_4=0$, $a_3=0$ and $a_2=1$ in Eqs. (3.96) and (3.97) we get

$$4a_0 + 3a_1 = -2 \qquad (3.102)$$
$$32a_0 + 12a_1 = -4 \qquad (3.103)$$

Solving Eqs. (3.102) and (3.103), we obtain $a_0 = 1/4$ and $a_1 = -1$. Therefore

$$a_0 = \frac{1}{4}, a_1 = -1, a_2 = 1, a_3 = 0, a_4 = 0$$

Hence

$$P(x) = \frac{1}{4}x^4 - x^3 + x^2 \Rightarrow P(2) = \frac{16}{4} - 8 + 4 = 0$$

**Answer: 0**

**5.** If $\theta$ is an angle of intersection of the curves $y = 3^{x-1}\log_e x$ and $y = x^x - 1$, then $2\cos\theta$ is equal to ________.

**Solution:** (1, 0) is a point of intersection of the curves. Now

$$y = 3^{x-1}\log_e x$$
$$\Rightarrow \frac{dy}{dx} = 3^{x-1}\log_e 3\log_e x + 3^{x-1}\times\frac{1}{x}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 0 + 1 = 1$$

Again

$$y = x^x - 1 \Rightarrow \frac{dy}{dx} = x^x(1+\log_e x)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1$$

Therefore the slopes of the two curves at (1, 0) are equal. Hence, they touch each other at (1, 0). This implies that $\cos\theta = 1$ or $2\cos\theta = 2$.

**Answer: 2**

**6.** The number of points on the curve $y^3 + 3x^2 = 12y$ where the tangent is horizontal is ________.

**Solution:** Differentiating the given curve we get

$$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\Rightarrow (4-y^2)\frac{dy}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow x = 0$$

$$\Rightarrow y^3 = 12y$$

$$\Rightarrow y = 0, \pm 2\sqrt{3}$$

Therefore the points are (0, 0) and $(0, \pm 2\sqrt{3})$.

**Answer: 3**

**7.** The height of a right circular cone of minimum volume that can be circumscribed about a sphere of radius $r$ is $kr$. Then the value of $k$ is ___________.

**Solution:** See Fig. 3.29. Here

$$OO_1 = OE = r \text{ (radius of the sphere)}$$

$$h = VO_1 \text{ (height of the cone)}$$

Therefore

$$VO = h - r$$

$\Delta VAO_1$ and $\Delta VOE$ are similar (see Fig. 3.29). Let $R = O_1A$ be the radius of the base of the cone. Therefore

$$\frac{R}{r} = \frac{h}{VE} = \frac{h}{\sqrt{VO^2 - OE^2}} = \frac{h}{\sqrt{(h-r)^2 - r^2}} = \frac{h}{\sqrt{h^2 - 2hr}}$$

So,

$$R^2 = \frac{h^2r^2}{h^2 - 2hr} = \frac{hr^2}{h-2r}$$

Now, $V$ = volume of the cone is given by

$$V = \frac{1}{3}\pi R^2 h = \frac{\pi h}{3}\left(\frac{hr^2}{h-2r}\right) = \frac{\pi r^2}{3}\left(\frac{h^2}{h-2r}\right)$$

Differentiating we get

$$\frac{dV}{dh} = \frac{\pi r^2}{3}\frac{[2h(h-2r)-h^2]}{(h-2r)^2} = \frac{\pi r^2}{3}\frac{(h^2-4hr)}{(h-2r)^2}$$

Therefore

$$\frac{dV}{dh} = 0 \Rightarrow h = 4r$$

Also $dV/dh$ changes sign from negative to positive at $h = 4r$. Hence $V$ is least or absolute minimum at $h = 4r = 4$ (radius of the sphere). Therefore $k = 4$.

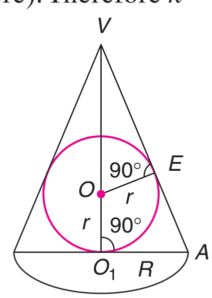


**FIGURE 3.29** Integer answer type question 7.

**Answer: 4**

**8.** Let $x, y$ be positive numbers such that their sum is 5. If the product of square of one number and cube of the other is maximum, then the greater of the numbers is ____________.

**Solution:** We have $x + y = 5$. Let

$$P = x^2y^3 = x^2(5-x)^3$$

Differentiating we get

$$\frac{dP}{dx} = 2x(5-x)^3 - 3x^2(5-x)^2$$
$$= x(5-x)^2[2(5-x)-3x]$$
$$= x(5-x)^2(10-5x)$$
$$= 5x(2-x)(5-x)^2$$

Now

$$\frac{dP}{dx} = 0 \Rightarrow x = 0, 2, 5$$

$x, y$ are positive $\Rightarrow x \neq 0, 5$. Therefore $x = 2$.

Also, at $x = 2$, $dP/dx$ changes sign from positive to negative. Hence $P$ is maximum at $x = 2$ and $y = 3$ and the maximum value of $P = 2^2 3^3 = 108$. Hence greater of the numbers is 3.

**Answer: 3**

**9.** An object moves on the curve $x^2 = 3y$. When $x = 3$, the $x$-coordinate of the object is increasing at the rate of 1 cm/s. At that moment, the rate of increase in $y$-coordinate is ________ cm/sec.

**Solution:** Differentiating $x^2 = 3y$ we get

$$\frac{dy}{dt} = \frac{2x}{3}\frac{dx}{dt}$$

$$\Rightarrow \left(\frac{dy}{dt}\right)_{x=3} = \frac{2(3)}{3}(1) = 2$$

**Answer: 2**

**10.** If the function

$$f(x) = \frac{x^2}{16} + \frac{1}{x}$$

attains its least value at $x_1$ and greatest value at $x_2$ on the interval $[1, 4]$, then $x_1 + x_2$ is equal to ________.

**Solution:** Differentiating we get

$$f'(x) = \frac{x}{8} - \frac{1}{x^2}$$

so that

$$f'(x) = 0 \Leftrightarrow x = 2$$

Also,

$$f''(2) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8} > 0$$

Therefore, $f$ is minimum at $x = 2$ and

$$f(2) = \frac{4}{16} + \frac{1}{2} = \frac{3}{4}$$

Also

$$f(1) = \frac{1}{16} + 1 = \frac{17}{16}$$

and $$f(4) = 1 + \frac{1}{4} = \frac{5}{4}$$

Hence $f(2)$ is least and $f(4)$ is greatest. Therefore

$$x_1 + x_2 = 2 + 4 = 6$$

**Answer: 6**

## SUMMARY

**1.** ***Geometrical meaning of dy/dx:*** Suppose $f(x)$ is a differentiable function and $P(x_1, y_1)$ a point on the graph of $y = f(x)$. Then $f'(x_1)$ or $(dy/dx)_{(x_1, y_1)}$ is the slope of the tangent to the curve at the point. Hence the equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = f'(x_1)(x - x_1)$$

**2.** If $f'(x_1) \neq 0$, then

$$-\frac{1}{f'(x_1)} = \frac{-1}{(dy/dx)_{(x_1, y_1)}}$$

is the slope of the normal to the curve at $P(x_1, y_1)$. The equation of the normal is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

**3.** Tangent at $(x, y)$ is parallel to $x$ − axis $\Leftrightarrow dy/dx = 0$ and tangent at $(x, y)$ is vertical $\Leftrightarrow f'(x)$ or $dy/dx$ is infinity with usual meaning. That is, the angle made by the tangent with $x$-axis is a right angle.

**4.** ***Angle of intersection:*** Angel of intersection of two curves at their point of intersection is defined to be the angle between the tangents drawn to the curves at their common point.

**5.** ***Orthogonal curves:*** Two curves are said to be orthogonal curves if their angle of intersection is a right angle.

**6.** ***Angle of intersection (formula):*** Let $C_1$ and $C_2$ be two curves represented by the functions $y = f(x)$ and $y = g(x)$, respectively. Suppose $P$ is *a* common point for $C_1$ and $C_2$. Let

$$m_1 = \left(\frac{df}{dx}\right)_p \quad \text{and} \quad m_2 = \left(\frac{dg}{dx}\right)_p$$

If $\theta$ the acute angle of intersection of $C_1$ and $C_2$ at $P$, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

**7.** **(i)** The two curves touch each other at $P \Leftrightarrow m_1 = m_2$.
**(ii)** The two curves cut orthogonally at $P \Leftrightarrow m_1 m_2 - 1$.

**8.** ***Lengths of tangent, normal, sub-tangent and sub-normal:*** Let $P(x_1, y_1)$ be a point on the curve $y = f(x)$ and $f$ is differentiable. Suppose the tangent and normal meet the $x$-axis in $T$ and $N$, respectively. Draw $PG$ perpendicular to $x$-axis. Then $PT$, $PN$, $TG$ and $GN$ are called lengths of the tangent, normal, sub-tangent and sub-normal, respectively, at $P$.

$$PT = |y_1|\sqrt{1 + \frac{1}{\sqrt{[f'(x_1)]^2}}}$$

$$PN = |y_1|\sqrt{1 + [f'(x_1)]^2}$$

$$TG = \left|y_1 \cdot \frac{1}{f'(x_1)}\right|$$

$$GN = |y_1 f'(x_1)|$$

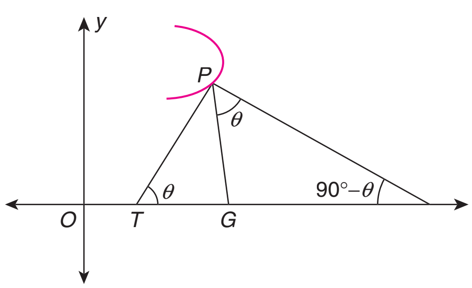


FIGURE 3.30

***Note:*** (Sub-Tangent)(Sub-normal) $= y_1^2$

**9.** ***Rate measure:*** Let $f$ be a function defined on an interval $(a, b)$ and $x_0 \in (a, b)$. Then the quotient

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

where $h$ may take positive and negative values is called average change of $f$ in $(x_0, x_0 + h)$. If

$$\lim_{x \to x_0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is a finite number, which is denoted by $f'(x_0)$, that number is called rate of change of $f$ at $x_0$.

**10.** ***Velocity and acceleration:*** Suppose a particle moving in a straight line covers a distance $s(t)$ in time $t$. Then the velocity $v$ of the particle is $ds/dt$ and its acceleration is

$$\frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

**11.** ***Angular velocity and angular acceleration:*** Suppose a particle is moving on a plane curve. Let $\theta$ be the angle made by $OP$ ("$O$" is the origin) with the $x$-axis at time $t$. Then $d\theta/dt$ is called angular velocity and $d^2\theta/dt^2$ is called angular acceleration.

## Mean Value Theorems

**12.** ***Rolle's theorem:*** If a function $f$ is continuous on closed interval $[a, b]$, differentiable in $(a, b)$ and $f(a) = f(b)$ then there exists $c \in (a, b)$ such that $f'(c) = 0$. Geometrically, the tangent at the point $(c, f(c))$ is parallel to the $x$-axis.

**13.** ***Rolle's theorem for polynomials:*** In between any two zeros of a polynomial, there lies a zero of its derivative polynomial. If a polynomial has $n$ zeros, then its derivative has $(n - 1)$ zeros.

**14.** ***Lagrange's mean value theorem (LMVT):*** If $f$ is continuous on closed $[a, b]$ and differentiable in $(a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Geometrically, the tangent at $(c, f(c))$ is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$.

**15.** ***Other forms of LMVT:***

**(i)** In the above statement, if $b$ is replaced by $a + h$, then there exists $\theta \in (0, 1)$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(a + \theta h)$$

or $\quad f(a+h) = f(a) + hf'(a + \theta h)$

where $\theta \in (0, 1)$.

**(ii)** In (i), if $a$ is replaced by $x$, then

$$f(x+h) = f(x) + h\, f'(x + \theta h)$$

where $\theta \in (0, 1)$.

**16.** ***Deductions from LMVT:***

**(i)** If $f$ is continuous on $[a, b]$, differentiable in $(a, b)$ and $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is a constant function on $[a, b]$.

**(ii)** Suppose $f$ is differentiable in $(a, b)$ and $f'(x) = 0\ \forall\ x \in (a, b)$. Then $f$ is constant in

$(a, b)$. This result fails if the domain of f is not an interval. For example consider the function

$$f(x) = \begin{cases} 0, & x \in (0,1) \\ 1, & x \in (2,3) \end{cases}$$

**(iii)** Suppose $f$ and $g$ are continuous on $[a, b]$ and differentiable in $(a, b)$. Further suppose that $f'(x) = g'(x)\ \forall\ x \in (a,b)$. Then $f$ and $g$ will differ by a constant for all $x \in [a,b]$.

**(iv)** Let $f:[a,b] \to \mathbb{R}$ be a differentiable function in $(a,b)$. Then

(a) $f$ is increasing on $[a, b]$ if and only if $f'(x) \geq 0$ for all $x \in (a,b)$.

(b) $f$ is decreasing on $[a, b]$ if and only if $f'(x) \leq 0\ \forall\ x \in (a,b)$.

Note that $f$ is strictly increasing $\Leftrightarrow f'(x) > 0\ \forall\ x \in (a,b)$ and strictly decreasing $\Leftrightarrow f'(x) < 0\ \forall\ x \in (a,b)$.

***Note:*** To prove functional inequality $f(x) > g(x)$ on an interval $I$, show that $f'(x) - g'(x) > 0$ for $x \in I$.

## Maxima And Minima

**17.** Let $f:[a,b] \to \mathbb{R}$ be a function and $a < c < b$.

**(i)** Suppose, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and $f(x) \leq f(c)\ \forall\ x \in (c-\delta,\ c+\delta)$. Then, we say that $f$ has **local maximum** at $x = c$ and $f(c)$ is called local maximum value of $f$.

**(ii)** Suppose $f(x) \geq f(c)\ \forall\ x \in (c-\delta, c+\delta)$. Then, we say that $f$ has **local minimum** at $x = c$ and $f(c)$ is called local minimum value of $f$.

**(iii)** If $f$ has either local maximum or local minimum at $x = c$, then $f$ is said to have **local extremum** at $c$.

**18.** ***Necessary Condition for an Extremum:*** Suppose $f:[a,b] \to \mathbb{R}$ and $c \in (a,b)$. If $f$ is differentiable at $c$ and $c$ is a point of local extremum, then $f'(c) = 0$.

***Note:***

**(i)** A function having local extremum at a point need not be differentiable. For example, $f(x) = |x|$ is not differentiable at $x = 0$, but $f(0)$ is the minimum value.

**(ii)** $f(x) = x^3$ is differentiable at $x = 0$, $f'(0) = 0$ but $x = 0$ is not a point of extremum.

**19.** ***First Derivative Test:*** Suppose $f:[a,b] \to \mathbb{R}$ is continuous, $a < c < b$ and $f$ is differentiable in $(a, c)$ and $(c, b)$. Let $\delta > 0$ be such that $(c - \delta, c + \delta) \subset (a,b)$. Then,

**(i)** $f$ has local maximum at $c$ if $f'(x) \geq 0\ \forall\ x \in (c-\delta, c)$ and $f'(x) \leq 0\ \forall\ x \in (c, c+\delta)$.

**(ii)** $f$ has local minimum at $c$ if $f'(x) \leq 0\ \forall\ x \in (c-\delta, c)$ and $f'(x) \geq 0\ \forall\ x \in (c, c+\delta)$.

**20.** ***Critical Point:*** A point $x_0$ in the domain of a function $f$ is said to be a critical point for $f$ if either $f'(x_0)$ exists and is equal to zero or $f'(x_0)$ does not exists, but $f'(x)$ exists in a neighbourhood of $x_0$ (in other words $f'$ is discontinuous at $x_0$).

**21.** ***Testing a differentiable function for a local extremum with first derivative:***

**Procedure:**

**Step 1:** Find $f'$.

**Step 2:**

**(i)** Equate $f'(x) = 0$ and obtain values of $x$.

**(ii)** Find the values of $x$ in the domain of $f$ at which $f'$ is discontinuous.

The values of $x$ obtained in 2(a) and 2(b) are the critical points.

**Step 3:** Let $x_0$ be a critical point of $f$. Then

**(i)** If $f'(x)$ changes sign from positive to negative when $x$ passes through the values less than $x_0$ to the values greater than $x_0$ (at $x_0$, $f'(x_0) = 0$ or $f'(x)$ is discontinuous), then at $x_0$, $f$ has local maximum.

**(ii)** If $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$, then $f$ has local minimum at $x_0$.

**(iii)** If $f'(x)$ keeps the same sign for $x < x_0$ and $x > x_0$ then $f$ has no extremum value at $x_0$.

The following table enables the reader about the character of a critical point.

| ***Sign of the derivative when x passes through the values at a critical point $x_0$*** | | | ***Character of the critical point $x_0$*** |
|---|---|---|---|
| ***$x < x_0$*** | ***$x = x_0$*** | ***$x > x_0$*** | |
| + | $f'(x_0) = 0$ or $f'(x)$ is discontinuous at $x_0$ | – | Local maximum at $x_0$ |
| – | $f'(x_0) = 0$ or $f'(x)$ is discontinuous at $x_0$ | + | Local minimum at $x_0$ |
| + | $f'(x_0) = 0$ or $f'(x)$ is discontinuous at $x_0$ | + | Neither maximum nor minimum at $x_0$ (actually, $f$ increases) |

*(Continued)*

| *Sign of the derivative when x passes through the values at a critical point $x_0$* | | | *Character of the critical point $x_0$* |
|---|---|---|---|
| $x < x_0$ | $x = x_0$ | $x > x_0$ | |
| – | $f'(x_0) = 0$ or $f'(x)$ is discontinuous at $x_0$ | – | Neither maximum nor minimum at $x_0$; the function decreases |

**21.** ***Sufficient Conditions for Extremum (Second derivative test):*** Let $f:[a, b] \to \mathbb{R}$ be a differentiable function and $a < x_0 < b$. Suppose $f'(x_0)$ exists and is equal to zero and $f''(x_0) \neq 0$. Then $f$ has local maximum at $x_0$ if $f''(x_0) < 0$ and has local minimum at $x_0$ if $f''(x_0) > 0$

***Note:*** If $f''(x_0) = 0$ then nothing can be said about $f$ at $x_0$. Hence, whenever $f''(x_0) = 0$, go for the first derivative test.

**22.** ***Darbaux Theorem or Intermediate Value Theorem for the Derivative:*** Let $f:[a, b] \to \mathbb{R}$ be a differentiable function [here $f'(a)$ means $f'(a + 0)$ and $f'(b)$ means $f'(b - 0)$]. Then $f'$ assumes every value between $f'(a)$ and $f'(b)$.

**23.** ***Computing the greatest and least values of a function on can interval [a, b].***

**Step 1:** Find all the critical points of $f$ in $(a, b)$.
**Step 2:** Suppose $x_1, x_2, \ldots x_n$ are the critical points, of $f$ in $(a, b)$ at which $f$ is extremum.
**Step 3:** Consider the set $S = \{f(a), f(x_1), f(x_2), \ldots, f(x_n), f(b)\}$. The maximum element of $S$ is the greatest value and the minimum is the least value of $f$ on the interval $[a, b]$.

## L'Hospital's Rule

**24.** Suppose

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

$f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

exists and is equal to $f'(a)/g'(a)$.

**Another form:** If $f$ and $g$ are differentiable in a deleted neighbourhood of $a$, are continuous at $a$, $f(a) = g(a) = 0$ and $\lim_{x \to a} f'(x)/g'(x) = l$, then $\lim_{x \to a} f(x)/g(x)$ exists and is equal to $l$.

For the other indeterminate forms, refer L'Hospital's Rule-I and-II.

# EXERCISES

## *Single Correct Choice Type Questions*

**1.** Rolle's theorem is **not** applicable to

(A) $f(x) = 9x^3 - 4x$ on the interval $\left[-\frac{2}{3}, \frac{2}{3}\right]$

(C) $f(x) = x^{2/3} - 2x^{1/3}$ on $[0, 8]$

(B) $f(x) = x^3 - 3x^2 + x + 1$ on $\left[1, 1+\sqrt{2}\right]$

(D) $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$

**2.** The number of zeros of the function $f(x) = 5x^3 - 2x^2 + 3x - 4$ belonging to $(0, 1)$ is

(A) 1 (B) 2
(C) 3 (D) cannot be determined

(**Hint:** Use Intermediate Value Theorem.)

**3.** The value of $c$ in the Rolle's theorem for $f(x) = x^2 - 2x + 3$ on the interval $[-1, 3]$ is

(A) 0 (B) 1
(C) $-\frac{1}{2}$ (D) 2

**4.** The point on the graph of the function $y = x^2 + x + 3$ between $x = 1$ and $x = 2$ where the tangent is parallel to the chord joining the points $(1, 5)$ and $(2, 9)$ is

(A) $(1, 7)$ (B) $\left(\frac{3}{2}, \frac{27}{2}\right)$

(C) $\left(\frac{3}{2}, \frac{27}{4}\right)$ (D) $\left(1, \frac{27}{2}\right)$

**5.** $f$ is differentiable for all real $x$ and $f(2) = -3$. Suppose $1 < f'(x) < 2$ for $2 < x < 5$. Then

(A) $0 < f(5) < \frac{1}{2}$ (B) $0 < f(5) < 1$

(C) $0 < f(5) < 2$ (D) $0 < f(5) < 3$

6. Angle of intersection of the curves $y = x^2$ and $x = (5/3)\cos t$, $y = (5/4)\sin t$ is

(A) $\text{Tan}^{-1}\left(\frac{1}{2}\right)$ (B) $\text{Tan}^{-1}\left(\frac{31}{2}\right)$

(C) $\text{Tan}^{-1}\left(\frac{41}{2}\right)$ (D) $\text{Tan}^{-1}\left(\frac{21}{2}\right)$

7. The length of the sub-tangent at any point for the curve $x = a\,(2\cos t - \cos 2t)$, $y = a\,(2\sin t - \sin 2t)$ is

(A) $\left|y\cot\frac{3t}{2}\right|$ (B) $\left|y\cot\frac{1}{2}t\right|$

(C) $\left|y\tan\frac{t}{2}\right|$ (D) $\left|y\tan\frac{3t}{2}\right|$

8. The equation of the tangent to the curve $x = 1 + 2\log_e \cot\theta$, $y = \tan\theta + \cot\theta$ at $\theta = \pi/4$ is

(A) $x + y = 1$ (B) $y = 2$

(C) $x = 1$ (D) $x - y = 1$

9. The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on the coordinate axes is equal to

(A) $a$ (B) $a/2$

(C) $2a$ (D) $\sqrt{a}$

10. The length of the segment of the tangent to the curve

$$y = \frac{a}{2}\log_e\left(\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right) - \sqrt{a^2-x^2}$$

at any point contained between the $y$-axis and the point of tangency

(A) is proportional to abscissa

(B) is proportional to ordinate

(C) is of constant length

(D) is of length $a^2$

11. The $x$-intercept of the tangent at any point $(x, y)$ of the curve

$$\frac{a}{x^2} + \frac{b}{y^2} = 1$$

is proportional to

(A) $x^3$ (B) $x^2$

(C) $y^2$ (D) $y^3$

12. The normal to the curve $x^2 - y^2 = a^2$ at a point $P(x_0, y_0)$ meets the $x$-axis at a point $A$. These $AP$ is equal to

(A) $a$ (B) $\sqrt{x_0^2 + y_0^2}$

(C) $2a$ (D) $\frac{1}{2}a$

13. Equation to the tangent at a point on the curve $x^2 = 4a$ whose abscissa is $2am$ is

(A) $mx = y + am^2$ (B) $my = x + am^2$

(C) $mx + y = am^2$ (D) $x + my = am^2$

14. The point of intersection of the tangents to the curve

$$y = \sin\left(2x - \frac{\pi}{3}\right)$$

at the points with the abscissa $x_1 = 0$ and $x_2 = 5\pi/12$ is

(A) $\left(\frac{3}{2}, 1\right)$ (B) $\left(1+\sqrt{3}, 1\right)$

(C) $\left(1+\frac{\sqrt{3}}{2}, 1\right)$ (D) $\left(1-\frac{\sqrt{3}}{2}, 1\right)$

15. The point on the curve

$$y = \frac{1}{1+x^2}$$

at which the tangent is parallel to $x$-axis is

(A) $\left(1, \frac{1}{2}\right)$ (B) $\left(2, \frac{1}{5}\right)$

(C) $\left(-1, \frac{1}{2}\right)$ (D) $(0, 1)$

16. The length of the sub-tangent at any point on the curve $y = ae^{bx}$ is

(A) $\frac{1}{|a|}$ (B) $\frac{1}{|b|}$

(C) $\frac{1}{|ab|}$ (D) $|ab|$

17. The tangent at any point of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the $x$-axis in $P$ and $y$-axis in $Q$. Then the mid-point of the segment $PQ$ lies on the curve whose equation is

(A) $x^2 + y^2 = a^2/4$ (B) $x^2 + y^2 = 2a^2$

(C) $x^2 - y^2 = a^2$ (D) $xy = a^2$

18. The maximum value of $xy$ $(x > 0, y > 0)$ subject to the condition $x + y = 16$ is

(A) 16 (B) 32

(C) 64 (D) 128

19. A particle is moving in a straight line such that its distances $s$ at any time $t$ is given by

$$s=\frac{1}{4}t^4-2t^3+4t^2-7$$

The minimum value of its acceleration attains when $t$ is equal to

(A) $2-\frac{2}{\sqrt{3}}$ (B) 2

(C) $2+\frac{2}{\sqrt{3}}$ (D) 3

**20.** The function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ attains its maximum at $x_1$ and minimum at $x_2$ such that $x_1^2 = x_2$. Then the value of $a$ is

(A) 2 (B) 3

(C) 1 (D) 4

**21.** The number of maximum and minimum values of the function

$$f(x)=\frac{40}{3x^4+8x^3-18x^2+60}$$

is

(A) 5 (B) 4

(C) 3 (D) 2

**22.** If $f(x) = x^2 + b/x$ has a local minimum at $x = 2$, then $b$ equals

(A) 4 (B) 8

(C) 12 (D) 16

**23.** $f(x)=\begin{cases}-2x & \text{for } x\le 0\\ 2x+3 & \text{for } x>0\end{cases}$

Then, at $x = 0$

(A) $f$ has maximum value

(B) $f$ has minimum value

(C) $f$ has no extremum value

(D) $f$ is decreasing for $x \ge 0$

**24.** If $A > 0$, $B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is

(A) 1 (B) 3/4

(C) 4/3 (D) 1/3

**25.** If $x > 0$, then the maximum value of $\log_e x/x$ is

(A) $e$ (B) $1/e$

(C) 1 (D) $2/e$

**26.** The function

$$y=\frac{ax+b}{(x-1)(x-4)}$$

has extremum at $(2, -1)$. Then, the maximum value of $y$ is

(A) 1 (B) 4

(C) $-2$ (D) $-1$

**27.** A line drawn through the point (1, 4) meets the positive coordinate axes. Then, the minimum value of the sum of the intercepts on the axes is

(A) 9 (B) 8

(C) 10 (D) 12

**28.** The efficiency $E$ of a screw jack is given by

$$E=\frac{\tan\theta}{\tan(\theta+\alpha)}$$

where $\alpha$ is constant. Then the maximum efficiency of the screw jack is

(A) $\frac{1-\sin\alpha}{1+\cos\alpha}$ (B) $\frac{1-\sin\alpha}{1+\sin\alpha}$

(C) $\frac{1-\cos\alpha}{1+\sin\alpha}$ (D) $\frac{1-\cos\alpha}{1+\cos\alpha}$

**29.** An open tank with fixed volume is to be constructed on a square base. If the material to be used is minimum, then the edge of the square is

(A) 2 times the depth

(B) 3 times the depths

(C) half of the depth

(D) 2/3 times the depth

**30.** The equation of the line through the point (3, 4) which forms a triangle of maximum area with positive coordinate axes is

(A) $4x + 3y - 24 = 0$ (B) $2x + 3y - 18 = 0$

(C) $3x + 4y - 25 = 0$ (D) $x + 3y - 15 = 0$

**31.** The value of $a$ for which the difference of the roots of the equation $ax^2+(a-1)x+2=0\ (a\ne 0)$ is minimum is

(A) 5 (B) 1/5

(C) 4 (D) 1/4

**32.** $\lim_{x\to 0}\frac{\sin x+\log_e(1-x)}{x^2}=$

(A) $-1/2$ (B) 1

(C) 1/2 (D) $-1$

**33.** The radius of a right circular cylinder is increasing at the rate of 2 cm/sec whereas its height is decreasing at the rate 3 cm/sec. When the radius is 3 cm and height is 5 cm, the volume of the cylinder would change at the rate of

(A) $87\pi$ cubic cm/sec (B) $27\pi$ cubic cm/sec
(C) $33\pi$ cubic cm/sec (D) $15\pi$ cubic cm/sec

**34.** Which one of the following statements is true?

(A) $f(x) = (1 - x^{2/3})^{3/2}$ is not differentiable at $x = 0$, but has maximum at $x = 0$
(B) $x^{2/3}$ is not differentiable at $x = 0$, but has maximum at $x = 0$
(C) $x^3$ is differentiable at $x = 0$ and has maximum at $x = 0$
(D) $x^{1/3}$ is not differentiable at $x = 0$, but has local extremum value at $x = 0$

**35.** $H$ is the height of a right circular cylinder of greatest lateral surface area that can inscribed in a given sphere of radius of $R$. Then

(A) $H = 2R$ (B) $H = R\sqrt{2}$
(C) $H = R\sqrt{3}$ (D) $H = \dfrac{\sqrt{3}R}{2}$

**36.** $H$ is the height of a right circular cylinder of greatest volume that can be inscribed in a sphere of radius $R$. Then

(A) $H = \dfrac{2R}{3}$ (B) $H = 2\sqrt{2}R$
(C) $H = \dfrac{2R}{\sqrt{3}}$ (D) $H = \dfrac{R}{\sqrt{3}}$

**37.** Let $P(a, 0)$ be a point on the positive $x$-axis. Then the abscissa of the point on the curve $y^2 = 2px$ which is closest to $P$ is

(A) $a - p$ (B) $a - (p/2)$
(C) $a + (p/2)$ (D) $a + p$

**38.** If $P(1) = 0$ and $P'(x) > P(x)$ for all $x \geq 1$, then

(A) $P(x) > 0\ \forall\ x > 0$ (B) $P(x) > 0$ for $x \in (0, 1)$
(C) $P(x) > 0$ for all $x > 1$ (D) $P(x) < 0$ for all $x < 1$

[**Hint:** Consider $h(x) = e^{-x}P(x)$ and show that $h$ is increasing for $x \geq 1$.]

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**39.** $\displaystyle\lim_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan(\pi x/2)} =$

(A) $e$ (B) 1
(C) $1/e$ (D) $\sqrt{e}$

**40.** $\displaystyle\lim_{x\to\infty} x^{2/x} =$

(A) 0 (B) 1
(C) $e^2$ (D) $e$

## *Multiple Correct Choice Type Questions*

**1.** Which of the following are true?

(A) In the interval $[0, 2\pi]$, the function $f(x) = 2\sin x + \cos 2x$ is maximum at $x = \pi/6, 5\pi/6$ and minimum at $x = \pi/2, 3\pi/2$
(B) The function $f(x) = x^6$ has minimum value at $x = 0$
(C) The function $f(x) = (x - 1)^3$ is neither maximum nor minimum at $x = 1$
(D) The function $f(x) = x^3 - 3x + 3$ has greatest value at $x = -3$

**2.** $h$ and $r$ are the height and the radius of the base of right circular cylinder of constant volume $V$ with minimum total surface area. Then

(A) $r = \sqrt{V/2\pi}$ (B) $r = \sqrt[3]{V/2\pi}$
(C) $h = 2r$ (D) $h = 3r$

**3.** Let $f(x) = e^{-x^2}$ (this curve is called Gaussian curve). Then

(A) $f(x)$ increases for $x < 0$
(B) $f(x)$ decreases for $x > 0$
(C) $f(x)$ is maximum at $x = 0$
(D) The points $(\pm 1/\sqrt{2}, e^{-1/2})$ are points of inflexion of the curve $y = e^{-x^2}$

**4.** Let $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$. Then

(A) $f$ increases for $x \geq 1$
(B) $f$ decreases for $x \leq 1$
(C) $f$ has minimum at $x = 1$
(D) $f$ has maximum at $x = 1$

**5.** Let $f(x) = \sin x + ax + b$. Then

(A) $f$ has critical points if $-1 \leq a \leq 1$
(B) $f$ is increasing for $a \geq 1$
(C) $f$ is decreasing if $a < -1$
(D) $f$ is neither increasing nor decreasing for any value of $a$

6. Let $f(x) = (4a - 3)(x + \log 5) + (a - 7)\sin x$. Then $f(x)$ has
   (A) critical points, if $-4/3 \le a \le 2$
   (B) no critical points if $a < -4/3$
   (C) critical points if $a > 2$
   (D) If $f(x)$ has critical points and $a = -4/3$, then $\cos x = -1$

7. The function

$$f(x) = \frac{|x-1|}{x^2},\ x \neq 0$$

is monotonically decreasing in
   (A) $(0, 1)$ (B) $(1, \infty)$
   (C) $(0, 2)$ (D) $(2, \infty)$

8. If the curves $y = x^2 + ax + b$ and $y = cx - x^2$ touch each other at the point $(1, 0)$, then
   (A) $a + b = 2$ (B) $a + c = -2$
   (C) $a + b + c = 0$ (D) $b - c = 1$

9. The abscissa of a point on the curve $(a + x)^2 = xy$, the normal at which cuts of numerically equal intercepts on the axes of the coordinates is
   (A) $-\sqrt{2}a$ (B) $\sqrt{2}a$
   (C) $-a/\sqrt{2}$ (D) $a/\sqrt{2}$

10. Which of the following inequalities are true?
   (A) $\tan x > x + \dfrac{x^3}{3}$ if $x \in \left(0, \dfrac{\pi}{2}\right)$
   (B) $e^x \ge 1 + x$ for all real $x$
   (C) $e^x < ex$ for $x > 1$
   (D) $x - \dfrac{x^3}{3} < \mathrm{Tan}^{-1} x < x - \dfrac{x^3}{6}$ for $0 < x \le 1$

11. Which of the following are true?
   (A) The function $f(x) = x^5 + 2x^3 + x$ increases everywhere
   (B) The function $g(x) = x^3 + 2x^2 - 5$ increases in $(-\infty, \infty)$
   (C) The function $h(x) = 2x/\log x$ decreases in $(0,1) \cup (1, e)$ and increases in $(e, +\infty)$
   (D) The function $\phi(x) = x^3 - ax$ increases on the entire line if $a \le 0$

12. The function $f(x) = x(x+1)^3(x-3)^2$ has
   (A) no extremum value at $x = -1$
   (B) local minimum at $x = \dfrac{1}{4}(3 - \sqrt{17})$
   (C) local maximum at $x = \dfrac{1}{4}(3 + \sqrt{17})$
   (D) local minimum at $x = 3$

13. Which of the following are true?
   (A) The function

$$f(x) = \begin{cases} x \sin \dfrac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   vanishes at infinite number of values of $x$ in $(0, 1)$
   (B) If $\alpha < \beta$, then $\mathrm{Tan}^{-1}\beta - \mathrm{Tan}^{-1}\alpha < \beta - \alpha$
   (C) The value of $c$ in the Cauchy's mean value theorem for the quotient

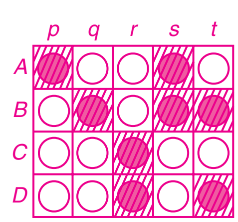
$$\frac{x^2 - 2x + 3}{x^3 - 7x^2 + 20x - 5}$$

   on the interval $[1, 4]$ is 2
   (D) $\lim\limits_{x \to 0} [\log_e(1 + \sin^2 x)\cot^2 \log_e(1 + x)]$ is equal to 1

## *Matrix-Match Type Questions*

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

**Example:** If the correct matches are (A) → (p), (s), (B) → (q), (s), (t), (C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t) then the correct darkening of bubbles will look as follows:



**1.** Match the items of Column I to those of Column II

| *Column I* | *Column II* |
|---|---|
| (A) The number of minima of the function $10x^6 - 24x^5 + 15x^4 + 40x^2 + 108$ is | (p) 1 |
| (B) The number of local maxima of $f(x) = x + \sin x$ is | (q) 2 |
| (C) The minimum value of $f(x) = x^8 + x^6 - x^4 - 2x^3 - x^2 - 2x + 9$ is | (r) 5 |
| (D) Let $P(4, -4)$ and $Q(9, 6)$ be two points on the curve $y^2 = 4x$. $R(x, y)$ is a point on the curve lying between the points $P$ and $Q$. If the area of $\Delta RPQ$ is largest, then the value of $2y - 4x$ is | (s) 0 |

**2.** Match the items of Column I with those of Column II

| *Column I* | *Column II* |
|---|---|
| (A) For the curve $y = x^n$ ($n$ is a positive integer) a normal is drawn at the point $(h, h^n)$ which intersects the $y$-axis at $(0, k)$. If $\lim_{h\to 0} k = 1/2$, then the value of $n$ is | (p) 4 |
| (B) Suppose $f$ is differentiable for all real $x$ and $f'(x) \le 2$ for all $x$. If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ is equal to (**Hint:** Use Lagrange's mean value theorem for $f(x)$ on $[1, 2]$ and $[2, 4]$) | (q) 7 |
| | (r) 2 |
| (C) The number of points on the curve $y = \dfrac{x}{1 - x^2}$ at which the slope is 1 is | (s) 3 |
| (D) The minimum value of the function $\dfrac{x^2 - 3x + 2}{x^2 + 2x + 1}$ is | (t) $-1/24$ |

**3.** Match the items of Column I with those of Column II

| *Column I* | *Column II* |
|---|---|
| (A) $f$ is differentiable in $(1, a)$ and continuous on $(1, a)$ and $f'(2) = \dfrac{f(a) - f(1)}{a - 1}$. Then the value of $a$ is | (p) 1 |
| (B) The integer part of $c$, where $c$ is in the Lagrange's mean value theorem, for $f(x) = x + \sqrt{x}$ on the interval $[1, 4]$ is | (q) 2 |
| (C) If $P(h, k)$ is the point on the curve $y = 4x - x^2$ which is nearest to the point $Q(2, 4)$, then $k - h$ is equal to | (r) 3 |
| (D) $f$ is continuous on $[a, b]$, differentiable in $(a, b)$. If $f(a) = a$ and $f(b) = b$, then there exist $c_1$ and $c_2$ in $(a,b)$ such that $f'(c_1) + f'(c_2)$ is equal to (**Hint:** Take $c = \dfrac{a + b}{2}$ and use LMVT on $[a, c]$ and $[c, b]$) | (s) 4 |
| | (t) 9/4 |

**4.** Consider the function $f(x) = 2\sin(2\sin x) + 2\cos^2 x$ for $0 \le x \le 2\pi$. Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) The number of critical values of $f$ is | (p) 4 |
| (B) $f$ is maximum at $x$ is equal to | (q) $\pi/6, 5\pi/6$ |
| (C) $f$ is minimum at $x$ is equal to | (r) $\pi/2, 3\pi/2$ |
| (D) The sum of the greatest and least value of $f$ on $[0, 2\pi]$ is | (s) 1 |
| | (t) 5/2 |

## Comprehension-Type Questions

**1. Passage:** The greatest or least values of a function $f(x)$ on an interval $[a, b]$ may be attained either at a critical point of $f$ or at the end points of the interval. Answer the following three questions.

**(i)** The greatest value of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on the interval $[-2, 5/2]$ is

(A) 10 (B) 5

(C) 1 (D) 8

**(ii)** The least value of the function $\sin x \sin 2x$ on $\mathbb{R}$ is

(A) $-\dfrac{2}{3\sqrt{3}}$ (B) $-\dfrac{4}{3\sqrt{3}}$

(C) $-\dfrac{1}{3\sqrt{3}}$ (D) 0

(**Hint:** Since $\sin 2x$ is even and is of least period $2\pi$, it is enough to consider $[0, \pi]$)

**(iii)** The sum of the least and greatest values of

$$f(x) = \frac{1-x+x^2}{1+x-x^2}$$

on $[0, 1]$ is

(A) 1 (B) 3/5

(C) 8/5 (D) 2/5

**2.** **Passage:** The perpendicular distance of a straight line $ax + by + c = 0$ from a point $(x_1, y_1)$ in the $xy$-plane is given to be $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$. Answer the following two questions.

**(i)** If $OT$ and $ON$ ($O$ is the origin) are the perpendiculars drawn from $O$ onto the tangent and normal to the curves $x = a\sin^3 t$ and $y = a\cos^3 t$, then $4OT^2 + ON^2$ is equal to

(A) $\sqrt{2}a^2$ (B) $2a^2$

(C) $a^2$ (D) $\sqrt{3}a^2$

**(ii)** $P$ is a point on the curve $2x = a(3\cos\theta = \cos 3\theta)$ $2y = a(3\sin\theta = \sin 3\theta)$. If $p$ is the length of the perpendicular from the origin $O$ onto the tangent at $P$ to the curve, then $3p^2 + 4a^2$ is equal to

(A) $3OP^2$ (B) $2OP^2$

(C) $4OP^2$ (D) $OP^2$

**3.** **Passage:** Using the first derivative test for a function to determine extrema, answer the following questions.

**(i)** The minimum value of

$$f(x) = \frac{50}{3x^4 + 8x^3 - 18x^2 + 60}$$

is

(A) 5/6 (B) –2/3

(C) 50/53 (D) 1

**(ii)** The minimum value of the function $e^{x^2} - 1$ is

(A) 1 (B) 0 (C) –1 (D) $e - 1$

**(iii)** The function $y = f(x)$ is parametrically represented by the equations

$$x = t^5 + 5t^3 - 30t + 7$$
$$y = 4t^3 - 3t^2 - 18t + 3$$

where $|t| < 2$. Then $y = f(x)$ has maximum value 14 at $x$ equals

(A) 21 (B) 11

(C) 31 (D) –21

**4.** Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$$

where $0 < a < 2$. Answer the following two questions.

**(i)** Which of the following is true?

(A) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$

(B) $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$

(C) $f'(1)f'(-1) = (2-a)^2$

(D) $f'(1)f'(-1) = (2+a)^2$

**(ii)** Which of the following is true?

(A) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$

(B) $f(x)$ is increasing on $(-1, 1)$ and has local maximum at $x = 1$

(C) $f(x)$ is increasing on $(-1, 1)$ and has no local extremum at $x = 1$

(D) $f(x)$ is decreasing on $(-1, 1)$ and has no local extremum at $x = 1$

**(IIT-JEE 2008)**

**5.** **Passage:** If $f: \mathbb{R} \to \mathbb{R}$ is continuous and assuming positive and negative values, then $f(x) = 0$ has a root in $\mathbb{R}$. For example, if $f$ is continuous and positive at some point and its minimum value is negative, then $f(x) = 0$ has a real root. Now consider the function $f(x) = ke^x - x$ for all real $x$ and $k$ is a real constant. Answer the following questions.

**(i)** The line $y = x$ meets $y = ke^x$ for $k \le 0$ at

(A) no point (B) one point

(C) two points (D) more than two points

**(ii)** The positive value of $k$ for which $ke^x - x = 0$ has only one root is

(A) $1/e$ (B) 1

(C) $e$ (D) $\log_e 2$

**(iii)** For $k > 0$, the set of all positive values of $k$ for which $ke^x - x = 0$ has two distinct roots is

(A) $\left(0, \frac{1}{e}\right)$ (B) $\left(\frac{1}{e}, 1\right)$ (C) $\left(\frac{1}{e}, \infty\right)$ (D) $(0, 1)$

**(IIT-JEE 2007)**

## *Assertion–Reasoning Type Questions*

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both Statements I and II are true and Statement II is a correct explanation for Statement I.
(B) Both Statements I and II are true but Statement II is not a correct explanation for Statement I.
(C) Statement I is true and Statement II is false.
(D) Statement I is false and Statement II is true.

1. **Statement I:** The function $f(x) = 2 - (x-1)^{2/3}$ attains maximum value 2 at $x = 1$.
   **Statement II:** If $f'$ does not exist at a point $x_0$ but $f'$ exists in a neighbourhood of $x_0$ and changes its sign at $x_0$, then $x_0$ is a point of local extremum.

2. **Statement I:** On the interval $[0, \pi/2]$, the maximum value of $\sin x$ is 1.
   **Statement II:** $\sin x$ is strictly increasing on $[0, \pi/2]$.

3. **Statement I:** $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{x} = 1$
   **Statement II:** If $\lim\limits_{x\to\infty} f(x) = \infty = \lim\limits_{x\to\infty} g(x)$ and $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists (finitely or infinitely), then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

4. **Statement I:** Both $\sin x$ and $\cos x$ are decreasing functions on $(\pi/2, \pi)$.
   **Statement II:** If $f$ is decreasing on $(a, b)$, then its derivative also decreasing on $(a, b)$.
   [**Hint:** Derivative of $\cos x$ increases in $(\pi/2, \pi)$]

5. Consider $f(x) = 2 + \cos x$, $x \in \mathbb{R}$.
   **Statement I:** For each real $t$, there is a point $c \in [t, t+\pi]$ such that $f'(c) = 0$.
   **Statement II:** For each real $t$, $f(t) = f(t + 2\pi)$.

6. **Statement I:** The function

$$f(x) = \begin{cases} 2 - x^2\left(2 + \sin\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

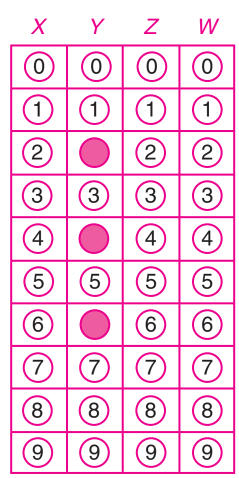
   has local maximum at $x = 0$.

   **Statement II:** If $f(x)$ continuous on $(a, b)$ and has a local maximum at the point $c \in (a, b)$, then in a sufficiently small neighbourhood of $c$, $f(x)$ increases for $x < c$ and decreases for $x > c$.

   ***Note:*** Statement II is wrong according to the function given in Statement 1.

## *Integer Answer Type Questions*

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number $Y$ is 246, then the bubbles under $Y$ labeled as 2, 4, 6 are to be darkened.



1. If $\lim_{x\to 0}\dfrac{e^{ax}-e^{-2ax}}{\log_e(1+x)}=ka$, then the value of $k$ is _______.

2. The maximum value of the function $f(x) = 2x^3 - 15x^2+36x-48$ on the set $A=\{x\,|x^2+20\le 9x|\}$ is _______.

   [**Hint:** $A = [4, 5]$]

3. If the tangents to the curve $y = x^2 - 5x + 6$ drawn at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ pass through the point $M(1,1)$, then $x_1 + x_2 + y_1 + y_2$ is equal to _______.

4. The area of the triangle bounded by the coordinate axes and the tangent to the curve $y = x/(2x - 1)$ at the point with abscissa $x_1 = 1$ is _______.

5. If the interval $[a, b]$ is the image of the interval $[-1, 3]$ under the mapping (function) $f(x) = 4x^3 - 12x$ then $|b/a|$ is equal to _______.

6. The greatest value of $f(x)=|x^2-5x+6|$ on the interval $[0, 2.4]$ is_______.

7. The greatest value of the function $f(x)=\sqrt{x(10-x)}$ is _______.

8. If the Lagrange's mean value theorem is applicable for

$$f(x)=\begin{cases}3 & \text{for } x=0\\ -x^2+3x+a & \text{for } 0<x<1\\ mx+b & \text{for } 1\le x\le 2\end{cases}$$

   on the interval [0, 2], then the value of $a + b + m$ is_______.

9. If $f$, $g$ and $h$ are continuous on $[a, b]$ and differentiable in $(a, b)$, then there exists $\theta\in(a,b)$ such that the value of the determinant $\begin{vmatrix} f(a) & f(b) & f'(\theta)\\ g(a) & g(b) & g'(\theta)\\ h(a) & h(b) & h'(\theta)\end{vmatrix}$ is _______.

10. Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable function such that $f(a) = 0$, $f(b) = 2$, $f(c)=-1$, $f(d)=2$, $f(e)=0$ where $a<b<c<d<e$. Then the minimum number of zeros of the function

$$g(x)=[f'(x)]^2+f(x)f''(x)$$

   in the interval $[a, e]$ is _______.

   [**Hint:** First note that $g(x)=\dfrac{d}{dx}(f(x)f'(x))$. Use Rolle's theorem and Intermediate value theorem for continuous functions.]

## ANSWERS

### *Single Correct Choice Type Questions*

1. (D)
2. (A)
3. (B)
4. (C)
5. (D)
6. (C)
7. (A)
8. (B)
9. (A)
10. (C)
11. (A)
12. (B)
13. (A)
14. (C)
15. (D)
16. (B)
17. (A)
18. (C)
19. (B)
20. (A)
21. (C)
22. (D)
23. (B)
24. (D)
25. (B)
26. (D)
27. (A)
28. (B)
29. (A)
30. (A)
31. (B)
32. (A)
33. (C)
34. (A)
35. (B)
36. (C)

37. (A)
38. (C)
39. (C)
40. (B)

## Multiple Correct Choice Type Questions

1. (A), (B), (C)
2. (B), (C)
3. (A), (B), (C), (D)
4. (A), (B), (C)
5. (A), (B), (C)
6. (A), (B), (D)
7. (A), (D)
8. (B), (C), (D)
9. (C), (D)
10. (A), (B), (D)
11. (A), (B), (C), (D)
12. (A), (B), (C), (D)
13. (A), (B), (C), (D)

## Matrix-Match Type Questions

1. (A) $\rightarrow$ (p); (B) $\rightarrow$ (s); (C) $\rightarrow$ (r); (D) $\rightarrow$ (p)
2. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (t)
3. (A) $\rightarrow$ (r); (B) $\rightarrow$ (q); (C) $\rightarrow$ (q); (D) $\rightarrow$ (q)
4. (A) $\rightarrow$ (p); (B) $\rightarrow$ (q); (C) $\rightarrow$ (r); (D) $\rightarrow$ (t)

## Comprehension Type Questions

1. (i) (D); (ii) (B); (iii) (C)
2. (i) (C); (ii) (C)
3. (i) (B); (ii) (B); (iii) (C)
4. (i) (A); (ii) (A)
5. (i) (B); (ii) (A); (iii) (A)

## Assertion–Reasoning Type Questions

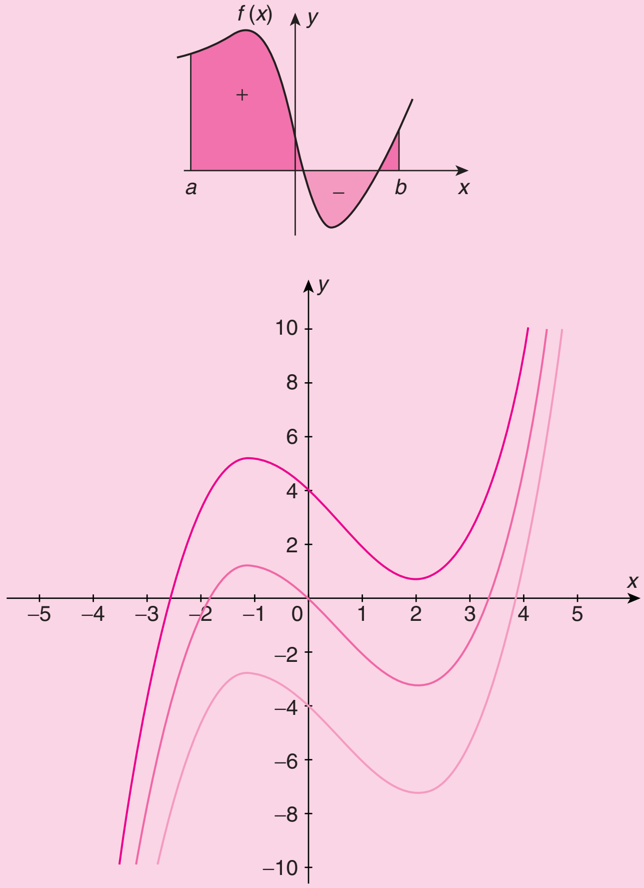
1. (A)
2. (A)
3. (B)
4. (C)
5. (B)
6. (C)

## Integer Answer Type Questions

1. 3
2. 7
3. 8
4. 2
5. 9
6. 6
7. 5
8. 8
9. 0
10. 6

# Indefinite Integral

# 4




Indefinite Integral

*Contents*



**Indefinite integration**, also known as **antidifferentiation**, is the reversing of the process of differentiation. Given a function $f$, one finds a function $F$ such that $F' = f$. Finding an antiderivative is an important process in calculus. It is used as a method to obtain the area under a curve and to obtain many physical and electrical equations that scientists and engineers use everyday.

## 4.1 | Introduction

After carefully studying Chapter 3 on differentiation, a natural question that arises is: Given a function on an internal $I$, can we find a function $g$ defined on $I$ such that the derivative of $g$ is $f$? That is, is there a function $g$ such that $g' = f$ on $I$. In most of the cases, the answer is yes. In general, the statement that "if $f(x)$ is a function then there exists a function $g$ such that $g'(x) = f(x)$" is not true. For example, if $f(x)$ is equal to $a$, $b$ or $c$ according as $x$ is less than, equal to or greater than 0, then there is no function $g$ such that $g'(x) = f(x)$ unless $a = b = c$ for which the reason is that $f$ is discontinuous at 0. However, if $f$ is a continuous function on an interval $I$, then there is always a function $g$ such that $g' = f$ on $I$ (which will be proved in Chapter 5). The process of finding $g$ such that $g' = f$ is called integration. It is in this sense, we say that integration is the *inverse process* of differentiation. Let us begin with the following definition.

**DEFINITION 4.1** **Antiderivative** Suppose $f$ is defined on an interval $I$ of $\mathbb{R}$. Then a $F$ function defined on $I$ is said to be an *antiderivative* or *primitive* of $f$ if $F$ is differentiable on $I$ and $F'(x) = f(x)\ \forall\ x \in I$.

### Examples

1. Let $f(x) = x$ on (0, 1). Take $F(x) = x^2/2\ \forall\ x \in (0,1)$. Then $F'(x) = x = f(x)\ \forall\ x \in (0,1)$ so that $F(x) = x^2/2$ is a primitive of $f(x) = x$ on (0,1).
2. Let $f$ and $F$ be as in (i) and $k$ be any constant. Define $g(x) = F(x) + k\ \forall\ x \in (0,1)$. Then clearly $g'(x) = F'(x) + 0 = f(x)\ \forall\ x \in (0,1)$. Thus, $g$ is also a primitive of $f$ on (0,1).
3. Let $f(x) = \sin x$, $x \in (0, \pi/2)$ and $F(x) = -\cos x$ on $(0, \pi/2)$. Then $F'(x) = -(-\sin x) = \sin x = f(x)$ so that $-\cos x$ is a primitive of $\sin x$ on $(0, \pi/2)$.
4. Let $f(x) = \cos x$ on $(0, \pi/2)$ and $F(x) = \sin x$ on $(0, \pi/2)$ so that $F'(x) = \cos x = f(x)$ and hence $\sin x$ is a primitive of $\cos x$ on $(0, \pi/2)$.
5. Let $F(x) = \log_e x$ on $(0, \infty)$ and $f(x) = 1/x$ on $(0, \infty)$ so that $F'(x) = 1/x = f(x)$ on $(0, \infty)$ and hence $\log x$ is a primitive of $1/x$ on $(0, \infty)$.
6. Let $f(x) = e^x$ and $F(x) = e^x$ on $\mathbb{R}$ so that $F'(x) = e^x = f(x)$ and hence $e^x$ is a primitive of $e^x$ on $\mathbb{R}$.
7. Let $I \subset \mathbb{R}$ be an interval and $f(x) = 0\ \forall\ x \in I$. Suppose $F(x) = k$ (constant) for $x \in I$. Now, $F'(x) = 0$ for all $x \in I$ implies that $F(x)$ is primitive of $f(x)$ on $I$.

***Note:*** If $f$ and $F$ are defined on $I$ and $F'(x) = f(x)$, then

$$\frac{d}{dx}(F(x) + k) = F'(x) = f(x)$$

(where $k$ is a constant) which shows that if $F$ is a primitive of $f$ on $I$, then $F + k$ is also a primitive of $f$ on $I$. Hence, if $f$ has a primitive then it has infinitely many primitives.

In point (7) above we have proved that if $F(x) = k$ (constant) on an interval, then $F'(x) = 0$ on $I$ so that on the interval $I$, constant function is a primitive of zero function. The converse of this result is also true. This, we establish in the following theorem.

**THEOREM 4.1** A function $F$ on an interval $I$ is a primitive of the zero function on $I$ if and only if $F$ is a constant function on $I$.

**PROOF** Suppose $F$ is a constant function say $k$ on the interval $I$. Then $F(x) = k\ \forall\ x \in I$ so that $F'(x) = 0\ \forall\ x \in I$. Thus $F$ is a primitive of the zero function on $I$. Now, suppose $F$ is a primitive of the zero function on $I$. Then $F'(x) = 0\ \forall\ x \in I$. Suppose $a, b \in I$ and $a < b$ so that $[a, b] \subset I$. Then $F$ is differentiable and hence is continuous on $[a, b]$. Therefore by Lagrange's mean value theorem, there exists $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) = 0$$

Consequently, $F(b) = F(a)$. This being true for every $a, b \in I$, it follows that $F$ is a constant function on $I$. ∎

***Note:*** In Theorem 4.1, if $I$ is not an interval the result may not be true. For example, take $I = (-1, 0) \cup (0, 1)$ and

$$F(x) = \begin{cases} -1 & \text{for } -1 < x < 0 \\ 1 & \text{for } 0 < x < 1 \end{cases}$$

so that $F$ is not a constant function on $I$, but $F'(x) = 0$ for $x \in I$.

**COROLLARY 4.1** Let $f$, $F$ and $G$ be functions defined on an interval $I$, and $F$ and $G$ be primitives of $f$ on $I$. Then $F$ and $G$ differ by a constant. That is, $F - G$ is a constant function on $I$.

**PROOF** Write $\phi = F - G$. Then

$$\phi'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \ \forall\, x \in I$$

Hence $\phi$ is a primitive of the zero function on $I$. So by Theorem 4.1, $\phi$ is a constant function, say $k$ on $I$. Then

$$F(x) - G(x) = \phi(x) = k \ \ \forall\, x \in I$$

■

***Note:*** From Corollary 4.1, we have the following: If $F$ is a primitive of $f$ on an interval $I$, then $\{F + k \mid k \in \mathbb{R}\}$ is the set of all primitives of $f$ on $I$.

**DEFINITION 4.2** Let $f$ be defined on interval $I$ and $F$ be a primitive of $f$ on $I$. If $k$ is any constant, then $F + k$ is called an indefinite integral of $f$ on $I$ and is denoted by $\int f(x)\, dx$. The sign $\int$ is the integral sign and $x$ is only a dummy variable. It can be replaced by any other variable or $\int f$ can also be used. Thus

$$\int f = \int f(t)dt = \int f(x)dx = F(x) + k$$

where $k$ is any constant. In this case $f$ is called the *integrand*, $\int f$ is the *indefinite integral* (or simply integral of $f$) and $k$ is called the *constant of integration*.

***Note:***

1. Here onwards constant of integration is denoted by "$c$" instead of $k$.
2. By definition, $\int f(x)\, dx$ is differentiable and $\dfrac{d}{dx}\left(\int f(x)dx\right) = f(x)$ on $I$.
3. If $f$ is differentiable on $I$, then $\int f'(x)dx = f(x) + c$.

From the definition of integral of a function we can list out integrals of some known differentiable functions which are called *standard integrals*. We would list them in separate sections, so that the student can have a ready reference. Even though the following theorem is a consequence of the differentiation, for completeness sake, we state and prove it.

**THEOREM 4.2** Suppose $f$ and $g$ are defined on an interval $I$.

**(i)** If $f$ and $g$ have integrals on $I$, then $f + g$ also has integral on $I$ and

$$\int (f+g)dx = \int f(x)dx + \int g(x)dx + c$$

**(ii)** If $f$ has integral on $I$ and $\alpha$ is any constant, then $\alpha f$ has integral on $I$ and

$$\int \alpha f(x)dx = \alpha \int f(x)dx + c$$

**PROOF** **(i)** Write $F(x) = \int f(x)dx$ and $G(x) = \int g(x)dx$. Then $F'(x) = f(x)$ and $G'(x) = g(x)$ so that

$$(F+G)'(x) = F'(x) + G'(x) = f(x) + g(x)$$

Hence,

$$\int (f+g)(x)dx = (F+G)(x) + c$$

$$= F(x)+G(x)+c$$
$$= \int f(x)dx + \int g(x)dx + c$$

**(ii)** We have

$$\int (\alpha f)(x)dx = \int (\alpha F)'(x)dx = \alpha F(x)+c = \alpha \int f(x)dx + c$$

***Note:***

1. If $f_1, f_2, \ldots, f_n$ have integrals on $I$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are constants, then $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$ has integral on $I$ and
$$\int (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n)\, dx = \alpha_1 \int f_1(x)dx + \alpha_2 \int f_2(x) + \cdots + \alpha_n \int f_n(x)dx + c$$
2. We have
$$\int f - g)(x)dx = \int (f + (-g))(x)dx$$
$$= \int f(x)dx + \int (-g)(x)dx + c$$
$$= \int f(x)dx - \int g(x)dx + c$$

**THEOREM 4.3 (METHOD OF SUBSTITUTION)** Suppose $I$ and $J$ are intervals, $g: J \to I$ is differentiable and $f: I \to \mathbb{R}$ has integral with primitive $F$. Then $(f \circ g)\cdot g' : J \to \mathbb{R}$ has an integral and

$$\int ((f \circ g)\cdot g')(x)dx = \int f(g(x))g'(x)dx = F(g(x)) + c$$

**PROOF** By hypothesis

$$\int f(t)dt = F(t) + c$$

Now $F: I \to \mathbb{R}$ is differentiable, being a primitive of $f$, and $g$: $J \to I$ is differentiable. Hence $F \circ g : J \to \mathbb{R}$ is differentiable and

$$(F \circ g)'(x) = F'(g(x))g'(x)$$
$$= f(g(x))g'(x) \quad [\because F'(t) = f(t)]$$
$$= (f \circ g)(x)g'(x) \tag{4.1}$$

Hence

$$\int (f \circ g)(x)g'(x)dx = \int f(g(x))g'(x)dx$$
$$= \int (F \circ g)'(x)dx \quad \text{[using Eq. (4.1)]}$$
$$= (F \circ g)(x) + c$$
$$= F(g(x)) + c$$

***Note:***

1. In practice, to evaluate $\int f(t)dt$ on $I$, we try to find a function $g: J \to I$ and $\int ((f \circ g)\cdot g')(x)\,dx$ on $J$ which is nothing but $\int f(t)\,dt$ by taking $t = g(x)\ \forall\, x \in J$. That is why evaluating an integral using Theorem 4.3 is called the **method of substitution**.
2. In practice, even though the constant of integration $c$ is not written, we assume its presence.

Some useful consequences of method of substitution, the method of Integration by Parts and various methods of integration are discussed appropriately in the following sections. Since this chapter is very important, in these sections we will give examples in the form of "Single Correct Answer Type" for students to practice the questions section-wise. A consolidated section with more problems would be presented in the "Worked-Out Problems" section.

## 4.2 | Examples on Direct Integration Using Standard Integrals

In this section we will be solving examples based on direct integration using the standard integrals. These integrals and some important formulae are given below.

### *Standard Integrals*

**A.1** $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

**A.2** $\int \dfrac{1}{x} dx = \log_e |x| + c$

**A.3** $\int a^x dx = \dfrac{a^x}{\log_e a} + c \quad (a > 0)$

**A.4** $\int e^x dx = e^x + c$

**A.5** $\int \sin x dx = -\cos x + c$

**A.6** $\int \cos x dx = \sin x + c$

**A.7** $\int \sec^2 x dx = \tan x + c$

**A.8** $\int \mathrm{cosec}^2 x = -\cot x + c$

**A.9** $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \mathrm{Sin}^{-1}\dfrac{x}{a} + c = -\mathrm{Cos}^{-1}\dfrac{x}{a} + c \quad (a > 0)$

**A.10** $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\mathrm{Tan}^{-1}\dfrac{x}{a} + c = -\dfrac{1}{a}\mathrm{Cot}^{-1}\dfrac{x}{a} \quad (a > 0)$

**A.11** $\int \sinh x\, dx = \cosh x + c$

**A.12** $\int \cosh x\, dx = \sinh x + c$

In A.11 and A.12, $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

**A.13** $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \left\{ \begin{array}{ll} \mathrm{Cosh}^{-1}\dfrac{x}{a} + c & \text{if } a < x < \infty \\ -\mathrm{Cosh}^{-1}\left(-\dfrac{x}{a}\right) + c & \text{if } -\infty < x < -a \end{array} \right\} = \log(x + \sqrt{x^2 - a^2}) + c$ on any interval contained in $(-\infty, -a) \cup (-a, a) \cup (a, \infty)$

**A.14** $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \mathrm{Sinh}^{-1}\dfrac{x}{a} + c = \log_e(x + \sqrt{x^2 + a^2}) + c$

**A.15** $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\log_e\left|\dfrac{x-a}{x+a}\right| + c$

**A.16** $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\log\left|\dfrac{x+a}{x-a}\right| + c$

**A.17** $\int (f_1(x) \pm f_2(x))\, dx = \int f_1(x)\, dx \pm \int f_2(x)\, dx + c$

**A.18** $\int (kf(x))\, dx = k\int f(x) dx$ where $k$ is a real constant

**A.19** $\int f(ax+b)\, dx = \dfrac{1}{a}F(ax+b)$ where $\int f(t)dt = F(t) + c$

***Note:*** A.19 gives you that in all the other formulae A.1—A.18 if $x$ is replaced by $px+q$ on the left-hand side then you have to divide the right-hand side by $p$, that is, the coefficient of $x$.

## Example 4.1

If

$$\int (4x^3 - 5x^2 + 2x + 1)dx = x^4 - ax^3 + x^2 + x + c$$

then $a$ is equal to

(A) $\frac{5}{3}$ (B) $\frac{3}{5}$ (C) $-\frac{5}{3}$ (D) $-\frac{3}{5}$

***Solution:*** Since $\int x^n dx = \frac{x^n + 1}{n+1} + c \quad (n \neq -1)$, we have

$$\int (4x^3 - 5x^2 + 2x + 1)dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + x + c$$

$$= x^4 - \frac{5}{3}x^3 + x^2 + x + c$$

Therefore

$$a = \frac{5}{3}$$

**Answer: (A)**

***Note:*** Here, the single constant $c$ is the sum of all the arbitrary constants arising in the individual integrals.

## Example 4.2

If

$$\int \frac{x^2 + 5x - 1}{\sqrt{x}} dx = ax^{5/2} + bx^{3/2} + c\sqrt{x} + k$$

then $a+b+c$ is equal to

(A) $\frac{16}{15}$ (B) $\frac{26}{15}$ (C) 2 (D) $-\frac{26}{15}$

***Solution:*** We have

$$\int \frac{x^2 + 5x - 1}{\sqrt{x}} dx = \int (x^{3/2} + 5x^{1/2} - x^{-1/2})\, dx$$

$$= \frac{x^{(3/2)+1}}{(3/2)+1} + 5 \cdot \frac{x^{(1/2)+1}}{(1/2)+1} - \frac{x^{-(1/2)+1}}{-(1/2)+1} + k$$

$$= \frac{2}{5}x^{5/2} + \frac{10}{3}x^{3/2} - 2x^{1/2} + k$$

Therefore

$$a+b+c = \frac{2}{5} + \frac{10}{3} - 2 = \frac{6+50-30}{15} = \frac{26}{15}$$

**Answer: (B)**

## Example 4.3

If

$$\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^3 dx = \frac{x^{n_1}}{2} + (\log_e |x|)^{n_2} + \frac{9}{4}x^{n_3} + \frac{9}{2}x^{n_4} + c$$

then $n_1 + n_2 + n_3 + n_4$ equals

(A) 5 (B) 6 (C) 4 (D) $\frac{9}{4}$

***Solution:*** We have

$$\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^3 dx = \int \left(x + \frac{1}{x} + 3x^{1/3} + 3x^{-1/3}\right) dx$$

$$= \frac{1}{2}x^2 + \log_e |x| + 3 \cdot \frac{x^{4/3}}{4/3} + 3 \cdot \frac{x^{2/3}}{2/3} + c$$

$$= \frac{1}{2}x^2 + \log_e |x| + \frac{9}{4}x^{4/3} + \frac{9}{2}x^{2/3} + c$$

Therefore

$$n_1 + n_2 + n_3 + n_4 = 2 + 1 + \frac{4}{3} + \frac{2}{3} = 5$$

**Answer: (A)**

## Example 4.4

$\int\left(3x^{-1/3}+\frac{1}{2}x^{-1/2}+x^{5/4}\right)dx$ is equal to

(A) $\frac{9}{2}x^{2/3}+\frac{1}{2}\sqrt{x}+\frac{4}{9}x^{9/4}+c$

(B) $\frac{9}{2}x^{2/3}+\sqrt{x}+\frac{9}{4}x^{9/4}+c$

(C) $\frac{9}{4}x^{2/3}+\frac{1}{2}\sqrt{x}+\frac{4}{9}x^{9/4}+c$

(D) $\frac{9}{2}x^{2/3}+\sqrt{x}+\frac{4}{9}x^{9/4}+c$

***Solution:*** We have

$$\int\left(3x^{-1/3}+\frac{1}{2}x^{-1/2}+x^{5/4}\right)dx = 3\cdot\frac{x^{-(1/3)+1}}{-(1/3)+1}+\frac{1}{2}\cdot\frac{x^{-(1/2)+1}}{-(1/2)+1}+\frac{x^{(5/4)+1}}{(5/4)+1}+c$$

$$=\frac{9}{2}x^{2/3}+x^{1/2}+\frac{4}{9}x^{9/4}+c$$

**Answer: (D)**

## Example 4.5

$\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx$ is equal to

(A) $1+\sin x+c$ (B) $x-\cos x+c$

(C) $x+\cos x+c$ (D) $1-\cos x+c$

***Solution:*** We have

$$\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx = \int\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos2\frac{x}{2}\right)dx$$

$$=\int(1+\sin x)\,dx$$

$$=x-\cos x+c$$

**Answer: (B)**

## Example 4.6

$\int\left(3\sin\frac{x}{3}-4\sin^3\frac{x}{3}\right)dx$ is equal to

(A) $\cos x + c$ (B) $\sin x + c$

(C) $-\cos x + c$ (D) $-\sin x + c$

***Solution:*** We have

$$\int\left(3\sin\frac{x}{3}-4\sin^3\frac{x}{3}\right)dx = \int\sin3\left(\frac{x}{3}\right)dx$$

$$=\int\sin x dx$$

$$=-\cos x+c$$

**Answer: (C)**

## Example 4.7

$\int(\cosh^2 x-\sinh^2 x)^2 dx$ equals

(A) $\frac{1}{2}\sinh 2x+c$ (B) $\frac{1}{2}\cosh 2x+c$

(C) $-\frac{1}{2}\cosh 2x+c$ (D) $x+c$

***Solution:*** We have

$$\int(\cosh^2 x-\sinh^2 x)^2 dx = \int\left(\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2\right)^2 dx$$

$$=\int 1^2\,dx \quad [\because (a+b)^2-(a-b)^2=4ab]$$

$$=x+c$$

**Answer: (D)**

## Example 4.8

$\int\left(\frac{1}{x^2}+\frac{4}{x\sqrt{x}}+2\right)dx$ equals

(A) $\frac{1}{x}-\frac{8}{\sqrt{x}}+2x+c$ (B) $-\frac{1}{x}-\frac{8}{\sqrt{x}}+x+c$

(C) $-\frac{1}{x}-\frac{8}{\sqrt{x}}+2x+c$ (D) $\frac{1}{x}+\frac{8}{\sqrt{x}}-2x+c$

***Solution:*** We have

$$\int\left(\frac{1}{x^2}+\frac{4}{x\sqrt{x}}+2\right)dx=\int(x^{-2}+4x^{-3/2}+2)\,dx$$

$$=\frac{x^{-2+1}}{-2+1}+4\frac{x^{-3/2+1}}{-(3/2)+1}+2x+c$$

$$=-\frac{1}{x}-\frac{8}{\sqrt{x}}+2x+c$$

**Answer: (C)**

## Example 4.9

$\int\left(x^2+\frac{1}{\sqrt[3]{x}}\right)^2 dx$ is equal to

(A) $\frac{x^5}{5}+\frac{1}{4}x^{8/3}+3x^{1/3}+c$

(B) $\frac{x^5}{5}+\frac{1}{2}x^{8/3}+2x^{1/3}+c$

(C) $\frac{x^5}{5}+\frac{3}{8}x^{8/3}+3x^{1/3}+c$

(D) $\frac{x^5}{5}+\frac{3}{4}x^{8/3}+3x^{1/3}+c$

***Solution:*** We have

$$\int\left(x^2+\frac{1}{\sqrt[3]{x}}\right)^2 dx=\int(x^4+2x^{5/3}+x^{-2/3})\,dx$$

$$=\frac{x^5}{5}+2\cdot\frac{x^{(5/3)+1}}{(5/3)+1}+\frac{x^{-(2/3)+1}}{-(2/3)+1}+c$$

$$=\frac{x^5}{5}+\frac{3}{4}x^{8/3}+3x^{1/3}+c$$

**Answer: (D)**

## Example 4.10

$\int\frac{dx}{\sqrt{x+1}-\sqrt{x}}$ is equal to $a\,(x+1)\sqrt{x+1}+bx\sqrt{x}+c$

where

(A) $a+b=\frac{2}{3}$ (B) $\frac{a}{b}=1$

(C) $a-b=\frac{2}{3}$ (D) $a+b=\frac{1}{3}$

***Solution:*** We have

$$\int\frac{1}{\sqrt{x+1}-\sqrt{x}}\,dx=\int(\sqrt{x+1}+\sqrt{x})\,dx$$

$$=\frac{(x+1)^{(1/2)+1}}{(1/2)+1}+\frac{x^{(1/2)+1}}{(1/2)+1}+c$$

$$=\frac{2}{3}(x+1)^{3/2}+\frac{2}{3}x^{3/2}+c$$

Therefore

$$\frac{a}{b}=\frac{2/3}{2/3}=1$$

**Answer: (B)**

## Example 4.11

If

$$\int\left(\frac{3x^2-2x+1}{\sqrt{x}}\right)dx=k\sqrt{x}(9x^2-10x+15)+c$$

then $k$ equals

(A) $\frac{2}{13}$ (B) $\frac{13}{2}$ (C) $\frac{15}{2}$ (D) $\frac{2}{15}$

***Solution:*** We have

$$\int\left(\frac{3x^2-2x+1}{\sqrt{x}}\right)dx=\int(3x^{3/2}-2x^{1/2}+x^{-1/2})\,dx$$

$$=\frac{6}{5}x^{5/2}-\frac{4}{3}x^{3/2}+2x^{1/2}+c$$

$$=\frac{2}{15}\sqrt{x}(9x^2-10x+15)+c$$

**Answer: (D)**

## Example 4.12

If $\int \cos x \cos 2x \, dx = a \sin x + b \sin 3x + c$, then

(A) $a = b = \dfrac{1}{2}$ (B) $a = b = \dfrac{1}{3}$

(C) $a + b = \dfrac{1}{2}$ (D) $\dfrac{a}{b} = 3$

***Solution:*** We have

$$\int \cos x \cos 2x \, dx = \int \frac{\cos x + \cos 3x}{2} dx$$

$$= \frac{1}{2}\left[\sin x + \frac{1}{3}\sin 3x\right] + c$$

Therefore $a = 1/2, b = 1/6$ so that

$$\frac{a}{b} = 3$$

**Answer: (D)**

## Example 4.13

If $\int \cos x \cos 2x \cos 5x \, dx = \dfrac{1}{a}\sin 2x + \dfrac{1}{b}\sin 4x + \dfrac{1}{c}\sin 6x + \dfrac{1}{d}\sin 8x + k$, then $a, b, c, d$ are in

(A) AP (B) GP (C) HP (D) $ac = bd$

***Solution:*** We have

$$\int \cos x \cos 2x \cos 5x \, dx = \int \frac{1}{2}(\cos x + \cos 3x) \cos 5x \, dx$$

$$= \frac{1}{2}\int \cos x \cos 5x dx + \frac{1}{2}\int \cos 3x \cos 5x dx$$

$$= \frac{1}{2}\int \frac{1}{2}(\cos 4x + \cos 6x)\, dx + \frac{1}{2}\int \frac{1}{2}(\cos 2x + \cos 8x)\, dx$$

$$= \frac{1}{4}\int (\cos 2x + \cos 4x + \cos 6x + \cos 8x)\, dx$$

$$= \frac{1}{4}\left[\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right] + k$$

Therefore

$$a = 8, b = 16, c = 24, d = 32$$

So $a, b, c, d$ are in AP.

**Anwser: (A)**

## Example 4.14

$\int \left(\dfrac{\cos 8x - \cos 7x}{1 + 2\cos 5x}\right) dx$ is equal to

(A) $\dfrac{1}{3}\cos 3x + \dfrac{1}{2}\cos 2x + c$

(B) $\dfrac{1}{3}\sin 3x - \dfrac{1}{2}\sin 2x + c$

(C) $\dfrac{1}{3}\cos 3x - \dfrac{1}{2}\cos 2x + c$

(D) $\dfrac{1}{3}\sin 3x + \dfrac{1}{2}\sin 2x + c$

***Solution:*** We have

$$\int \frac{(\cos 8x - \cos 7x)}{1 + 2\cos 5x} dx = \int \frac{(\cos 8x - \cos 7x)\sin 5x}{\sin 5x + \sin 10x} dx$$

$$= \int -\frac{2\sin\frac{15x}{2}\sin\frac{x}{2}\sin 5x}{2\sin\frac{15x}{2}\cos\frac{5x}{2}} dx$$

$$= -\int \frac{\sin\frac{x}{2}\left(2\sin\frac{5x}{2}\cos\frac{5x}{2}\right)}{\cos\frac{5x}{2}} dx$$

$$= -\int \left(2\sin\frac{x}{2}\sin\frac{5x}{2}\right) dx$$

$$= -\int (\cos 2x - \cos 3x)\, dx$$

$$= \int (\cos 3x - \cos 2x)\, dx$$

$$= \frac{1}{3}\sin 3x - \frac{1}{2}\sin 2x + c$$

**Answer: (B)**

## Example 4.15

If $\int (\sec x + \tan x)^2 dx = 2\tan x + f(x) + c$, then $f(x)$ is equal to

(A) $2\sec x - x$ (B) $x - 2\sec x$
(C) $2\sec x + x$ (D) $x + 2\sec x$

***Solution:*** We have

$$\int (\sec x + \tan x)^2 dx = \int (\sec^2 x + 2\sec x \tan x + \tan^2 x)\, dx$$

$$= \int (2\sec^2 x - 1 + 2\sec x \tan x)\, dx$$

$$= 2\tan x - x + 2\sec x + c \quad \left(\because \int \sec x \tan x\, dx = \sec x + c\right)$$

Therefore

$$f(x) = 2\sec x - x$$

**Answer: (A)**

## Example 4.16

$\int \sec^2 x \operatorname{cosec}^2 x\, dx$ equals

(A) $\sec x + \operatorname{cosec} x + c$
(B) $\sec x - \operatorname{cosec} x + c$
(C) $\tan x - \cot x + c$
(D) $\tan x + \cot x + c$

***Solution:*** We have

$$\int \sec^2 x \operatorname{cosec}^2 x\, dx = \int \frac{dx}{\sin^2 x \cos^2 x}$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$$

$$= \tan x - \cot x + c$$

**Answer: (C)**

## Example 4.17

$\int \frac{x^2 + \cos^2 x}{(1+x^2)\sin^2 x} dx$ is equal to

(A) $\mathrm{Tan}^{-1} x + \cot x + c$ (B) $\mathrm{Tan}^{-1} x - \cot x + c$
(C) $\mathrm{Cot}^{-1} x - \tan x + c$ (D) $-\mathrm{Tan}^{-1} x - \cot x + c$

***Solution:*** We have

$$\int \frac{x^2 + \cos^2 x}{(1+x^2)\sin^2 x} dx = \int \frac{x^2 + 1 - \sin^2 x}{(1+x^2)\sin^2 x} dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{1+x^2}\right) dx$$

$$= \int \operatorname{cosec}^2 x - \int \frac{dx}{1+x^2}$$

$$= -\cot x - \mathrm{Tan}^{-1} x + c$$

**Answer: (D)**

## Example 4.18

$\int \left(\frac{2^x - 5^x}{10^x}\right) dx$ is equal to

(A) $\frac{2^{-x}}{\log_e 2} - \frac{5^{-x}}{\log_e 5} + c$ (B) $\frac{2^x}{\log_e 2} - \frac{5^x}{\log_e 5} + c$

(C) $\frac{2^x}{\log_e 2} + \frac{5^x}{\log_e 5} + c$ (D) $\frac{5^{-x}}{\log_e 5} - \frac{2^{-x}}{\log_e 2} + c$

***Solution:*** We have

$$\int \frac{2^x - 5^x}{10^x} dx = \int \left[\left(\frac{2}{10}\right)^x - \left(\frac{5}{10}\right)^x\right] dx$$

$$= \int \left[\left(\frac{1}{5}\right)^x - \left(\frac{1}{2}\right)^x\right] dx$$

$$= \frac{\left(\frac{1}{5}\right)^x}{\log_e\left(\frac{1}{5}\right)} - \frac{\left(\frac{1}{2}\right)^x}{\log_e\left(\frac{1}{2}\right)} + c \quad \left(\because \int a^x dx = \frac{a^x}{\log_e a} + c \text{ (see A.3)}\right)$$

$$= \frac{-5^{-x}}{\log_e 5} + \frac{2^{-x}}{\log_e 2} + c$$

**Answer: (A)**

## Example 4.19

If $\int \frac{x+3}{(x+2)^2}\,dx = f(x) - \frac{1}{x+2} + c$, then $f(x)$ is equal to

(A) $\log_e (x+2)$ (B) $\log_e |x+2|$
(C) $\log_e |x+1|$ (D) $x+2$

***Solution:*** We have

$$\int \frac{x+3}{(x+2)^2}\,dx = \int \frac{x+2+1}{(x+2)^2}\,dx$$

$$= \int \left( \frac{1}{x+2} + \frac{1}{(x+2)^2} \right) dx$$

$$= \log_e |x+2| - \frac{1}{x+2} + c$$

Therefore, $f(x) = \log_e |x+2|$.

**Answer: (B)**

## Example 4.20

$$\int \frac{dx}{\sqrt{ax+b} - \sqrt{ax+c}} = \frac{2}{3k}[(ax+b)^{3/2} + (ax+c)^{3/2}] + C$$

where $k$ is equal to

(A) $a(b-c)$ (B) $a(c-b)$
(C) $b(c-a)$ (D) $b(a-c)$

***Solution:*** We have

$$\int \frac{dx}{\sqrt{ax+b} - \sqrt{ax+c}} = \int \frac{\sqrt{ax+b} + \sqrt{ax+c}}{(ax+b)-(ax+c)}\,dx$$

$$= \frac{1}{b-c} \int [(ax+b)^{1/2} + (ax+c)^{1/2}]\,dx$$

$$= \frac{1}{b-c} \left[ \frac{(ax+b)^{(1/2)+1}}{\left(\frac{1}{2}+1\right)a} + \frac{(ax+c)^{(1/2)+1}}{\left(\frac{1}{2}+1\right)a} \right] + C$$

$$= \frac{2}{3a\,(b-c)} \left[ (ax+b)^{3/2} + (ax+c)^{3/2} \right] + C$$

Therefore

$$k = a(b-c)$$

**Answer: (A)**

## Example 4.21

$\int \frac{dx}{x^2+2x+3}$ equals

(A) $\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$

(B) $\frac{1}{\sqrt{2}} \mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$

(C) $\frac{1}{\sqrt{2}} \mathrm{Tan}^{-1}\left(\frac{x+1}{2}\right) + c$

(D) $\frac{1}{\sqrt{2}} \log_e \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| + c$

***Solution:*** We have

$$\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{(x+1)^2+2}$$

$$= \frac{1}{\sqrt{2}} \mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c \quad \text{(See A.10)}$$

**Answer: (B)**

## Example 4.22

$\int \frac{dx}{\sqrt{3+2x-x^2}}$ is equal to

(A) $\sin(x-2) + c$ (B) $\sin\left(\frac{x-2}{2}\right) + c$

(C) $\frac{1}{2} \mathrm{Sin}^{-1}\left(\frac{x-1}{2}\right) + c$ (D) $\mathrm{Sin}^{-1}\left(\frac{x-1}{2}\right) + c$

***Solution:*** We have

$$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}}$$

$$= \mathrm{Sin}^{-1}\left(\frac{x-1}{2}\right) + c \quad \text{(See A.9)}$$

**Answer: (D)**

### Example 4.23

$\int \cos^3 x\, dx$ is equal to

(A) $\frac{1}{4}\left(\frac{1}{3}\sin 3x + \sin x\right) + c$

(B) $\frac{1}{4}\left(\frac{1}{3}\sin 3x + 3\sin x\right) + c$

(C) $\frac{1}{3}\sin 3x + \frac{1}{4}\sin x + c$

(D) $\frac{1}{4}\left(\frac{1}{3}\cos 3x + 3\cos x\right) + c$

***Solution:*** We have

$$\int \cos^3 x\, dx = \frac{1}{4}\int (\cos 3x + 3\cos x) dx$$

$$= \frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c$$

**Answer: (B)**

### Example 4.24

$\int \frac{dx}{\sqrt{5+2x+x^2}}$ is equal to

(A) $\log_e((x+1)+\sqrt{x^2+2x+5})+c$

(B) $\text{Cosh}^{-1}\frac{x+1}{2}+c$

(C) $\log_e((x+1)-\sqrt{x^2+4})+c$

(D) $\text{Sinh}^{-1}\left(\frac{x+1}{4}\right)+c$

***Solution:*** We have

$$\int \frac{dx}{\sqrt{5+2x+x^2}} = \int \frac{dx}{\sqrt{2^2+(x+1)^2}}$$

$$= \log_e[(x+1)+\sqrt{x^2+2x+5}]+c$$

(See A.14)

**Answer: (A)**

### Example 4.25

If $\int \frac{dx}{x^2+2x-8} = \frac{1}{a}\log_e\left|\frac{x-2}{x+4}\right| + c$ then $a$ is equal to

(A) 4 (B) 2

(C) 6 (D) 1

***Solution:*** We have

$$\int \frac{dx}{x^2+2x-8} = \int \frac{dx}{(x+1)^2-3^2}$$

$$= \frac{1}{2(3)}\log_e\left|\frac{x+1-3}{x+1+3}\right| + c \quad \text{(See A.15)}$$

$$= \frac{1}{6}\log_e\left|\frac{x-2}{x+4}\right| + c$$

**Answer: (C)**

## 4.3 | Integration by Substitution

The formula we use is

$$\int f(g(x))g'(x)\,dx = \int f(t)\,dt$$

where $t = g(x)$. After evaluating $\int f(t)\,dt$, we return to the old variable $x$.

***Note:***

1. In some of the problems, where more number of substitutions are used, it is difficult to return to the old variable. In such cases we leave the answer in the final variable.
2. There are no fixed rules for substitution. One can get perfection only by practice.

The following three are important consequences.

**1.** $\int (f(x))^n f'(x)\,dx = \dfrac{(f(x))^{n+1}}{n+1} + c \quad (n \neq -1)$

**2.** $\int \dfrac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + c$

**3.** $\int \dfrac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

Before going for worked-out problems, we obtain *some more standard formulae*. In B.1–B.3 we will use point (2) above.

**B.1** 
$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int -\frac{\sin x}{\cos x} dx \\ &= -\log_e |\cos x| + c \\ &= \log_e |\sec x| + c\end{aligned}$$

**B.2** $\int \cot x\,dx = \int \dfrac{\cos x}{\sin x}\,dx = \log_e |\sin x| + c$

**B.3** 
$$\begin{aligned}\int \sec x\,dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \log_e |\sec x + \tan x| + c \\ &= \log_e \left|\frac{1+\sin x}{\cos x}\right| + c\end{aligned}$$

**B.4** $\int \text{cosec } x\,dx = \log_e |\text{cosec}\, x - \cot x| + c = \log_e \left|\tan \dfrac{x}{2}\right| + c$

We now use B.3 and B.4 in the evaluation of the following integrals. After this we would present more examples in "Single Answer Type Questions."

## Illustration 1

Evaluate $\int \dfrac{dx}{a\sin x + b\cos x}$.

***Solution:*** Let

$$I = \int \frac{dx}{a\sin x + b\cos x}$$

Put $a = r \cos\,\alpha$, $b = r \sin\,\alpha$ so that $r = \sqrt{a^2 + b^2}$ and $\alpha$ is given by the equations $\cos\alpha = a/r$ and $\sin\alpha = b/r$. Therefore

$$\begin{aligned}I &= \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)} \\ &= \frac{1}{r}\int \text{cosec}\,(x+\alpha)\,dx \\ &= \frac{1}{\sqrt{a^2+b^2}} \log_e \left|\tan\left(\frac{x}{2} + \frac{\alpha}{2}\right)\right| + c\end{aligned}$$

***Note:***

**1.** If we put $a = r\sin\,\alpha$ and $b = r\cos\,\alpha$, then the integral will be transformed to $(1/r)\int \sec(x-\alpha)\,dx$.

**2.** Another method of evaluation of the integral is: Write $\sin x$ and $\cos x$ in terms of $\tan x/2$ and then substitute $t = \tan(x/2)$.

**3.** To evaluate $\int \frac{dx}{a\sin x + b\cos x + c}$, the only method is the substitution $t = \tan(x/2)$.

## Illustration 2

Evaluate $I = \int \frac{dx}{\sin(x-a)\sin(x-b)} \quad (a \neq b)$

***Solution:*** We have

$$I = \int \frac{\sin[(x-a)-(x-b)]}{\sin(b-a)\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(b-a)}\int [\cot(x-b)-\cot(x-a)]\,dx$$

$$= \frac{1}{\sin(b-a)}\log_e\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + c \quad \text{(See B.2)}$$

***Note:*** In a similar way,

$$\int \frac{dx}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(b-a)}\int [\tan(x-a)-\tan(x-b)]\,dx$$

$$= \frac{1}{\sin(b-a)}\log_e\left|\frac{\sec(x-a)}{\sec(x-b)}\right| + c$$

### 4.3.1 Examples on Integration by Substitutions

## Example 4.26

If $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = a\sin\sqrt{x} + c$, then $a$ equals

(A) $\sqrt{2}$ (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) 2

***Solution:*** Let

$$I = \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

Put $t = \sqrt{x}$ so that

$$dt = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2t}\,dx$$

$$\Rightarrow dx = 2t\,dt$$

So

$$I = \int \frac{\cos t}{t}(2t)\,dt$$

$$= 2\int \cos t\,dt$$

$$= 2\sin t + c$$

$$= 2\sin\sqrt{x} + c$$

**Answer: (D)**

## Example 4.27

$\int (x^4+1)^{1/3}x^7\,dx = k(x^4+1)^{4/3}(4x^4-3)+c$, where $k$ is equal to

(A) $\frac{1}{112}$ (B) $\frac{1}{28}$

(C) $\frac{3}{112}$ (D) $\frac{1}{56}$

***Solution:*** Let

$$I = \int (x^4+1)^{1/3}x^7\,dx$$

Put $x^4 + 1 = t^3$ so that

$$4x^3\,dx = 3t^2\,dt$$

Therefore,

$$I = \int t(t^3-1)\frac{3}{4}t^2\,dt$$
$$= \frac{3}{4}\int (t^6 - t^3)\,dt$$
$$= \frac{3}{4}\left(\frac{t^7}{7} - \frac{t^4}{4}\right) + c$$
$$= \frac{3}{112}(t^4)(4t^3-7)+c$$
$$= \frac{3}{112}(x^4+1)^{4/3}[4(x^4+1)-7]+c$$
$$= \frac{3}{112}(x^4+1)^{4/3}(4x^4-3)+c$$

**Answer: (C)**

## Example 4.28

$\int x^2\sqrt{1-x}\,dx = -\frac{1}{51}(1-x)^{3/2}f(x)+c$ where $f(x)$ is equal to

(A) $15x^2+12x+8$ (B) $15x^2+12x+4$

(C) $15x^2-12x+8$ (D) $15x^2-12x+4$

***Solution:*** Put $\sqrt{1-x} = t$ so that $x = 1-t^2$ and $dx = -2t\,dt$. Therefore

$$I = \int x^2\sqrt{1-x}\,dx$$
$$= \int (1-t^2)^2(-2t^2)\,dt$$
$$= -2\int (t^6 - 2t^4 + t^2)\,dt$$
$$= -2\left[\frac{t^7}{7} - \frac{2}{5}t^5 + \frac{1}{3}t^3\right]+c$$
$$= -\frac{2}{105}(15t^7 - 42t^5 + 35t^3)+c$$
$$= -\frac{1}{51}[15(1-x)^{7/2} - 42(1-x)^{5/2} + 35(1-x)^{3/2}]+c$$
$$= -\frac{1}{51}(1-x)^{3/2}[15(1-x)^2 - 42(1-x)+35]+c$$
$$= -\frac{1}{51}(1-x)^{3/2}[15x^2+12x+8]+c$$

Therefore

$$f(x) = 15x^2+12x+8$$

**Answer: (A)**

## Example 4.29

A particle is moving in a straight line with acceleration $a = \sin 2t + t^2$ feet/s$^2$. At time $t = 0$, its velocity is 3 feet/s. The distance travelled by particle between $t = 0$ and $t = \pi/2$ is

(A) $\frac{\pi}{192}(\pi^3+336)$ ft (B) $\frac{\pi}{288}(\pi^3+192)$ ft

(C) $\pi\left(\frac{\pi^3+288}{192}\right)$ ft (D) $\frac{\pi^3+192}{288}$ ft

***Solution:*** Let $v$ denote the velocity of the particle at time $t$. Therefore

$$\frac{dv}{dt} = a = \sin 2t + t^2$$
$$\Rightarrow v = -\frac{1}{2}\cos 2t + \frac{1}{3}t^3 + c_1$$

Therefore

$$3 = v(0) = -\frac{1}{2} + c_1$$
$$\Rightarrow c_1 = \frac{7}{2}$$

Therefore

$$v = -\frac{1}{2}\cos 2t + \frac{1}{3}t^3 + \frac{7}{2}$$

Hence at $t = \pi/2$,

$$v\left(\frac{\pi}{2}\right) = \frac{1}{2} + \frac{\pi^3}{24} + \frac{7}{2} = 4 + \frac{\pi^3}{24} \tag{4.2}$$

Now

$$s = \int v\,dt = \int\left(-\frac{1}{2}\cos 2t + \frac{1}{3}t^3 + \frac{7}{2}\right)dt$$

$$= -\frac{1}{4}\sin 2t + \frac{1}{12}t^4 + \frac{7}{2}t + c_2$$

So

$$s(0) = c_2$$

and $$s\left(\frac{\pi}{2}\right) = -\frac{1}{4}\sin\pi + \frac{1}{12}\left(\frac{\pi}{2}\right)^4 + \frac{7}{2}\left(\frac{\pi}{2}\right) + c_2$$

Therefore

$$s\left(\frac{\pi}{2}\right) - s(0) = \frac{1}{12}\left(\frac{\pi}{2}\right)^4 + \frac{7\pi}{4}$$

$$= \frac{\pi^4 + 336\pi}{192}$$

**Answer: (A)**

## Example 4.30

$\int \sec^5 x \tan x \, dx$ is equal to

(A) $\frac{1}{5}\tan^5 x + c$ (B) $\frac{1}{5}\sec^5 x + c$

(C) $\frac{1}{5}\cos^5 x + c$ (D) $\frac{1}{5}\cot^5 x + c$

***Solution:*** Let $I = \int \sec^5 x \tan x \, dx$. Put $\sec x = t$ so that

$$(\sec x \tan x)\, dx = dt$$

Therefore

$$I = \int \sec^4 x(\sec x \tan x)\, dx$$

$$= \int t^4 \, dt$$

$$= \frac{1}{5}t^5 + c$$

$$= \frac{1}{5}\sec^5 x + c$$

**Answer: (B)**

## Example 4.31

If $\int (x^4 + x^2 + 1)^{49}(2x^3 + x)\, dx = \frac{1}{a}(x^4 + x^2 + 1)^{50} + c$, then $a$ is equal to

(A) 50 (B) 49

(C) 100 (D) 98

***Solution:*** Let

$$I = \int (x^4 + x^2 + 1)^{49}(2x^3 + x)\, dx$$

Put $x^4 + x^2 + 1 = t$. Therefore

$$2(2x^3 + x)\, dx = dt$$

Hence

$$I = \int t^{49}\left(\frac{1}{2}\right) dt$$

$$= \frac{1}{2} \times \frac{t^{50}}{50} + c$$

$$= \frac{1}{100}(x^4 + x^2 + 1)^{50} + c$$

This implies $a = 100$.

**Answer: (C)**

## Example 4.32

$\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx \; (x > 0)$ equals

(A) $\frac{1}{6(m+1)}(2x^{3m} + 3x^{2m} + 6x^m)^{(1+m)/m} + c$

(B) $\frac{1}{m+1}(2x^{2m} + 3x^m + 6)^{(1+m)/m} + c$

(C) $\frac{1}{m+1}(x^{3m} + x^{2m} + x^m)^{(1+m)/m} + c$

(D) $\frac{1}{6m+1}(x^{3m} + x^{2m} + x^m)^{(1+m)/m} + c$

**(IIT-JEE 2002)**

***Solution:*** Let

$$I = \int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m}\,dx$$
$$= \int (x^{2m} + x^m + 1)(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} x^{m-1}\,dx$$

Put $2x^{3m} + 3x^{2m} + 6x^m = t$. Therefore

$$6m(x^{3m-1} + x^{2m-1} + x^{m-1})\,dx = dt$$
$$6m(x^{2m} + x^m + 1)\,x^{m-1}dx = dt$$

So

$$I = \int t^{1/m}\left(\frac{1}{6m}\right)dt$$
$$= \frac{1}{6m}\,\frac{t^{(1/m)+1}}{(1/m)+1} + c$$
$$= \frac{1}{6(m+1)}(2x^{3m} + 3x^{2m} + 6x^m)^{(1+m)/m} + c$$

**Answer: (A)**

## Example 4.33

$$\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}} =$$

(A) $\dfrac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (B) $\dfrac{\sqrt{2x^4-2x^2+1}}{x^3}+c$

(C) $\dfrac{\sqrt{2x^4-2x^2+1}}{x}+c$ (D) $\dfrac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

**(IIT-JEE 2006)**

***Solution:*** Let

$$I = \int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$$
$$= \int \frac{x^2-1}{x^5\sqrt{2-\dfrac{2}{x^2}+\dfrac{1}{x^4}}}\,dx$$

Put $2-\dfrac{2}{x^2}+\dfrac{1}{x^4} = t^2$. Then

$$\left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx = 2t\,dt$$
$$\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx = \frac{1}{2}t\,dt$$

Therefore

$$I = \int \frac{\dfrac{1}{x^3}-\dfrac{1}{x^5}}{\sqrt{2-\dfrac{2}{x^2}+\dfrac{1}{x^4}}}\,dx$$
$$= \int \frac{1}{t}\cdot\left(\frac{1}{2}t\right)dt$$
$$= \frac{1}{2}t + c$$
$$= \frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}} + c$$
$$= \frac{\sqrt{2x^4-2x^2+1}}{2x^2} + c$$

**Answer: (D)**

## Example 4.34

Let $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ where $n \ge 2$ is integer and

$$g(x) = \underbrace{(f\circ f\circ\cdots\circ f)(x)}_{f \text{ occurs } n \text{ times}}$$

Then $\int x^{n-2} g(x)\,dx$ equals

(A) $\dfrac{1}{n(n-1)}(1+nx^n)^{1-(1/n)}+k$

(B) $\dfrac{1}{n-1}(1+nx^n)^{1-(1/n)}+k$

(C) $\dfrac{1}{n(n+1)}(1+nx^n)^{1+(1/n)}+k$

(D) $\dfrac{1}{n+1}(1+nx^n)^{1+(1/n)}+k$

**(IIT-JEE 2007)**

***Solution:*** We have

$$(f\circ f)(x) = f(f(x))$$
$$= f\left(\frac{x}{(1+x^n)^{1/n}}\right)$$

$$= \frac{\dfrac{x}{(1+x^n)^{1/n}}}{\left(1+\dfrac{x^n}{(1+x^n)}\right)^{1/n}}$$

$$= \frac{x}{(1+2x^n)^{1/n}}$$

By induction, we can show that

$$\underbrace{(f \circ f \circ \cdots \circ f)(x)}_{f \text{ occurs } m \text{ times}} = \frac{x}{(1+mx^n)^{1/n}}, \quad m \geq 2$$

Therefore

$$g(x) = \frac{x}{(1+nx^n)^{1/n}} \tag{4.3}$$

Let

$$I = \int x^{n-2} g(x)\, dx$$

$$= \int x^{n-2} \cdot \left[\frac{x}{(1+nx^n)^{1/n}}\right] dx \quad \text{[By Eq. (4.3)]}$$

$$= \int \frac{x^{n-1}}{(1+nx^n)^{1/n}}\, dx \tag{4.4}$$

Put $1 + nx^n = t$. Therefore

$$n^2 x^{n-1}\, dx = dt$$

From Eq. (4.4), we have

$$I = \int t^{-1/n} \left(\frac{1}{n^2}\right) dt$$

$$= \frac{1}{n^2} \frac{t^{-(1/n)+1}}{-(1/n)+1} + c$$

$$= \frac{t^{1-(1/n)}}{n(n-1)} + c$$

$$= \frac{(1+nx^n)^{1-(1/n)}}{n(n-1)} + c$$

**Answer: (A)**

## Example 4.35

Let

$$I = \int \frac{e^x}{e^{4x}+e^{2x}+1}\, dx, \quad J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\, dx$$

Then for any arbitrary constant $c$, the value of $J - I$ equals

(A) $\dfrac{1}{2}\log\left(\dfrac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$

(B) $\dfrac{1}{2}\log\left(\dfrac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

(C) $\dfrac{1}{2}\log\left(\dfrac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$

(D) $\dfrac{1}{2}\log\left(\dfrac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+c$

**(IIT-JEE 2008)**

***Solution:*** We have

$$J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}\, dx = \int \frac{e^{3x}}{e^{4x}+e^{2x}+1}\, dx$$

Therefore

$$J - I = \int \frac{e^{3x}-e^x}{e^{4x}+e^{2x}+1}\, dx$$

$$= \int \frac{(e^{2x}-1)\, e^x}{e^{4x}+e^{2x}+1}\, dx$$

$$= \int \frac{t^2-1}{t^4+t^2+1}\, dt \quad \text{where } t = e^x$$

$$= \int \frac{1-\dfrac{1}{t^2}}{t^2+\dfrac{1}{t^2}+1}\, dt$$

$$= \int \frac{dz}{z^2-1} \quad \text{where } z = t + \frac{1}{t}$$

$$= \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c \quad \text{(See A.15)}$$

$$= \frac{1}{2}\log\left|\frac{t+\dfrac{1}{t}-1}{t+\dfrac{1}{t}+1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right| + c$$

**Answer: (C)**

## Example 4.36

$$\int \frac{dx}{1+e^x} =$$

(A) $\log_e\left(\frac{e^x+1}{e^x}\right)+c$ (B) $\log_e\left(\frac{e^x}{e^x+1}\right)+c$

(C) $x = \log_e(e^x+1)+c$ (D) $e^x + x + c$

***Solution:*** We have

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$= -\int \frac{-e^{-x}}{e^{-x}+1}\,dx$$

$$= -\log_e(e^{-x}+1)+c \quad \text{(See B.2)}$$

$$= -\log_e\left(\frac{1+e^x}{e^x}\right)+c$$

$$= \log_e\left(\frac{e^x}{1+e^x}\right)+c$$

**Answer: (B)**

## Example 4.37

$$\int \frac{x^2-1}{(x^4+3x^2+1)\,\mathrm{Tan}^{-1}\left(\frac{x^2+1}{x}\right)}\,dx \text{ is equal to}$$

(A) $\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)+c$

(B) $\log_e\left(\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)\right)+c$

(C) $\log_e\left(\tan\left(\frac{x^2+1}{x}\right)\right)+c$

(D) $\left(x+\frac{1}{x}\right)\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)+c$

***Solution:*** Let $I$ be the given integral. Then

$$I = \int \frac{x^2-1}{[(x^2+1)^2+x^2]\,\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)}\,dx$$

Dividing both numerator and denominator by $x^2$ we get

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)}\,dx$$

Put $\mathrm{Tan}^{-1}[x+(1/x)] = t$. Then

$$\frac{1-(1/x^2)}{\left(x+\frac{1}{x}\right)^2+1}\,dx = dt$$

Therefore

$$I = \int \frac{dt}{t} = \log_e|t|+c$$

$$= \log_e\left(\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)\right)+c$$

**Answer: (B)**

## Example 4.38

$$\int \frac{\log_e x}{x\sqrt{1+\log_e x}}\,dx =$$

(A) $(1+\log_e x)^{3/2}+c$

(B) $\frac{2}{3}(1+\log_e x)(\log_e x-2)+c$

(C) $\frac{2}{3}(1+\log_e x)^{1/2}(\log_e x-5)+c$

(D) $\frac{2}{3}(1+\log_e x)^{1/2}(\log_e x-2)+c$

***Solution:*** Let

$$I = \int \frac{\log_e x}{x\sqrt{1+\log_e x}}\,dx$$

Put $\sqrt{1+\log_e x} = t$. Therefore

$$\log_e x = t^2-1 \quad \text{and} \quad \frac{1}{x}dx = (2t)\,dt$$

So

$$I = \int \frac{t^2-1}{t}(2t)\,dt$$

$$= 2\int (t^2-1)\,dt$$

$$= 2\left(\frac{1}{3}t^3 - t\right) + c$$

$$= \frac{2t}{3}(t^2 - 3) + c$$

$$= \frac{2}{3}(1+\log_e x)^{1/2}(1+\log_e x - 3)$$

$$= \frac{2}{3}(1+\log_e x)^{1/2}(\log_e x - 2) + c$$

**Answer: (D)**

## Example 4.39

The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}\,dx$ is

(A) $\sin x - 6\,\mathrm{Tan}^{-1}(\sin x) + c$

(B) $\sin x - 2(\sin x)^{-1} + c$

(C) $\sin x - 2(\sin x)^{-1} - 6\,\mathrm{Tan}^{-1}(\sin x) + c$

(D) $\sin x - 2(\sin x)^{-1} + 5\,\mathrm{Tan}^{-1}(\sin x) + c$

***Solution:*** Let

$$I = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}\,dx$$

$$= \int \frac{[(1-\sin^2 x) + (1-\sin^2 x)^2]\cos x}{\sin^2 x + \sin^4 x}\,dx$$

$$= \int \frac{(2 - 3\sin^2 x + \sin^4 x)\cos x}{\sin^2 x + \sin^4 x}\,dx$$

$$= \int \frac{2 - 3t^2 + t^4}{t^2 + t^4}\,dt \quad \text{where } t = \sin x$$

$$= \int \frac{t^4 - 3t^2 + 2}{t^2(t^2+1)}\,dt$$

$$= \int \left(1 + \frac{2}{t^2} - \frac{6}{t^2+1}\right)dt \quad \text{(See Partial Fractions, Vol. 1)}$$

$$= t - \frac{2}{t} - 6\,\mathrm{Tan}^{-1} t + c$$

$$= \sin x - \frac{2}{\sin x} - 6\,\mathrm{Tan}^{-1}(\sin x) + c$$

**Answer: (C)**

## Example 4.40

$$\int \frac{x-1}{(x+1)\sqrt{x(x^2+x+1)}}\,dx \text{ is}$$

(A) $\mathrm{Tan}^{-1}\left(\frac{x^2+x+1}{x}\right) + c$

(B) $2\mathrm{Tan}^{-1}\left(\frac{x^2+x+1}{x}\right) + c$

(C) $\mathrm{Tan}^{-1}\left(\frac{\sqrt{x^2+x+1}}{x}\right) + c$

(D) $2\mathrm{Tan}^{-1}\sqrt{x + \frac{1}{x} + 1} + c$

***Solution:*** Let

$$I = \int \frac{x-1}{(x+1)\sqrt{x(x^2+x+1)}}\,dx$$

$$= \int \frac{x^2-1}{(x+1)^2\sqrt{x(x^2+x+1)}}\,dx$$

$$= \int \frac{x^2-1}{(x^2+2x+1)\sqrt{x(x^2+x+1)}}\,dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x} + 2\right)\sqrt{x + \frac{1}{x} + 1}}\,dx$$

Put $x + (1/x) + 1 = t^2$ so that

$$\left(1 - \frac{1}{x^2}\right)dx = (2t)\,dt$$

Therefore

$$I = \int \frac{1}{(t^2+1)\,t}(2t)\,dt$$

$$= 2\int \frac{dt}{1+t^2}$$

$$= 2\,\mathrm{Tan}^{-1}(t) + c \quad \text{(See A.10)}$$

$$= 2\,\mathrm{Tan}^{-1}\sqrt{x + \frac{1}{x} + 1} + c$$

**Answer: (D)**

## Example 4.41

If $\int \frac{x+1}{x(1+xe^x)^2}\,dx = \log_e \left|\frac{xe^x}{1+xe^x}\right| + f(x) + c$, then $f(x)$ is

(A) $\frac{1}{1+xe^x}$ (B) $\frac{x}{1+xe^x}$

(C) $\frac{xe^x}{1+x}$ (D) $\frac{xe^x}{1+e^x}$

***Solution:*** Let

$$I = \int \frac{x+1}{x(1+xe^x)^2}\,dx$$

Put $1 + xe^x = t$. Therefore

$$e^x(x+1)\,dx = dt$$

So,

$$I = \int \frac{e^x(x+1)}{xe^x(1+xe^x)^2}\,dx$$

$$= \int \frac{dt}{(t-1)\,t^2}$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t} - \frac{1}{t^2}\right) dt \quad \text{(By Partial Fractions)}$$

$$= \log_e |t-1| - \log_e |t| + \frac{1}{t} + c$$

$$= \log_e |xe^x| - \log_e |1+xe^x| + \frac{1}{1+xe^x} + c$$

$$= \log_e \left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + c$$

So

$$f(x) = \frac{1}{1+xe^x}$$

**Answer: (A)**

## Example 4.42

$$\int \frac{(x-x^3)^{1/3}}{x^4}\,dx =$$

(A) $\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3} + c$ (B) $-\frac{3}{8}\left(\frac{1}{x^2}+1\right)^{4/3} + c$

(C) $-\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3} + c$ (D) $-\frac{3}{4}\left(1-\frac{1}{x^2}\right)^{4/3} + c$

***Solution:*** We have

$$I = \int \frac{(x-x^3)^{1/3}}{x^4}\,dx$$

$$= \int \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}\,dx$$

Put

$$\frac{1}{x^2} - 1 = t^3$$

$$\Rightarrow -\frac{2}{x^3}\,dx = 3\,t^2\,dt$$

$$\Rightarrow -\frac{1}{x^3}\,dx = -\frac{3}{2}\,t^2\,dt$$

Therefore

$$I = \int t\left(-\frac{3}{2}t^2\right) dt$$

$$= -\frac{3}{2}\int t^3\,dt$$

$$= -\frac{3}{2} \times \frac{1}{4} t^4 + c$$

$$= -\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3} + c$$

**Answer: (C)**

## Example 4.43

If $y(x-y)^2 = x$, then $\int \frac{dx}{x-3y}$ equals

(A) $\frac{x}{2}\log_e\{(x-y)^2-1\}+c$

(B) $\frac{1}{2}\log_e\{(x-y)^2-1\}+c$

(C) $x+\frac{1}{2}\log_e\{(x-y)^2+1\}+c$

(D) $\log_e\{(x-y)^2-1\}+c$

***Solution:*** Put $x - y = t$ so that $y(x-y)^2 = x$ becomes $(x-t)t^2 = x$. Therefore

$$x = \frac{t^3}{t^2-1} \Rightarrow dx = \frac{t^2(t^2-3)}{(t^2-1)^2}dt$$

Also

$$x-3y = \frac{t^3}{t^2-1} - 3(x-t)$$

$$= \frac{t^3}{t^2-1} - 3\left(\frac{t^3}{t^2-1} - t\right)$$

$$= \frac{t^3-3t}{t^2-1}$$

$$= \frac{t(t^2-3)}{t^2-1}$$

Hence

$$\int \frac{dx}{x-3y} = \int \frac{t^2-1}{t(t^2-3)} \cdot \frac{t^2(t^2-3)}{(t^2-1)^2}dt$$

$$= \int \frac{t}{t^2-1}dt$$

$$= \frac{1}{2}\log_e(t^2-1)+c$$

$$= \frac{1}{2}\log_e\{(x-y)^2-1\}+c$$

**Answer: (B)**

## Example 4.44

If $\int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2} \frac{dx}{x} = 2\text{Cos}^{-1}\sqrt{x} - \phi(x) + c$, then $\phi(x)$ equals

(A) $\log_e\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)$ (B) $\frac{1}{2}\log_e\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)$

(C) $2\log_e\left(\frac{1-\sqrt{1-x}}{\sqrt{x}}\right)$ (D) $2\log_e\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)$

***Solution:*** Let

$$I = \int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2} dx$$

Put $x = \cos^2 2\theta$ so that $dx = -2\sin 4\theta\, d\theta$. Therefore

$$I = \int \left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{1/2} \frac{1}{\cos^2 2\theta}(-2\sin 4\theta)\, d\theta$$

$$= \int \frac{\tan\theta}{\cos^2 2\theta}(-4\sin 2\theta\cos 2\theta)\, d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta\cos 2\theta}(-8\sin\theta\cos\theta)\, d\theta$$

$$= \int -\frac{8\sin^2\theta}{\cos 2\theta}\, d\theta$$

$$= -8\int \frac{1-\cos^2\theta}{2\cos^2\theta - 1}\, d\theta$$

$$= 4\int \frac{2\cos^2\theta-1-1}{2\cos^2\theta-1}\, d\theta$$

$$= 4\int\left(1-\frac{1}{\cos 2\theta}\right) d\theta$$

$$= 4\int (1-\sec 2\theta)\, d\theta$$

$$= 4\left[\theta - \frac{1}{2}\log_e|\sec 2\theta + \tan 2\theta|\right]+c \quad \text{(See B.3)}$$

$$= 4\left(\frac{1}{2}\text{Cos}^{-1}\sqrt{x}\right) - 2\log_e\left|\frac{1}{\sqrt{x}} + \frac{\sqrt{1-x}}{\sqrt{x}}\right| + c$$

$$(\because \sqrt{x} = \cos 2\theta)$$

$$= 2\text{Cos}^{-1}\sqrt{x} - 2\log_e\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)+c \quad (\because 0 < x < 1)$$

Therefore

$$\phi(x) = 2\log_e\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)$$

**Answer: (D)**

## Example 4.45

Let $I_1 = \int \frac{x^2+1}{(x^3+3x+1)^{1/3}}\,dx = \frac{1}{2}(x^3+3x+1)^{2/3} + c$

and $I_2 = \int \frac{\sin 2x}{1+\sin^2 x}\,dx = \log_e(1+\sin^2 x) + c$

Then

(A) both $I_1$ and $I_2$ are correct
(B) both $I_1$ and $I_2$ are not correct
(C) $I_1$ is correct and $I_2$ is not
(D) $I_2$ is correct and $I_1$ is not

***Solution:*** We have

$$I_1 = \int \frac{x^2+1}{(x^3+3x+1)^{1/3}}\,dx$$

Put $x^3 + 3x + 1 = t^3$. Therefore

$$3(x^2+1)\,dx = (3t^2)\,dt$$
$$(x^2+1)\,dx = t^2\,dt$$

This implies

$$\begin{aligned} I_1 &= \int \frac{1}{t}(t^2)\,dt \\ &= \frac{1}{2}t^2 + c \\ &= \frac{1}{2}(x^3+3x+1)^{2/3} + c \end{aligned}$$

So $I_1$ is correct. Now

$$\begin{aligned} I_2 &= \int \frac{\sin 2x}{1+\sin^2 x}\,dx \\ &= \int \frac{f'(x)}{f(x)}\,dx \quad \text{where } f(x) = 1+\sin^2 x \\ &= \log_e|f(x)| + c \quad \text{(See B.2)} \\ &= \log_e(1+\sin^2 x) + c \end{aligned}$$

Hence $I_2$ is also correct.

**Answer: (A)**

## Example 4.46

Which of the following are true?

(A) $\int \left(\log_e x + \frac{1}{\log_e x}\right)\frac{dx}{x} = \frac{1}{2}(\log_e x)^2 - \log_e x + c$

(B) $\int \frac{1+\log_e x}{3+x\log_e x}\,dx = \log_e|3+x\log_e x| + c$

(C) $\int \frac{\log_e x}{x}\,dx = \log_e|\log_e x| + c$

(D) $\int \frac{\log_{10} x}{x}\,dx = \frac{1}{2}\log_e|\log_e x| + c$

***Solution:*** Let

$$I_1 = \int \left(\log_e x + \frac{1}{\log_e x}\right)\frac{dx}{x}$$

Put $\log_e x = t$ so that $(1/x)dx = dt$. Therefore

$$\begin{aligned} I_1 &= \int \left(t + \frac{1}{t}\right)dt \\ &= \frac{1}{2}t^2 + \log_e|t| + c \\ &= \frac{1}{2}(\log_e x)^2 - \log_e|\log_e x| + c \end{aligned}$$

Therefore (A) is false. Let

$$I_2 = \int \frac{1+\log_e x}{3+x\log_e x}\,dx$$

Put $3 + x\log_e x = t$. Therefore

$$\left(\log_e x + x\left(\frac{1}{x}\right)\right)dx = dt$$
$$(1+\log_e x)dx = dt$$

So

$$\begin{aligned} I_2 &= \int \frac{dt}{t} \\ &= \log_e|t| + c \\ &= \log_e|3+x\log_e x| + c \end{aligned}$$

So (B) is true. Now let

$$I_3 = \int \frac{\log_e x}{x}\,dx$$

$$= \int t\,dt \quad \text{where } t = \log_e x$$
$$= \frac{1}{2}t^2 + c$$
$$= \frac{1}{2}(\log_e x)^2 + c$$

Hence (C) is false. Finally let

$$I_4 = \int \frac{\log_{10} x}{x}\,dx$$
$$= \int \frac{\log_e x \log_{10} e}{x}\,dx$$
$$= \frac{1}{2}(\log_e x)^2 \cdot \log_{10} e + c$$

So (D) is false.

**Answer: (B)**

## Example 4.47

$\int \frac{e^x - 1}{e^x + 1}\,dx = 2f(x) - x + c$ where $f$ equals

(A) $(e^x + 1)$ (B) $e^x$
(C) $e^x - 1$ (D) $\log_e(1 + e^x)$

***Solution:*** We have

$$\int \frac{e^x - 1}{e^x + 1}\,dx = \int \frac{(e^x - 1)\,e^x}{e^x\,(e^x + 1)}\,dx$$
$$= \int \frac{t-1}{t(t+1)}\,dt \quad \text{where } t = e^x$$
$$= \int \left(\frac{1}{t+1} - \frac{1}{t(t+1)}\right) dx$$
$$= \int \left(\frac{1}{t+1} - \left(\frac{1}{t} - \frac{1}{t+1}\right)\right) dt$$
$$= \int \left(\frac{2}{t+1} - \frac{1}{t}\right) dt$$
$$= 2\log_e|t+1| - \log_e|t| + c$$
$$= 2\log_e(1+e^x) - x + c \quad (\because t = e^x > 0)$$

**Answer: (D)**

## Example 4.48

$\int \frac{dx}{\sqrt{1+x^2}\sqrt{\log_e(x+\sqrt{1+x^2})}}$ is

(A) $2\sqrt{\log_e(x+\sqrt{1+x^2})} + c$

(B) $\sqrt{\log_e(x+\sqrt{1+x^2})} + c$

(C) $\frac{1}{2}\log_e(x+\sqrt{1+x^2}) + c$

(D) $\frac{1}{2}\sqrt{\log_e(x+\sqrt{1+x^2})} + c$

***Solution:*** Let

$$I = \int \frac{dx}{\sqrt{1+x^2}\sqrt{\log_e(x+\sqrt{1+x^2})}}$$

Put $\log_e(x+\sqrt{1+x^2}) = t^2$. Then

$$\frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) dx = (2t)\,dt$$
$$\frac{1}{\sqrt{1+x^2}}\,dx = (2t)\,dt$$

Therefore

$$I = \int \frac{1}{t}(2t)\,dt = 2t + c = 2\log_e(x+\sqrt{1+x^2}) + c$$

**Answer: (A)**

## Example 4.49

If $y = \sqrt{x^2 - x + 1}$ and for $n \ge 1$, $I_n = \int x^n/y\,dx$ and $aI_3 + bI_2 + cI_1 = x^2 y$, then $(a, b, c)$ is equal to

(A) $\left(\frac{3}{2}, \frac{1}{2}, -1\right)$ (B) $(1, -1, 1)$

(C) $\left(3, -\frac{5}{2}, 2\right)$ (D) $\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$

**Solution:** We have

$$x^2 y = a\int \frac{x^3}{y}\,dx + b\int \frac{x^2}{y}\,dx + c\int \frac{x}{y}\,dx$$

Differentiating both sides with respect $x$, we have

$$2xy + x^2 \frac{dy}{dx} = \frac{ax^3}{y} + \frac{bx^2}{y} + \frac{cx}{y}$$

$$2xy^2 + x^2 y\left(\frac{dy}{dx}\right) = ax^3 + bx^2 + cx \qquad (4.5)$$

But

$$y^2 = x^2 - x + 1 \Rightarrow y\frac{dy}{dx} = \frac{1}{2}(2x-1)$$

Substituting the value of $y^2 = x^2 - x + 1$ and $y\dfrac{dy}{dx} = \dfrac{1}{2}(2x-1)$ in Eq. (4.5) we obtain

$$2x(x^2 - x + 1) + x^2\frac{(2x-1)}{2} = ax^3 + bx^2 + cx$$

$$\Rightarrow 3x^3 - \frac{5}{2}x^2 + 2x = ax^3 + bx^2 + cx$$

Therefore

$$a = 3,\ b = -\frac{5}{2},\ c = 2$$

**Answer: (C)**

## Example 4.50

$\displaystyle\int \frac{\tan x}{1 + \tan x + \tan^2 x}\,dx$ is equal to

(A) $x + \dfrac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\dfrac{(2\tan x + 1)}{\sqrt{3}} + c$

(B) $x - \dfrac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\dfrac{2\tan x + 1}{\sqrt{3}}\right) + c$

(C) $x + \dfrac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\dfrac{\tan x + 1}{\sqrt{3}}\right) + c$

(D) $x - \dfrac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\dfrac{\tan x + 1}{\sqrt{3}}\right) + c$

**Solution:** Let

$$I = \int \frac{\tan x}{1 + \tan x + \tan^2 x}\,dx$$

$$= \int \frac{\tan x}{\sec^2 x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \tan x - \sec^2 x}{\sec^2 x + \tan x}\,dx$$

$$= \int \left(1 - \frac{\sec^2 x}{\tan x + \sec^2 x}\right)dx$$

$$= x - \int \frac{\sec^2 x}{1 + \tan x + \tan^2 x}\,dx$$

$$= x - \int \frac{dt}{1 + t + t^2} \quad \text{where } t = \tan x$$

$$= x - \int \frac{dt}{[t + (1/2)]^2 + (3/4)}$$

$$= x - \frac{1}{(\sqrt{3}/2)}\mathrm{Tan}^{-1}\left(\frac{t + (1/2)}{(\sqrt{3}/2)}\right) + c \qquad \text{(See A.10)}$$

$$= x - \frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + c$$

**Answer: (B)**

## Example 4.51

$\displaystyle\int \frac{\tan\left(\frac{\pi}{4} - x\right)\sec^2 x}{\sqrt{\tan^3 x + \tan^2 x + \tan x}}\,dx$ equals

(A) $-2\,\mathrm{Tan}^{-1}\sqrt{1 + \tan x + \cot x} + c$

(B) $2\,\mathrm{Tan}^{-1}\sqrt{\tan x + \cot x} + c$

(C) $\mathrm{Tan}^{-1}\sqrt{\tan x + \cot x} + c$

(D) $-2\,\mathrm{Tan}^{-1}\sqrt{\sec x + \tan x} + c$

**Solution:** Let

$$I = \int \frac{\tan\left(\frac{\pi}{4} - x\right)\sec^2 x}{\sqrt{\tan^3 x + \tan^2 x + \tan x}}\,dx$$

$$= \int \left(\frac{1 - \tan x}{1 + \tan x}\right)\frac{\sec^2 x}{\sqrt{\tan^3 x + \tan^2 + \tan x}}\,dx$$

$$= \int \frac{1 - t}{1 + t}\frac{dt}{\sqrt{t^3 + t^2 + t}}$$

$$= -2\,\mathrm{Tan}^{-1}\sqrt{1+t+\frac{1}{t}}+c \quad \text{(See Example 4.40)}$$

$$= -2\,\mathrm{Tan}^{-1}\sqrt{1+\tan x+\cot x}+c$$

**Answer: (A)**

## Example 4.52

$\displaystyle\int \frac{2+\sqrt{x}}{(x+\sqrt{x}+1)^2}\,dx$ equals

(A) $\dfrac{2}{x+\sqrt{x}+1}+c$ (B) $\dfrac{x}{x+\sqrt{x}+1}+c$

(C) $\dfrac{2x}{1+\sqrt{x}-x}+c$ (D) $\dfrac{2x}{1+\sqrt{x}+x}+c$

***Solution:*** Let

$$I = \int \frac{2+\sqrt{x}}{(x+\sqrt{x}+1)^2}\,dx$$

Dividing numerator and denominator by $x^2$ we get

$$I = \int \frac{(2/x^2)+(1/x\sqrt{x})}{\left(1+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}\right)^2}\,dx$$

Now, put $1+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}=t$. Therefore

$$\left(-\frac{1}{2}\cdot\frac{1}{x\sqrt{x}}-\frac{1}{x^2}\right)dx = dt$$

$$\Rightarrow \left(\frac{1}{x\sqrt{x}}+\frac{2}{x^2}\right)dx = -2\,dt$$

So

$$I = \int \frac{1}{t^2}(-2)\,dt$$

$$= \frac{2}{t}+c$$

$$= \frac{2}{1+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}}+c$$

$$= \frac{2x}{x+\sqrt{x}+1}+c$$

**Answer: (D)**

## Example 4.53

$\displaystyle\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}\,dx$ is equal to

(A) $\dfrac{x^2+2x}{(x^5+x^3+1)^2}+c$

(B) $\dfrac{x^{10}}{2(x^5+x^3+1)^2}+c$

(C) $\log_e(x^5+x^3+1)+\sqrt{2x^7+5x^4}+c$

(D) $\log_e(2x^7+5x^4+\sqrt{x^5+x^3+1})+c$

***Solution:*** Let

$$I = \int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}\,dx$$

Dividing numerator and denominator with $x^{15}$ we get

$$I = \int \frac{\left(\dfrac{2}{x^3}+\dfrac{5}{x^6}\right)}{\left(1+\dfrac{1}{x^2}+\dfrac{1}{x^5}\right)^3}\,dx$$

Put $1+\dfrac{1}{x^2}+\dfrac{1}{x^5}=t$. Then

$$\left(-\frac{2}{x^3}-\frac{5}{x^6}\right)dx = dt$$

So

$$I = \int \frac{1}{t^3}(-dt)$$

$$= \frac{1}{2t^2}+c$$

$$= \frac{1}{2}\left(\frac{1}{\left(1+\dfrac{1}{x^2}+\dfrac{1}{x^5}\right)^2}\right)+c$$

$$= \frac{x^{10}}{2(x^5+x^3+1)^2}+c$$

**Answer: (B)**

## Example 4.54

If

$$\int \frac{3x^4-1}{(x^4+x+1)^2}\,dx = \int f_1(x)\,dx - \int \frac{dx}{x^4+x+1} + c$$

then $f_1(x)$ is

(A) $\dfrac{4x^3+1}{(x^4+x+1)^2}$ (B) $\dfrac{x^3(4x+1)}{(x^4+x+1)^2}$

(C) $\dfrac{x(4x^3-1)}{(x^4+x+1)^2}$ (D) $\dfrac{x(4x^3+1)}{(x^4+x+1)^2}$

***Solution:*** We know that

$$\frac{d}{dx}(x^4+x+1) = 4x^3+1$$

Therefore

$$\int \frac{3x^4-1}{(x^4+x+1)^2}\,dx = \int \frac{x(4x^3+1)-(x^4+x+1)}{(x^4+x+1)^2}\,dx$$

$$= \int \frac{x(4x^3+1)}{(x^4+x+1)^2}\,dx - \int \frac{dx}{x^4+x+1}$$

Therefore

$$f_1(x) = \frac{x(4x^3+1)}{(x^4+x+1)^2}$$

**Answer: (D)**

## Example 4.55

$\displaystyle\int\left(\frac{x^2+\sin^2 x}{1+x^2}\right)\frac{dx}{\cos^2 x}$ is equal to

(A) $\tan x - \mathrm{Tan}^{-1} x + c$ (B) $\tan x + \mathrm{Sin}^{-1} x + c$

(C) $\tan x - \mathrm{Sec}^{-1} x + c$ (D) $\cot x - \mathrm{Cot}^{-1} x + c$

***Solution:*** We have

$$\int\left(\frac{x^2+\sin^2 x}{1+x^2}\right)\frac{dx}{\cos^2 x} = \int \frac{x^2+1-(1-\sin^2 x)}{1+x^2}\,\frac{dx}{\cos^2 x}$$

$$= \int \sec^2 x\,dx - \int \frac{dx}{1+x^2}$$

$$= \tan x - \mathrm{Tan}^{-1} x + c$$

**Answer: (A)**

## Example 4.56

If $f:\mathbb{R}\to\mathbb{R}$ is a function such that $f(0)=0, f'(0)=3$ and it satisfies the relation

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$$

for all $x, y \in \mathbb{R}$ then $\int f(x)\,dx$ equals

(A) $3x^2+3$ (B) $\dfrac{2x^2}{3}+3$

(C) $\dfrac{3}{2}x^2+c$ (D) $3x^2+c$

***Solution:*** Given that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$$

Substituting $y=0$ and replacing $x$ with $3x$, we have

$$f(x) = \frac{f(3x)+f(0)}{3} = \frac{f(3x)}{3} \quad [\because f(0)=0]$$

$$3f(x) = f(3x) \tag{4.6}$$

Now using Eq. (4.6) we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(3\dfrac{(x+h)}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{3f\left(\dfrac{x+h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{3\left(\dfrac{f(x)+f(h)}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)-f(0)}{h} \quad [\because f(0)=0]$$

$$= f'(0) = 3$$

Therefore $f$ is differentiable at any real $x$ and $f'(x) = f'(0) = 3$. Hence

$$f(x) = 3x + c$$

and $\qquad f(0) = 0 \Rightarrow c = 0$

So $f(x) = 3x$ and hence

$$\int f(x)\,dx = \frac{3}{2}x^2 + c$$

**Answer: (C)**

## Example 4.57

$$\int \frac{x^4+1}{x^6+1}\,dx \text{ is equal to}$$

(A) $\mathrm{Tan}^{-1}x - \frac{1}{3}\mathrm{Tan}^{-1}(x^3) + c$

(B) $\mathrm{Tan}^{-1}x + \frac{\mathrm{Tan}^{-1}(x^3)}{3} + c$

(C) $\mathrm{Tan}^{-1}x + \frac{\mathrm{Tan}^{-1}(x^2)}{2} + c$

(D) $\mathrm{Tan}^{-1}x - \frac{\mathrm{Tan}^{-1}(x^2) + c}{2}$

***Solution:*** We have

$$\int \frac{x^4+1}{x^6+1}\,dx = \int \frac{x^4+1}{(x^2+1)(x^4-x^2+1)}\,dx$$

$$= \int \frac{(x^4-x^2+1)+x^2}{(x^2+1)(x^4-x^2+1)}\,dx$$

$$= \int \frac{dx}{x^2+1} + \int \frac{x^2}{x^6+1}\,dx$$

$$= \mathrm{Tan}^{-1}x + \frac{1}{3}\int \frac{3x^2}{(x^3)^2+1}\,dx$$

$$= \mathrm{Tan}^{-1}x + \frac{1}{3}\int \frac{dt}{t^2+1} \quad \text{where } t = x^3$$

$$= \mathrm{Tan}^{-1}x + \frac{1}{3}\mathrm{Tan}^{-1}t + c$$

$$= \mathrm{Tan}^{-1}x + \frac{1}{3}\mathrm{Tan}^{-1}(x^3) + c$$

**Answer: (B)**

## Example 4.58

$$\int \frac{\sin^3 x dx}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}} \text{ equals}$$

(A) $\mathrm{Cos}^{-1}(\sec x + \cos x) + c$

(B) $\mathrm{Sin}^{-1}(\sec x + \cos x) + c$

(C) $\mathrm{Sec}^{-1}(\sec x + \cos x) + c$

(D) $\mathrm{Tan}^{-1}(\sec x + \tan x) + c$

***Solution:*** Let

$$I = \int \frac{\sin^3 x}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}}\,dx$$

Dividing numerator and denominator by $\cos^2 x$ we get

$$I = \int \frac{\sin^3 x \sec^2 x}{(\sec x+\cos x)\sqrt{\sec^2 x+1+\cos^2 x}}\,dx$$

Put $\sec x + \cos x = t$. Therefore

$$(\sec x \tan x - \sin x)\,dx = dt$$

$$\left(\frac{\sin x}{\cos^2 x} - \sin x\right)dx = dt$$

$$\frac{\sin x(1-\cos^2 x)}{\cos^2 x}\,dx = dt$$

$$\sin^3 x \sec^2 x\,dx = dt$$

Hence

$$I = \int \frac{dt}{t\sqrt{t^2-1}}$$

$$= \mathrm{Sec}^{-1}t + c$$

$$= \mathrm{Sec}^{-1}(\sec x + \cos x) + c$$

**Answer: (C)**

## Example 4.59

$$\int \frac{(1+x)\sin x}{(x^2+2x)\cos^2 x - (1+x)\sin 2x}\,dx = \frac{1}{2}\log_e\left|\frac{t+1}{t-1}\right| + c$$

where $t$ is

(A) $(x+1)\cos x - \sin x$ (B) $(x+1)\sin x - \cos x$

(C) $(x+1)\sin x + \cos x$ (D) $(x+1)\cos x + \sin x$

***Solution:*** Let $I$ be the given integral. Now

$$I = \int \frac{(1+x)\sin x}{(x+1)^2\cos^2 x - \cos^2 x + (1+x)(2\sin x\cos x)}\,dx$$
$$= \int \frac{(1+x)\sin x}{(x+1)^2\cos^2 x + \sin^2 x - 1 - 2(1+x)\sin x\cos x}\,dx$$
$$= \int \frac{(1+x)\sin x}{[(x+1)\cos x - \sin x]^2 - 1}\,dx$$

Put $(x+1)\cos x - \sin x = t$. Therefore

$$[\cos x + (x+1)(-\sin x) - \cos x]dx = dt$$
$$-(1+x)\sin x\,dx = dt$$

Hence

$$I = \int \frac{1}{t^2-1}(-1)\,dt$$
$$= \frac{1}{2}\int\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt$$
$$= \frac{1}{2}\log_e\left|\frac{1+t}{1-t}\right| + c$$

**Answer: (A)**

## Example 4.60

If $f$ is a real-valued function satisfying the relation

$$5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$$

for all $x \neq 0$, then $\int xf(x)\,dx$ is

(A) $\frac{1}{6}\left(\frac{5}{3}x^3 + 2x^2 + 3x\right) + c$

(B) $\frac{1}{16}\left(\frac{5}{3}x^3 - 2x^2 + 3x\right) + c$

(C) $\frac{1}{16}\left(\frac{5x^3}{3} + 2x^2 - 3x\right) + c$

(D) $\frac{1}{16}\left(\frac{5}{3}x^3 - 2x^2 - 3x\right) + c$

***Solution:*** We have

$$5f(x) + 3f\left(\frac{1}{x}\right) = x + 2 \tag{4.7}$$

Therefore

$$5f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x} + 2 \tag{4.8}$$

Solving Eqs. (4.7) and (4.8), we get

$$f(x) = \frac{1}{16}\left(5x - \frac{3}{x} + 4\right)$$

So

$$\int xf(x)\,dx = \frac{1}{16}\int (5x^2 + 4x - 3)\,dx$$
$$= \frac{1}{16}\left(\frac{5}{3}x^3 + 2x^2 - 3x\right) + c$$

**Answer: (C)**

## Example 4.61

For $0 < \theta < \pi/2$, if

$$f(\theta) = \sin^3\theta + \sin^3\theta\cos^2\theta + \sin^3\theta\cos^4\theta + \cdots + \infty$$

then $\int e^{\cos\theta} f(\theta)\,d\theta$ is equal to

(A) $-e^{\cos\theta}\sin\theta + c$ (B) $e^{\sin\theta}\cos\theta + c$
(C) $-e^{\cos\theta} + c$ (D) $e^{\cos\theta}(\sin\theta + \cos\theta) + c$

***Solution:*** We have

$$0 < \theta < \pi/2 \Rightarrow 0 < \cos^2\theta < 1$$

Since $f(\theta)$ is a sum of an infinite Geometric series with first term $\sin^3\theta$ and common ratio $\cos^2\theta$ $(0 < \cos^2\theta < 1)$, we have

$$f(\theta) = \frac{\sin^3\theta}{1-\cos^2\theta} = \sin\theta$$

Therefore

$$\int e^{\cos\theta} f(\theta) = \int e^{\cos\theta}\sin\theta\,d\theta$$
$$= \int e^t(-1)\,dt \quad \text{where } t = \cos\theta$$
$$= -e^t + c$$
$$= -e^{\cos\theta} + c$$

**Answer: (C)**

## Example 4.62

$\int \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx \ (x > 1)$ is equal to

(A) $\log_e(x^2+1) + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right) + c$

(B) $\log_e\left(\frac{x^2+1}{x^2-1}\right) + \log_e\left(\frac{x-1}{x+1}\right) + c$

(C) $\log_e\left(\frac{x^2+1}{x^2-1}\right) + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right) + c$

(D) $\log_e(x^2+1) + \frac{1}{x^2+1} + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right) + c$

***Solution:*** Let $I$ be the given integral. Then

$$I = \int \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2(x^2-1)} dx$$

$$= \int \frac{2x^3}{(x^2+1)^2} dx + \int \frac{dx}{x^2-1}$$

$$= \int \frac{x^2(2x)}{(x^2+1)^2} dx + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right)$$

$$= \int \frac{t-1}{t^2} dt + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right) \quad \text{where } t = x^2+1$$

$$= \int\left(\frac{1}{t} - \frac{1}{t^2}\right)dt + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right)$$

$$= \log_e(x^2+1) + \frac{1}{x^2+1} + \frac{1}{2}\log_e\left(\frac{x-1}{x+1}\right) + c$$

**Answer: (D)**

## Example 4.63

If $f(x) = \sqrt{x}$ and $g(x) = e^x - 1$, then $\int (f \circ g)(x)\, dx$ equals

(A) $2\sqrt{e^x - 1} - 2\,\mathrm{Tan}^{-1}(\sqrt{e^x-1}) + c$

(B) $2\,(e^x - 1) - 2\,\mathrm{Tan}^{-1}(\sqrt{e^x-1}) + c$

(C) $2\sqrt{e^x - 1} - 2\tan(\sqrt{e^x-1}) + c$

(D) $2\sqrt{e^x - 1} - 2\,\mathrm{Tan}^{-1}(e^x-1) + c$

***Solution:*** We have

$$\int (f \circ g)(x)\, dx = \int f(g(x))\, dx$$

$$= \int \sqrt{e^x - 1}\, dx$$

$$= \int \frac{2t^2}{1+t^2} dx \quad \text{where } t = \sqrt{e^x - 1}$$

$$= \int\left(2 - \frac{2}{1+t^2}\right) dt$$

$$= 2t - 2\,\mathrm{Tan}^{-1}(t) + c$$

$$= 2\sqrt{e^x-1} - 2\,\mathrm{Tan}^{-1}(\sqrt{e^x-1}) + c$$

**Answer: (A)**

## Example 4.64

$\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals

(A) $(x^4+1)^{1/4} + c$ (B) $-\frac{(x^4+1)^{1/4}}{x} + c$

(C) $-\frac{(x^4+1)^{3/4}}{x^3} + c$ (D) $\frac{(x^3+1)^{3/4}}{x^4} + c$

***Solution:*** Let

$$I = \int \frac{dx}{x^2(x^4+1)^{3/4}}$$

$$= \int \frac{dx}{x^5\left(1 + \frac{1}{x^4}\right)^{3/4}}$$

Put $t = 1 + (1/x^4)$. Then

$$dt = -\frac{4}{x^5} dx$$

Therefore

$$I = \int \frac{1}{t^{3/4}}\left(-\frac{1}{4}\right) dt$$

$$= -\frac{1}{4}\frac{t^{-(3/4)+1}}{-(3/4)+1} + c = -t^{1/4} + c$$

$$= -\left(1+\frac{1}{x^4}\right)^{1/4} + c \qquad = -\frac{(x^4+1)^{1/4}}{x} + c$$

**Answer: (B)**

## Example 4.65

If

$$\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx = -\frac{1}{2}\log_e|x+1| + \frac{1}{4}\log_e(x^2+1) + f(x) + c$$

then $f(x)$ equals

(A) $\frac{1}{2}\mathrm{Tan}^{-1}x + \frac{x}{1+x^2}$ (B) $\frac{3}{2}\mathrm{Tan}^{-1}x - \frac{x}{1+x^2}$

(C) $\frac{3}{2}\mathrm{Tan}^{-1}x + \frac{x}{1+x^2}$ (D) $\frac{3}{2}\log_e|x| + \frac{x}{1+x^2}$

***Solution:*** Let $I$ be the given integral. Now, we use partial fractions. Write

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Therefore

$$x^3+3x+2 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1)$$

So,

$$x = -1 \Rightarrow 4A = -2$$
$$\Rightarrow A = -\frac{1}{2}$$

Equating the coefficient of $x^4$ on both sides, we get

$$A + B = 0 \Rightarrow B = \frac{1}{2}$$

Again equating the coefficients of $x^3$, we get

$$B + C = 1 \Rightarrow C = \frac{1}{2}\left(\because B = \frac{1}{2}\right)$$

Similarly, equating the coefficients of $x^2$ we get

$$2A + B + C + D = 0$$
$$-1 + \frac{1}{2} + \frac{1}{2} + D = 0$$
$$D = 0$$

Now

$$x = 0 \Rightarrow 2 = A + C + E$$
$$\Rightarrow 2 = -\frac{1}{2} + \frac{1}{2} + E$$
$$\Rightarrow E = 2$$

Therefore

$$I = -\frac{1}{2}\int\frac{dx}{x+1} + \frac{1}{2}\int\frac{x+1}{x^2+1}\,dx + \int\frac{2}{(x^2+1)^2}\,dx$$

$$= -\frac{1}{2}\log_e|x+1| + \frac{1}{2}\int\frac{x}{x^2+1}\,dx + \frac{1}{2}\int\frac{dx}{x^2+1} + 2\int\frac{dx}{(x^2+1)^2} \tag{4.9}$$

$$= -\frac{1}{2}\log_e|x+1| + \frac{1}{4}\log_e(x^2+1) + \frac{1}{2}\mathrm{Tan}^{-1}x + I_1$$

Here

$$I_1 = 2\int\frac{dx}{(x^2+1)^2}$$
$$= 2\int\frac{\sec^2\theta}{\sec^4\theta}\,d\theta \quad \text{where } x = \tan\theta$$
$$= \int 2\cos^2\theta\,d\theta$$
$$= \int(1+\cos 2\theta)\,d\theta$$
$$= \theta + \frac{1}{2}\sin 2\theta$$
$$= \mathrm{Tan}^{-1}x + \frac{1}{2}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= \mathrm{Tan}^{-1}x + \frac{x}{1+x^2}$$

Therefore

$$I_1 = \mathrm{Tan}^{-1}x + \frac{x}{1+x^2}$$

Substituting the value of $I_1$ in Eq. (4.9), we have

$$I = -\frac{1}{2}\log_e|x+1| + \frac{1}{4}\log_e(x^2+1) + \frac{1}{2}\mathrm{Tan}^{-1}x + \mathrm{Tan}^{-1}x + \frac{x}{1+x^2} + c$$

Therefore

$$f(x)=\frac{1}{2}\mathrm{Tan}^{-1}x+\mathrm{Tan}^{-1}x+\frac{x}{1+x^2}$$

$$=\frac{3}{2}\mathrm{Tan}^{-1}x+\frac{x}{1+x^2}$$

**Answer: (C)**

## Example 4.66

If

$$\int\frac{\sin x}{\sin x-\cos x}\,dx=g(x)+\frac{x}{2}+c$$

then $g(x)$ equals

(A) $\frac{1}{2}\log_e|\sin x+\cos x|+c$

(B) $\frac{1}{2}\log_e|\sin x-\cos x|+c$

(C) $\sin x+\cos x+c$

(D) $\sin x-\cos x+c$

***Solution:*** We have

$$\int\frac{\sin x}{\sin x-\cos x}\,dx=\frac{1}{2}\int\frac{2\sin x}{\sin x-\cos x}\,dx$$

$$=\frac{1}{2}\int\frac{(\sin x+\cos x)+(\sin x-\cos x)}{\sin x-\cos x}\,dx$$

$$=\frac{1}{2}\int\frac{\sin x+\cos x}{\sin x-\cos x}\,dx+\frac{1}{2}\int 1dx$$

$$=\frac{1}{2}\log_e|\sin x-\cos x|+\frac{x}{2}+c$$

(See B.2 for the conversion of first integral.) Therefore

$$g(x)=\frac{1}{2}\log_e|\sin x-\cos x|$$

**Answer: (B)**

## Example 4.67

If

$$\int\frac{1-(\cot x)^{2010}}{\tan x+(\cot x)^{2011}}\,dx=\frac{1}{k}\log_e\left|(\sin x)^k+(\cos x)^k\right|+c$$

then $k$ is equal to

(A) 2010 (B) 2011

(C) 2012 (D) 2013

***Solution:*** Let

$$I=\int\frac{1-(\cot x)^{2010}}{\tan x+(\cot x)^{2011}}\,dx$$

$$=\int\frac{(\sin x)^{2010}-(\cos x)^{2010}}{(\sin x)^{2010}}\cdot\frac{(\sin x)^{2011}\cos x}{(\sin x)^{2012}+(\cos x)^{2012}}\,dx$$

$$=\int\frac{[(\sin x)^{2010}-(\cos x)^{2010}]\sin x\cos x}{(\sin x)^{2012}+(\cos x)^{2012}}\,dx$$

Put $t=(\sin x)^{2012}+(\cos x)^{2012}$. Then

$$dt=[2012(\sin x)^{2011}\cos x+2012(\cos x)^{2011}(-\sin x)]dx$$

$$=2012[(\sin x)^{2010}-(\cos x)^{2010}](\sin x\cos x)\,dx$$

Therefore

$$I=\int\frac{1}{t}\left(\frac{1}{2012}\right)dt$$

$$=\frac{1}{2012}\log_e|t|+c$$

$$=\frac{1}{2012}\log_e\left|(\sin x)^{2012}+(\cos x)^{2012}\right|+c$$

**Answer: (C)**

## Example 4.68

$\displaystyle\int\frac{1+x\sec^2 x}{x(xe^{\tan x}+1)}\,dx$ is equal to

(A) $\log_e\left|\frac{xe^{\tan x}}{xe^{\tan x}+1}\right|+c$ (B) $\log_e\left|\frac{(x+1)\,e^{\tan x}}{e^{x\tan x}+1}\right|+c$

(C) $\log_e\left|\frac{e^{x\tan x}+1}{xe^{\tan x}+1}\right|+c$ (D) $\log_e\left|\frac{xe^{\tan x}+1}{(x+1)\,e^{\tan x}+1}\right|+c$

***Solution:*** We have

$$\frac{d}{dx}(xe^{\tan x}+1)=e^{\tan x}+xe^{\tan x}\sec^2 x$$
$$=e^{\tan x}(1+x\sec^2 x)$$

Now, let

$$I=\int\frac{1+x\sec^2 x}{x(xe^{\tan x}+1)}dx$$
$$=\int\frac{e^{\tan x}(1+x\sec^2 x)}{xe^{\tan x}(xe^{\tan x}+1)}dx$$
$$=\int\frac{dt}{(t-1)t}\quad\text{where } t=xe^{\tan x}+1$$
$$=\int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$
$$=\log_e\left|\frac{t-1}{t}\right|+c$$
$$=\log_e\left|\frac{xe^{\tan x}}{xe^{\tan x}+1}\right|+c$$

**Answer: (A)**

## Example 4.69

$$\int\frac{x\cos x\log_e x-\sin x}{x(\log_e x)^2}dx \text{ equals}$$

(A) $\dfrac{\log_e x}{\sin x}+c$ (B) $\dfrac{\sin x}{\log_e x}+c$

(C) $\dfrac{\sin x}{x}+\dfrac{1}{2}(\log_e x)^2+c$ (D) $\sin x+\dfrac{1}{2}\log_e x+c$

***Solution:*** Let

$$I=\int\frac{x\cos x\log_e x-\sin x}{x(\log_e x)^2}dx$$
$$=\int\frac{\cos x\log_e x-(\sin x/x)}{(\log_e x)^2}$$

Now, put $t=\sin x/\log_e x$. Therefore

$$dt=\frac{\cos x\log_e x-(\sin x/x)}{(\log_e x)^2}dx$$

Hence

$$I=\int dt=t+c=\frac{\sin x}{\log_e x}+c$$

**Answer: (B)**

## Example 4.70

$$\int\frac{\sec x(2+\sec x)}{(1+2\sec x)^2}dx \text{ is equal to}$$

(A) $2\operatorname{cosec} x+\cot x+c$
(B) $(2\cot x+\operatorname{cosec} x)^{-1}+c$
(C) $2\operatorname{cosec} x-\cot x+c$
(D) $(2\operatorname{cosec} x+\cot x)^{-1}+c$

***Solution:*** Let

$$I=\int\frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2}dx$$
$$=\int\frac{2\cos x+1}{(2+\cos x)^2}dx$$

Dividing numerator and denominator by $\sin^2 x$ we get

$$I=\int\frac{2\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x}{(2\operatorname{cosec} x+\cot x)^2}dx$$
$$=-\int\frac{dt}{t^2}\quad\text{where } 2\operatorname{cosec} x+\cot x=t$$
$$=\frac{1}{t}+c$$
$$=\frac{1}{2\operatorname{cosec} x+\cot x}+c$$

**Answer: (D)**

## Example 4.71

If

$$\int\frac{x^{11}+x^6-x}{(6-3x^5-2x^{10})^{3/5}}dx=k\,(6x^5-3x^{10}-2x^{15})^{2/5}+c$$

then the value of $k$ is

(A) $\dfrac{1}{6}$ (B) $-\dfrac{1}{6}$

(C) $-\dfrac{1}{12}$ (D) $\dfrac{1}{12}$

**Solution:** Let

$$I=\int \frac{x^{11}+x^6-x}{(6-3x^5-2x^{10})^{3/5}}\,dx$$

Multiplying numerator and denominator with $x^3$ we get

$$I=\int \frac{x^{14}+x^9-x^4}{(6x^5-3x^{10}-2x^{15})^{3/5}}\,dx$$

Put $t=6x^5-3x^{10}-2x^{15}$. Then

$$dt=-30(x^{14}+x^9-x^4)dx$$

Therefore

$$\begin{aligned}I&=\int t^{-3/5}(-1/30)\,dt\\&=-\frac{1}{30}\left(\frac{t^{-(3/5)+1}}{-(3/5)+1}\right)+c\\&=-\frac{1}{12}t^{2/5}+c\\&=-\frac{1}{12}(6x^5-3x^{10}-2x^{15})^{2/5}+c\end{aligned}$$

Hence $k=-1/12$.

**Answer: (C)**

## Example 4.72

If $P(x)$ is a polynomial function such that $P(x)+P(2x)=5x^2-18$, then $\int P(e^x)\,dx$ equals

(A) $\frac{1}{2}e^{2x}-9x+c$ (B) $e^{2x}+9x+c$

(C) $\frac{1}{2}e^x-9x^2+c$ (D) $e^{2x}-x+c$

**Solution:** Since $P(x)+P(2x)$ is a quadratic expression, it follows that $P(x)$ must be a quadratic expression. Suppose

$$P(x)=ax^2+bx+c$$

Therefore

$$\begin{aligned}5x^2-18&=P(x)+P(2x)\\&=(ax^2+bx+c)+(4ax^2+2bx+c)\end{aligned}$$

Now equating the corresponding coefficients we get

$$5a=5,\ 3b=0 \text{ and } 2c=-18$$

$$\Rightarrow a=1,\ b=0,\ c=-9$$

Hence

$$P(x)=x^2-9$$

So

$$\begin{aligned}\int P(e^x)\,dx&=\int (e^{2x}-9)\,dx\\&=\frac{1}{2}e^{2x}-9x+c\end{aligned}$$

**Answer: (A)**

## Example 4.73

$\int e^x(1+x)\operatorname{cosec}^2(xe^x)\,dx$ is

(A) $-\cot[(x+1)e^x]+c$ (B) $-\cot(xe^x)+c$

(C) $\operatorname{cosec}(xe^x)+c$ (D) $\sin(xe^x)+c$

**Solution:** Put $t=xe^x$. Therefore

$$dt=e^x(x+1)\,dx$$

So

$$\begin{aligned}\int e^x(1+x)\operatorname{cosec}^2(xe^x)\,dx&=\int \operatorname{cosec}^2 t\,dt\\&=-\cot(xe^x)+c\end{aligned}$$

**Answer: (B)**

## Example 4.74

$\int \frac{(1+\log_e x)^2}{1+\log_e(x^{x+1})+(\log_e x^{\sqrt{x}})^2}\,dx$ equals

(A) $\log_e|1+x\log_e x|+c$ (B) $\log_e|x+\log_e x|+c$

(C) $\log_e|(1+x)\log_e x|+c$ (D) $\log_e|x\log_e x|+c$

**Solution:** Let $I$ be the given integral. Then

$$I=\int \frac{(1+\log_e x)^2}{1+(x+1)\log_e x+x(\log x)^2}\,dx$$

$$= \int \frac{(1+\log_e x)^2}{(1+\log_e x)(1+x\log_e x)}\,dx$$

$$= \int \frac{1+\log_e x}{1+x\log_e x}\,dx$$

$$= \int \frac{dt}{t} \quad \text{where } t = 1 + x\log_e x$$

$$= \log_e|1 + x\log_e x| + c$$

**Answer: (A)**

## Example 4.75

$$\int \frac{\sec x \operatorname{cosec} x}{\log_e(\tan x)}\,dx =$$

(A) $\log_e(\tan x) + x + c$

(B) $\log_e|\log_e \tan x| + c$

(C) $\tan(\log_e \tan x) + c$

(D) $\log_e|\tan(\log_e x)| + c$

***Solution:*** We have

$$I = \int \frac{\sec x \operatorname{cosec} x}{\log_e(\tan x)}\,dx$$

$$= \int \frac{dt}{t} \quad \text{where } t = \log_e(\tan x)$$

$$= \log_e|t| + c$$

$$= \log_e\left|\log_e|\tan x|\right| + c$$

**Answer: (B)**

## 4.4 | Integration by Parts

Let $u$ and $v$ be differentiable functions. Then it is known that (Chapter 2)

$$d(uv) = udv + vdu$$

so that on integration we have

$$uv = \int udv + \int vdu$$

and therefore, we have the following formula.

### Formula 1

$$\int udv = uv - \int vdu$$

This formula enables us to find the integral of product of two functions. Out of the two factors of the product, we consider one factor as $u$ and the second factor as the differential $dv$ of $v$.

If we consider $f(x) = u$ and $g(x)\,dx = dv$ so that $v = \int g(x)dx$, then we have the second formula.

### Formula 2

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left(\int g(x)\,dx\right)f'(x)\,dx$$

If we consider $f(x)$ as first function and $g(x)$ as the second function then Formula 2 can be written as

$$\int (\text{First function}) \times (\text{Second function})\,dx$$

$$= (\text{First function}) \times (\text{Integral of second function})$$

$$- \int \begin{matrix}(\text{Integral of second function}) \times \\ (\text{Derivative of first function})\,dx\end{matrix}$$

***Note:*** Throughout the chapter, we use Formula 1.

***Selecting u and dv***

Suppose the product contains two of the following functions: Inverse trigonometric, Logarithmic, Algebraic, Trigonometric and Exponential. According to the order mentioned, we choose $u$ and $dv$. This rule of selection can be easily remembered as given below.

**I** Inverse Trigonometric
**L** Logarithmic
**A** Algebraic
**T** Trigonometric
**E** Exponential

Using by parts, we obtain some standard formulae. We also show how to solve them for students to understand.

## Formula 3

$$\int \log_e x\, dx = x(\log_e x - 1) + c$$

***Solution:*** Whenever the integrand is a single function, take the second function as $1 = x^0$ which is algebraic. Therefore, in Formula 1, take $u = \log_e x$ and $dv = 1dx$.

Hence, $$\int \log x\, dx = (\log_e x)x - \int x\left(\frac{1}{x}\right) dx$$
$$= x \log_e x - x + c$$
$$x(\log_e x - 1) + c$$

## Formula 4

$$\int xe^x dx = e^x(x-1) + c$$

***Solution:*** Here $x$ is an algebraic function and $e^x$ is an exponential function. According to the rule mentioned above, A comes first and E comes next. Therefore, take $u = x$ and $dv = e^x\, dx$ so that $v = e^x$. Hence

$$\int xe^x dx = uv - \int vdu$$
$$= xe^x - \int e^x(1)\, dx$$
$$= xe^x - e^x + c$$
$$= e^x(x-1) + c$$

## Formula 5

$$\int x \sin x\, dx = \sin x - x\cos x + c$$

***Solution:*** $x$ is an algebraic function and $\sin x$ is a trigonometric function. So, take $u = x$ and $dv = \sin x\, dx$ so that $v = -\cos x$. Therefore

$$\int x \sin x\, dx = x(-\cos x) - \int(-\cos x)(1)\, dx$$
$$= -x\cos x + \sin x + c$$

## Formula 6

$$\int x \cos x\, dx = \cos x + x\sin x + c$$

***Solution:*** Take $u = x$ and $dv = \cos x\, dx$ so that $v = \sin x$. Therefore

$$\int x\cos x\, dx = x\sin x - \int(\sin x)(1)\, dx$$
$$= x\sin x - (-\cos x) + c$$
$$= \cos x + x\sin x + c$$

## Formula 7

$$\int \mathrm{Sin}^{-1} x\, dx = x\mathrm{Sin}^{-1} x + \sqrt{1-x^2} + c$$

***Solution:*** $\mathrm{Sin}^{-1} x$ is an inverse trigonometric function which comes first in the given order. So, take $u = \mathrm{Sin}^{-1} x$ and $dv = 1\, dx$ so that $v = x$. Now

$$\int \mathrm{Sin}^{-1} x\, dx = x\, \mathrm{Sin}^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$
$$= x\, \mathrm{Sin}^{-1} x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\, dx$$

$$= x\mathrm{Sin}^{-1}x + \frac{1}{2}(2\sqrt{1-x^2}) + c \qquad \text{(See B.3)}$$

$$= x\,\mathrm{Sin}^{-1}x + \sqrt{1-x^2} + c$$

**Try it out** $\int \mathrm{Cos}^{-1}x\,dx = \int\left(\frac{\pi}{2} - \mathrm{Sin}^{-1}x\right)dx = -(x\mathrm{Sin}^{-1}x + \sqrt{1-x^2}) + \frac{\pi}{2}x + c$

## Formula 8

$$\int \mathrm{Tan}^{-1}x\,dx = x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\log_e(1+x^2) + c$$

***Solution:*** Take $u = \mathrm{Tan}^{-1}x$ and $dv = 1dx$ so that $v = x$. Therefore

$$\int \mathrm{Tan}^{-1}x\,dx = x\,\mathrm{Tan}^{-1}x - \int \frac{x}{1+x^2}\,dx$$

$$= x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\log_e(1+x^2) + c \qquad \text{(See B.2)}$$

**Try it out** Since

$$\mathrm{Cot}^{-1}x + \mathrm{Tan}^{-1}x = \frac{\pi}{2}$$

Compute $\int \mathrm{Cot}^{-1}x\,dx$.

## Formula 9

$$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c$$

***Solution:*** Take $u = \sqrt{a^2-x^2}$ and $dv = 1dx$ so that $v = x$. Therefore

$$\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\left(\frac{1}{2\sqrt{a^2-x^2}}\right)(-2x)\,dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

Therefore

$$2\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\mathrm{Sin}^{-1}\frac{x}{a} + c \qquad \text{(See A.9)}$$

$$\int \sqrt{a^2-x^2} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\mathrm{Sin}^{-1}\frac{x}{a} + c$$

Similarly, we can prove the following.

## Formula 10

$$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\mathrm{Cosh}^{-1}\frac{x}{a} + c$$

## Formula 11

$$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\operatorname{Sinh}^{-1}\frac{x}{a} + c$$

**Try it out** Prove Formulae 10 and 11.

***Note:*** Formulae 9, 10 and 11 can be proved by substitution method using the substitutions $x = a\sin\theta$, $x = a\cosh\theta$ and $x = a\sinh\theta$, respectively. The reader is advised to prove them by the substitution method. The following formula is the most important consequence of by parts.

## Formula 12

$$\int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c$$

***Solution:*** By taking $u = f(x)$ and $dv = e^x\,dx$ we get

$$\int e^x f(x)\,dx = e^x f(x) - \int e^x f'(x)$$

$$\int e^x f(x)\,dx + \int e^x f'(x)\,dx = e^x f(x)+c$$

$$\int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c$$

## Examples

Some of the examples illustrating Formula 12 are as follows:

**1.** $\int e^x(\sin x+\cos x)\,dx = e^x\sin x + c$

**2.** $\int e^x(\tan x+\sec^2 x)\,dx = e^x\tan x + c$

**3.** $\int e^x\left(\operatorname{Sin}^{-1}x+\frac{1}{\sqrt{1-x^2}}\right)dx = e^x\operatorname{Tan}^{-1}x + c$

**4.** $\int e^x\left(\log x+\frac{1}{x}\right)dx = e^x\log x + c$

**5.** $\int e^x\left(\operatorname{Tan}^{-1}x+\frac{1}{1+x^2}\right)dx = e^x\operatorname{Tan}^{-1}x + c$

**6.** $\int \frac{e^x(x-1)}{x^2}\,dx = \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{e^x}{x}+c$

## Formula 13

*Generalized Formula for Integration by Parts*

When integration by parts is to be applied repeatedly to compute an integral, the following is very useful.

$$\int f(x)g(x)\,dx = f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - f'''(x)g_4(x) + \cdots + (-1)^n\int f^{(n)}(x)g_n(x) + c$$

where dashes denote the successive derivatives of $f(x)$ and suffixes denote the integrals. That is

$$g_1(x) = \int g(x)\,dx,\ g_2(x) = \int g_1(x)\,dx, \ldots, g_n(x) = \int g_{n-1}(x)\,dx$$

***Caution:*** Do not effect the simplification, until the last integral is computed.

We illustrate Formula 13 in the following illustration.

**Illustration 3**

Find out $\int x^4 e^{2x} dx$

***Solution:***

$$\int x^4 e^{2x} dx = x^4\left(\frac{e^{2x}}{2}\right) - 4x^3\left(\frac{e^{2x}}{4}\right) + 12x^2\left(\frac{e^{2x}}{8}\right) - 24x\left(\frac{e^{2x}}{16}\right) + \int 24\left(\frac{e^{2x}}{16}\right) dx$$

$$= \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3}{2}x^2 e^{2x} - \frac{3}{2}x e^{2x} + \frac{3}{4}e^{2x} + c$$

$$= e^{2x}\left[\frac{x^4}{2} - x^3 + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{3}{4}\right] + c$$

## 4.4.1 Examples

**Example 4.76**

$\int x^3 \log_e x dx$ is equal to

(A) $\frac{1}{4}x^4 \log_e x - \frac{1}{8}x^4 + c$ (B) $\frac{1}{4}x^4 + \frac{x^4}{8}\log_e x + c$

(C) $\frac{1}{4}x^4 \log_e x - \frac{x^4}{16} + c$ (D) $\frac{1}{4}x^4 \log_e x + \frac{x^4}{16} + c$

***Solution:*** $\log_e x$ is logarithmic and $x^3$ is algebraic. In the word ILATE, L comes first and A comes next. Therefore, take $u = \log_e x$ and $dv = x^3\, dx$ so that $v = x^4/4$. Therefore

$$\int x^3 \log_e x dx = \frac{x^4}{4}\log_e x - \int\left(\frac{x^4}{4}\right)\left(\frac{1}{x}\right)dx$$

$$= \frac{1}{4}x^4 \log_e x - \int \frac{x^3}{4} dx$$

$$= \frac{1}{4}x^4 \log_e x - \frac{x^4}{16} + c$$

**Answer: (C)**

**Example 4.77**

$\int \sin x \log_e(\tan x)\, dx =$

(A) $\log_e\left|\tan\frac{x}{2}\right| - \cos x \log_e(\tan x) + c$

(B) $-\cos x \log_e\left|\tan\frac{x}{2}\right| - \log_e\left|\tan\frac{x}{2}\right| + c$

(C) $\log\left|\tan\frac{x}{2}\right| + \cos x \log(\tan x) + c$

(D) $\log_e|\tan x| - \cos x \log\left|\tan\frac{x}{2}\right| + c$

***Solution:*** Let

$$I = \int \sin x \log_e(\tan x)\, dx$$

Take $u = \log_e(\tan x)$ and $dv = \sin x\, dx$ so that $v = -\cos x$. Therefore

$$I = [\log_e(\tan x)](-\cos x) - \int(-\cos x)\frac{\sec^2 x}{\tan x} dx$$

$$= -\cos x \log_e(\tan x) + \int \operatorname{cosec} x\, dx$$

$$= -\cos x \log_e(\tan x) + \log\left|\tan\frac{x}{2}\right| + c \quad \text{(See B.4)}$$

**Answer: (A)**

## Example 4.78

If

$$\int (x^3+1)\cos x\,dx = f_1(x)\sin x + f_2(x)\cos x + c$$

then $f_1(x)+f_2(x)$ is equal to

(A) $x^3-3x^2+6x-5$ (B) $x^3+3x^2+6x+5$
(C) $x^3-3x^2-6x+5$ (D) $x^3+3x^2-6x-5$

***Solution:*** Let

$$I = \int (x^3+1)\cos x\,dx$$
$$= \int x^3\cos x\,dx + \int \cos x\,dx$$
$$= \int x^3\cos x\,dx + \sin x \qquad (4.10)$$

Let

$$I_1 = \int x^3\cos x\,dx$$

Take $u = x^3$, $dv = \cos x\,dx$ so that $v = \sin x$. Therefore

$$I_1 = x^3\sin x - \int (3x^2)\sin x\,dx \qquad (4.11)$$

Let

$$I_2 = 3\int x^2\sin x\,dx$$
$$= 3[x^2(-\cos x)] - \int (2x)(-\cos x)\,dx$$
$$= -3x^2\cos x + 6\int x\cos x$$
$$= -3x^2\cos x + 6[\cos x + x\sin x] + c \qquad \text{(By Formula 6)}$$

Substituting the value $I_2$ in Eq. (4.11), we get

$$I_1 = x^3\sin x + 3x^2\cos x - 6(\cos x + x\sin x)$$

Therefore substituting the value of $I_1$ in Eq. (4.10), we have

$$I = x^3\sin x + 3x^2\cos x - 6(\cos x + x\sin x) + \sin x + c$$
$$= (\sin x)(x^3-6x+1) + (3x^2-6)\cos x + c$$

Therefore

$$f_1(x) = x^3-6x+1$$

and

$$f_2(x) = 3x^2-6$$

Hence

$$f_1(x)+f_2(x) = x^3+3x^2-6x-5$$

**Answer: (D)**

## Example 4.79

$$\int \cos(\log_e x)\,dx = f(x)[\cos(\log_e x)+\sin(\log_e x)] + c$$

where $f(x)$ equals

(A) $\frac{x}{2}$ (B) $\frac{x^2}{2}$
(C) $x$ (D) $x^2$

***Solution:*** Let

$$I = \int \cos(\log_e x)\,dx$$

Take $u = \cos(\log_e x)$ and $dv = dx$ so that $v = x$. Then

$$I = x\cos(\log_e x) - \int -x\frac{\sin(\log_e x)}{x}\,dx$$
$$= x\cos(\log_e x) + \int \sin(\log_e x) \qquad (4.12)$$

Now, let

$$I_1 = \int \sin(\log_e x)\,dx$$

Again using by parts we have

$$I_1 = x\sin(\log_e x) - \int x\frac{\cos(\log_e x)}{x}\,dx$$
$$= x\sin(\log_e x) - I$$

Substituting the value of $I_1$ in Eq. (4.12), we have

$$I = x\cos(\log_e x) + x\sin(\log_e x) - I$$

Therefore

$$2I = x[\cos(\log_e x) + \sin(\log_e x)]$$

So

$$f(x) = \frac{x}{2}$$

**Answer: (A)**

## Example 4.80

$$\int \sqrt[3]{x}\,(\log_e x)^2\,dx =$$

(A) $\frac{3}{4}(x)^{4/3}\left[(\log_e x)^2 - \frac{3}{2}\log_e x - \frac{9}{8}\right] + c$

(B) $\frac{3}{4}(x)^{4/3}\left[(\log_e x)^2 + \frac{3}{2}\log_e x + \frac{9}{8}\right] + c$

(C) $\frac{3}{4}(x)^{4/3}\left[(\log_e x)^2 - \frac{3}{2}\log_e x + \frac{9}{8}\right] + c$

(D) $\frac{3}{4}(x)^{1/3}\left[(\log_e x)^2 + \frac{3}{2}\log_e x - \frac{9}{8}\right] + c$

***Solution:*** We have

$$I = \int \sqrt[3]{x}(\log_e x)^2\,dx$$

Take $f(x) = \sqrt[3]{x}$, $g(x) = (\log_e x)^2$. Now take $u = (\log_e x)^2$, $dv = x^{1/3}\,dx$ so that

$$v = \frac{x^{(1/3)+1}}{(1/3)+1} = \frac{3}{4}x^{4/3}$$

Therefore

$$I = \frac{3}{4}x^{4/3}(\log_e x)^2 - \int \frac{3}{4}x^{4/3}\left(\frac{2\log_e x}{x}\right)dx$$

$$= \frac{3}{4}x^{4/3}(\log_e x)^2 - \frac{3}{2}\left[\int x^{1/3}\log_e x dx\right]$$

$$= \frac{3}{4}x^{4/3}(\log_e x)^2 - \frac{3}{2}\left[\frac{3}{4}x^{4/3}\log_e x - \int \frac{3}{4}x^{4/3}\cdot\frac{1}{x}dx\right]$$

$$= \frac{3}{4}x^{4/3}(\log_e x)^2 - \frac{9}{8}x^{4/3}\log_e x + \frac{9}{8}\cdot\frac{x^{(1/3)+1}}{(1/3)+1} + c$$

$$= \frac{3}{4}x^{4/3}\left[(\log_e x)^2 - \frac{3}{2}\log_e x + \frac{9}{8}\right] + c$$

**Answer: (C)**

## Example 4.81

$$\int \mathrm{Sin}^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx \text{ is equal to}$$

(A) $\frac{3}{2}\left[\left(\frac{2x+2}{3}\right)\mathrm{Tan}^{-1}\left(\frac{2x+2}{3}\right) - \log_e\left(\frac{1}{3}(2x+3)\right)\right] + c$

(B) $\frac{3}{2}\left[\frac{2x+2}{3}\mathrm{Tan}^{-1}\left(\frac{2x+2}{3}\right) - \frac{1}{3}\log_e\left(\frac{1}{3}\sqrt{4x^2+8x+3}\right)\right] + c$

(C) $\frac{3}{2}\left[\frac{2x+2}{3}\mathrm{Tan}^{-1}\left(\frac{2x+2}{3}\right) - \log_e\left(\frac{1}{3}\sqrt{4x^2+8x+13}\right)\right] + c$

(D) $\frac{3}{2}\left[\frac{2x+2}{3}\mathrm{Tan}^{-1}\sqrt{4x^2+8x+3} - \log_e\left(\frac{2x+2}{3}\right)\right] + c$

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***Solution:*** We have

$$I = \int \mathrm{Sin}^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$$

$$= \int \mathrm{Sin}^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right)dx$$

Now, put

$$\frac{2x+2}{3} = \tan\theta$$

Then

$$dx = \frac{1}{2}(3\sec^2\theta)\,d\theta$$

Hence

$$I = \int \mathrm{Sin}^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int \mathrm{Sin}^{-1}(\sin\theta)\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int \theta\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\left[\theta\tan\theta - \int\tan\theta\,d\theta\right] \quad \text{(using by parts)}$$

$$= \frac{3}{2}[\theta\tan\theta - \log|\sec\theta|] + c$$

$$= \frac{3}{2}\left[\left(\mathrm{Tan}^{-1}\left(\frac{2x+2}{3}\right)\right)\frac{2x+2}{3} - \log_e\frac{1}{3}\sqrt{4x^2+8x+13}\right] + c$$

because

$$\tan\theta = \frac{2x+2}{3}$$

$$\Rightarrow \sec\theta = \sqrt{1+\tan^2\theta}$$

$$= \sqrt{1+\left(\frac{2x+2}{3}\right)^2}$$

$$= \frac{1}{3}\sqrt{4x^2+8x+13}$$

**Answer: (C)**

## Example 4.82

$\int [\sin(\log_e x) + \cos(\log_e x)]\, dx$ is equal to

(A) $x \sin(\log_e x) + c$ (B) $x \cos(\log_e x) + c$

(C) $e^x \tan(\log_e x) + c$ (D) $e^{\sin(\log_e x)} \cos x + c$

***Solution:*** We have

$$I = \int (\sin(\log_e x) + \cos(\log_e x)\, dx$$

Put $\log_e x = t$. Therefore $x = e^t$ and $dx = e^t\, dt$. Now

$$I = \int (\sin t + \cos t)\, e^t\, dt$$

$$= \int e^t (\sin t + \cos t)\, dt$$

$$= e^t \sin t + c \quad \left(\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c\right)$$

$$= x \sin(\log_e x) + c$$

**Answer: (A)**

## Example 4.83

$\int x \operatorname{Sin}^{-1} x\, dx = \frac{x}{4}\sqrt{1-x^2} + f(x)\operatorname{Sin}^{-1} x$ where $f(x)$ is

(A) $\frac{2x^2+1}{4}$ (B) $\frac{2x^2+x}{4}$

(C) $\frac{2x^2-x}{4}$ (D) $\frac{2x^2-1}{4}$

***Solution:*** We have

$$I = \int x \operatorname{Sin}^{-1} x\, dx$$

Take $u = \operatorname{Sin}^{-1} x$, $dv = x\, dx$ so that $v = x^2/2$. Therefore

$$I = \frac{x^2}{2}\operatorname{Sin}^{-1} x - \int \frac{x^2}{2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2}\operatorname{Sin}^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2}\operatorname{Sin}^{-1} x + \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2}\operatorname{Sin}^{-1} x + \frac{1}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\operatorname{Sin}^{-1} x\right] - \frac{1}{2}\operatorname{Sin}^{-1} x$$

(By Formula 9)

$$= \frac{x}{4}\sqrt{1-x^2} + \left(\frac{x^2}{2} + \frac{1}{4} - \frac{1}{2}\right)\operatorname{Sin}^{-1} x + c$$

$$= \frac{x}{4}\sqrt{1-x^2} + \left(\frac{2x^2-1}{4}\right)\operatorname{Sin}^{-1} x + c$$

**Answer: (D)**

## Example 4.84

$\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}\, dx$ is equal to

(A) $\log_e |x\sin x + \cos x| + \frac{x^2}{x\tan x + 1} + c$

(B) $2\log_e |x\sin x + \cos x| - \frac{x^2}{x\tan x + 1} + c$

(C) $2\log_e |x\cos x + \sin x| + \frac{x^2}{\sec x + x\tan x} + c$

(D) $\log_e |x\cos x + \sin x| + \frac{x^2}{\sec^2 x + x\tan x} + c$

***Solution:*** Observe that

$$\frac{d}{dx}(x\tan x + 1) = x\sec^2 x + \tan x$$

Let

$$I = \int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}\, dx$$

Take

$$u = x^2, \quad dv = \frac{x\sec^2 x + \tan x}{(x\tan x + 1)^2}$$

so that

$$v = \frac{-1}{x\tan x + 1}$$

Therefore

$$I = x^2\left(\frac{-1}{x\tan x+1}\right) - \int \frac{-1}{x\tan x+1}(2x)\,dx$$

$$= \frac{-x^2}{x\tan x+1} + 2\int \frac{x}{x\tan x+1}\,dx$$

$$= \frac{-x^2}{x\tan x+1} + 2\int \frac{x\cos x}{x\sin x+\cos x}\,dx$$

$$= \frac{-x^2}{x\tan x+1} + 2\int \frac{dt}{t} \quad \text{where } t = x\sin x + \cos x$$

$$= \frac{-x^2}{x\tan x+1} + 2\log_e|x\sin x+\cos x| + c$$

**Answer: (B)**

## Example 4.85

$\int \frac{x+\sin x}{1+\cos x}\,dx$ is equal to

(A) $x\tan\frac{x}{2} + \sec\frac{x}{2} + c$ (B) $x\sec\frac{x}{2} + \tan\frac{x}{2} + c$

(C) $x\sin\frac{x}{2} + \cos\frac{x}{2} + c$ (D) $x\tan\frac{x}{2} + c$

***Solution:*** We have

$$\int \frac{x+\sin x}{1+\cos x}\,dx = \int \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} + c$$

**Answer: (D)**

## Example 4.86

If $\int x^4\cos x\,dx = f(x)\sin x + g(x)\cos x + c$, then $f(x) + g(x)$ is equal to

(A) $x^4 + 4x^3 - 12x^2 - 24x + 24$

(B) $x^4 - 4x^3 + 12x^2 - 24x + 6$

(C) $x^4 + 4x^3 + 12x^2 - 24x - 24$

(D) $x^4 - 4x^3 - 12x^2 + 24x - 24$

***Solution:*** Take $u = x^4$ and $v = \cos x$ and apply Formula 3. We get

$$\int x^4\cos x\,dx = x^4\sin x - 4x^3(-\cos x) + 12x^2(-\sin x) - 24x\cos x + 24\sin x + c$$

$$= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x + c$$

$$= (x^4 - 12x^2 + 24)\sin x + (4x^3 - 24x)\cos x + c$$

Therefore

$$f(x) + g(x) = x^4 + 4x^3 - 12x^2 - 24x + 24$$

**Answer: (A)**

## Example 4.87

$\int e^x \frac{(x^3+x+1)}{(x^2+1)^{3/2}}\,dx =$

(A) $\frac{e^x x}{x^2+1} + c$ (B) $\frac{xe^x}{\sqrt{x^2+1}} + c$

(C) $\frac{xe^x}{(x^2+1)^{3/2}} + c$ (D) $\frac{e^x}{\sqrt{x^2+1}} + c$

***Solution:*** We have

$$I = \int e^x \frac{(x^3+x+1)}{(x^2+1)^{3/2}}\,dx$$

$$= \int e^x\left(\frac{x(x^2+1)+1}{(x^2+1)^{3/2}}\right)dx$$

$$= \int e^x \left[ \frac{x}{\sqrt{x^2+1}} + \frac{1}{(x^2+1)^{3/2}} \right] dx$$

$$= \int e^x [f(x) + f'(x)]\, dx \quad \text{where } f(x) = \frac{x}{\sqrt{x^2+1}}$$

$$= e^x f(x) + c$$

$$= e^x \left( \frac{x}{\sqrt{x^2+1}} \right) + c$$

**Answer: (B)**

## Example 4.88

$$\int e^x \left( \frac{x+2}{x+4} \right)^2 dx =$$

(A) $\dfrac{xe^x}{x+4} + c$ (B) $\dfrac{xe^x}{(x+4)^2} + c$

(C) $\dfrac{(x+1)\,e^x}{x+4} + c$ (D) $\dfrac{(x+1)}{(x+4)^2} e^x + c$

***Solution:*** We have

$$I = \int e^x \left( \frac{x+2}{x+4} \right)^2 dx$$

$$= \int e^x \left( \frac{x(x+4)+4}{(x+4)^2} \right) dx$$

$$= \int e^x \left( \frac{x}{x+4} + \frac{4}{(x+4)^2} \right) dx$$

$$= \int e^x [f(x) + f'(x)]\, dx$$

where $f(x) = x/(x+4)$. Therefore

$$I = e^x \left( \frac{x}{x+4} \right) + c$$

**Answer: (A)**

## Example 4.89

$$\int e^x \frac{(x^2+1)}{(x+1)^2} dx =$$

(A) $\dfrac{xe^x}{x+1} + c$ (B) $\dfrac{e^x}{(x+1)^2} + c$

(C) $\left( \dfrac{x+1}{x-1} \right) e^x + c$ (D) $e^x \left( \dfrac{x-1}{x+1} \right) + c$

***Solution:*** We have

$$I = \int \frac{e^x (x^2+1)}{(x+1)^2} dx$$

$$= \int e^x \left( \frac{x^2-1+2}{(x+1)^2} \right) dx$$

$$= \int e^x \left( \frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right) dx$$

$$= \int e^x [f(x) + f'(x)]\, dx \quad \text{where } f(x) = \frac{x-1}{x+1}$$

$$= e^x f(x) + c$$

$$= e^x \left( \frac{x-1}{x+1} \right) + c$$

**Answer: (D)**

## Example 4.90

$$\int e^x \frac{(1-x)^2}{(1+x^2)^2} dx =$$

(A) $e^x \left( \dfrac{x}{1+x^2} \right) + c$ (B) $e^x \left( \dfrac{2x}{1+x^2} \right) + c$

(C) $e^x \left( \dfrac{1}{1+x^2} \right) + c$ (D) $e^x \left( \dfrac{2}{1+x^2} - \dfrac{x}{(1+x^2)^2} \right) + c$

***Solution:*** We have

$$\int e^x \frac{(1-x)^2}{(1+x^2)^2} dx = \int e^x \frac{(1+x^2-2x)}{(1+x^2)^2} dx$$

$$= \int e^x \left( \frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2} \right) dx$$

$$= \int e^x [f(x) + f'(x)]\, dx \quad \text{where } f(x) = \frac{1}{1+x^2}$$

$$= e^x f(x) + c$$

$$= \frac{e^x}{1+x^2} + c$$

**Answer: (C)**

## Example 4.91

$$\int e^x \frac{(x^2+3x+3)}{(x+2)^2}\,dx =$$

(A) $e^x\left(\dfrac{x}{x+2}\right)+c$ (B) $e^x\left(\dfrac{x+1}{(x+2)^2}\right)+c$

(C) $e^x\left(\dfrac{x+1}{x+2}\right)+c$ (D) $\dfrac{e^x x(x+1)}{x+2}+c$

***Solution:*** We have

$$\int e^x \frac{(x^2+3x+3)}{(x+2)^2}\,dx = \int e^x \frac{[(x+1)(x+2)+1]}{(x+2)^2}\,dx$$

$$= \int e^x \left[\frac{x+1}{x+2}+\frac{1}{(x+2)^2}\right]dx$$

$$= \int e^x [f(x)+f'(x)]\,dx \quad \text{where } f(x)=\frac{x+1}{x+2}$$

$$= e^x f(x)+c$$

$$= e^x\left(\frac{x+1}{x+2}\right)+c$$

**Answer: (C)**

## Example 4.92

$$\int e^x \frac{(x^3-x+2)}{(x^2+1)^2}\,dx =$$

(A) $\dfrac{xe^x}{x^2+1}+c$ (B) $\dfrac{(x+1)\,e^x}{x^2+1}+c$

(C) $\dfrac{x^2e^x}{x+1}+c$ (D) $\dfrac{x^2e^x}{x^2+1}+c$

***Solution:*** We have

$$\int e^x \frac{(x^3-x+2)}{(x^2+1)^2}\,dx = \int e^x \frac{[(x^2+1)(x+1)+1-2x-x^2]}{(x^2+1)^2}\,dx$$

$$= \int e^x \left(\frac{x+1}{x^2+1}+\frac{1-2x-x^2}{(x^2+1)^2}\right)dx$$

$$= \int e^x [f(x)+f'(x)]\,dx \quad \text{where } f(x)=\frac{x+1}{x^2+1}$$

$$= e^x f(x)+c$$

$$= e^x\left(\frac{x+1}{x^2+1}\right)+c$$

**Answer: (B)**

## Example 4.93

$$\int \frac{\log_e x}{(1+\log_e x)^2}\,dx = \frac{f(x)}{1+\log_e x}+c \quad \text{where } f(x) \text{ is}$$

(A) $x$ (B) $-x$

(C) $x^2$ (D) $-x^2$

***Solution:*** We have

$$I = \int \frac{\log_e x}{(1+\log_e x)^2}\,dx$$

Put $t=\log_e x$. Therefore, $e^t = x$ and $e^t\,dt = dx$. Hence

$$I = \int \frac{t}{(1+t)^2}(e^t)\,dt$$

$$= \int e^t \frac{(t+1-1)}{(t+1)^2}\,dt$$

$$= \int e^t\left(\frac{1}{t+1}-\frac{1}{(t+1)^2}\right)dt$$

$$= \int e^t[g(t)+g'(t)]\,dt \quad \text{where } g(t)=\frac{1}{t+1}$$

$$= e^t g(t)+c$$

$$= \frac{e^t}{t+1}+c$$

$$= \frac{x}{\log_e x+1}+c$$

so that $f(x)=x$.

**Answer: (A)**

## Example 4.94

$\int e^{\sin x}\frac{(x\cos^3 x-\sin x)}{\cos^2 x}dx$ is equal to

(A) $e^{\sin x}(x+\cos x)+c$  (B) $e^{\sin x}(x+\sin x)+c$
(C) $e^{\sin x}(x-\sec x)+c$  (D) $e^{\sin x}(x+\sec x)+c$

***Solution:*** Let

$$I=\int e^{\sin x}\left(\frac{x\cos^3 x-\sin x}{\cos^2 x}\right)dx$$

$$=\int e^{\sin x}(x\cos x-\sec x\tan x)\,dx$$

$$=\int (e^{\sin x}\cos x)\,x\,dx-\int e^{\sin x}\sec x\tan x dx$$

Using integration by parts, we get

$$I=xe^{\sin x}-\int e^{\sin x}dx-\left[e^{\sin x}\sec x-\int e^{\sin x}\cos x\sec x dx\right]+c$$

$$=xe^{\sin x}-\int e^{\sin x}dx-e^{\sin x}\sec x+\int e^{\sin x}dx+c$$

$$=xe^{\sin x}-e^{\sin x}\sec x+c$$

$$=e^{\sin x}(x-\sec x)+c$$

**Answer: (C)**

## Example 4.95

$\int \frac{e^{\sec x}\sin[x+(\pi/4)]}{\cos x(1-\sin x)}dx\left(0<x<\frac{\pi}{4}\right)$ is equal to

(A) $\frac{1}{\sqrt{2}}e^{\sec x}(\sec x-\tan x)+c$

(B) $\frac{1}{\sqrt{2}}e^{\sec x}(\sec x+\tan x)+c$

(C) $\frac{1}{\sqrt{2}}e^{\sec x}(\text{cosec}\, x+\cot x)+c$

(D) $\frac{1}{\sqrt{2}}e^{\sec x}(\text{cosec}\, x-\cot x)+c$

***Solution:*** We have

$$I=\int\frac{e^{\sec x}\sin[x+(\pi/4)]}{\cos x(1-\sin x)}dx$$

$$=\frac{1}{\sqrt{2}}\int\frac{e^{\sec x}(\sin x+\cos x)}{\cos x(1-\sin x)}dx$$

$$=\frac{1}{\sqrt{2}}\int e^{\sec x}\frac{(\sin x+\cos x)(1+\sin x)}{\cos x(1-\sin^2 x)}dx$$

$$=\frac{1}{\sqrt{2}}\int e^{\sec x}\frac{(\sin x+\sin^2 x+\cos x+\cos x\sin x)}{\cos^3 x}dx$$

$$=\frac{1}{\sqrt{2}}\int e^{\sec x}(\sec^2 x\tan x+\sec x\tan^2 x+\sec^2 x+\sec x\tan x)dx$$

$$=\frac{1}{\sqrt{2}}\int e^{\sec x}(\sec^2 x+\sec x\tan^2 x)\,dx$$

$$+\frac{1}{\sqrt{2}}\int e^{\sec x}\sec x\tan x(\sec x+1)\,dx$$

$$=\frac{1}{\sqrt{2}}\int\frac{d}{dx}(e^{\sec x}\tan x)\,dx+\frac{1}{\sqrt{2}}\int e^t(t+1)\,dt\quad(t=\sec x)$$

$$=\frac{1}{\sqrt{2}}[e^{\sec x}\tan x+t\,e^t]+c$$

$$=\frac{1}{\sqrt{2}}e^{\sec x}(\tan x+\sec x)+c$$

**Answer: (B)**

## Example 4.96

If $\lim_{x\to 0}\frac{f(x)}{x^2}$ exists finitely and $\lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{1/x}=e^3$, then $\int f(x)\log_e x\,dx$ is equal to

(A) $\frac{2}{3}x^3\left(\log_e x-\frac{1}{3}\right)+c$

(B) $\frac{x^3}{3}\left(\log_e x-\frac{1}{3}\right)+c$

(C) $\frac{2}{3}x^3(\log_e x+1)+c$

(D) $\frac{2}{3}x^3(\log_e x-1)+c$

***Solution:*** We have

$$\lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{1/x}=e^3$$

$$\Rightarrow \lim_{x\to 0}\left(1+x\left(1+\frac{f(x)}{x^2}\right)\right)^{1/x}=e^3$$

$$\Rightarrow \lim_{x\to 0}\left[\left(1+\frac{x^2+f(x)}{x}\right)^{\frac{x}{x^2+f(x)}}\right]^{\frac{x^2+f(x)}{x}\frac{1}{x}}=e^3$$

$$\Rightarrow \frac{x^2+f(x)}{x^2}=3$$

$$\Rightarrow f(x)=2x^2$$

Therefore

$$\int f(x)\log_e x dx = 2\int x^2 \log_e x dx$$

$$= 2\left[\frac{x^3}{3}\log_e x - \int \frac{x^3}{3}\cdot\frac{1}{x}dx\right] \quad \text{(By Parts)}$$

$$= \frac{2}{3}x^3 \log_e x - \frac{2}{9}x^3 + c$$

$$= \frac{2}{3}x^3\left(\log_e x - \frac{1}{3}\right) + c$$

**Answer: (A)**

## Example 4.97

For $0 < x < 1$, let

$$f(x) = \lim_{n\to\infty}(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})$$

then $\int \frac{f(x)}{1-x}\log_e x dx$ equals

(A) $\log_e\left(\frac{x}{1-x}\right) + c$

(B) $-\log_e\left(\frac{x}{1-x}\right) + \frac{\log_e x}{1-x} + c$

(C) $\frac{\log_e x}{1-x} + \log_e(1-x) + c$

(D) $x\log_e x + \log_e(1-x) + c$

***Solution:*** We have

$$f(x) = \lim_{n\to\infty}\frac{1}{1-x}[(1-x)(1+x)(1+x^2)\cdots(1+x^{2^n})]$$

$$= \lim_{n\to\infty}\frac{1}{1-x}[(1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^n})]$$

$$= \lim_{n\to\infty}\frac{1}{1-x}[(1-x^4)(1+x^4)\cdots(1+x^{2^n})]$$

$$= \lim_{n\to\infty}\left(\frac{1-x^{2^{n+1}}}{1-x}\right) = \frac{1}{1-x}$$

Therefore

$$f(x) = \frac{1}{1-x}$$

Let

$$I = \int \frac{f(x)}{1-x}\log_e x dx = \int \frac{\log_e x}{(1-x)^2}dx$$

$$= \left(\frac{1}{1-x}\right)\log_e x - \int \frac{1}{1-x}\cdot\frac{1}{x}dx$$

$$= \frac{\log_e x}{1-x} - \int\left(\frac{1}{1-x} + \frac{1}{x}\right)dx$$

$$= \frac{\log_e x}{1-x} - \log_e\left(\frac{x}{1-x}\right) + c$$

**Answer: (B)**

## Example 4.98

$$\int \frac{x^2\cos x}{(1+\sin x)^2}dx = \frac{-x^2}{1+\sin x} - 2x(\sec x - \tan x) + 2\log|f(x)| + c$$

where $f(x)$ is equal to

(A) $\sec x + \tan x$ (B) $x(\sec x + \tan x)$

(C) $1 + \sin x$ (D) $\tan x\,(\sec x + \tan x)$

***Solution:*** Let

$$I = \int \frac{x^2\cos x}{(1+\sin x)^2}dx$$

Take

$$u = x^2 \text{ and } dv = \frac{\cos x}{(1+\sin x)^2}dx$$

so that

$$v = -\frac{1}{1+\sin x}$$

Therefore

$$I = x^2\left(\frac{-1}{1+\sin x}\right) - \int \frac{-1}{1+\sin x}(2x)\,dx$$

$$= -\frac{x^2}{1+\sin x} + 2\int \frac{x}{1+\sin x}dx$$

$$= -\frac{x^2}{1+\sin x} + 2\int \frac{x(1-\sin x)}{\cos^2 x}dx$$

$$= -\frac{x^2}{1+\sin x} + 2\int x\sec^2 x dx - 2\int x\sec x\tan x\,dx$$

$$= -\frac{x^2}{1+\sin x} + 2\left[x\tan x - \int \tan x\,dx\right]$$

$$-2\left[x\sec x - \int \sec x\,dx\right]$$

$$= -\frac{x^2}{1+\sin x} + 2x\tan x - 2\int \tan x\, dx - 2x\sec x + 2\int \sec x\, dx$$

$$= -\frac{x^2}{1+\sin x} + 2x(\tan x - \sec x) + 2\log_e|\cos x|$$
$$+ 2\log_e|\sec x + \tan x| + c$$

$$= -\frac{x^2}{1+\sin x} + 2x(\tan x - \sec x)$$
$$+ 2\log_e|\cos x(\sec x + \tan x)| + c$$

$$= -\frac{x^2}{1+\sin x} + 2x(\tan x - \sec x) + 2\log_e|1+\sin x| + c$$

**Answer: (C)**

## Example 4.99

$\int \frac{\sec^2 x - 7}{\sin^7 x}\, dx$ is equal to

(A) $\cot x \operatorname{cosec}^7 x + c$ (B) $\tan x \operatorname{cosec}^6 x + c$

(C) $\sec^7 x \operatorname{cosec} x + c$ (D) $\tan x \operatorname{cosec}^7 x + c$

***Solution:*** Let

$$I = \int \frac{\sec^2 x - 7}{\sin^7 x}\, dx$$

$$= \int \sec^2 x \operatorname{cosec}^7 x\, dx - 7\int \operatorname{cosec}^7 x\, dx$$

$$= \tan x \operatorname{cosec}^7 x - \int \tan x(-7\operatorname{cosec}^6 x\, \operatorname{cosec} x \cot x)\, dx$$
$$-7\int \operatorname{cosec}^7 x\, dx$$

$$= \tan x \operatorname{cosec}^7 x + 7\int \operatorname{cosec}^7 x\, dx - 7\int \operatorname{cosec}^7 x\, dx + c$$

$$= \tan x \operatorname{cosec}^7 x + c$$

**Answer: (D)**

## Example 4.100

$\int (\sin 4x) e^{\tan^2 x} dx =$

(A) $-2\, e^{\tan^2 x} \cos^4 x + c$ (B) $2\, e^{\tan^2 x} \sec^4 x + c$

(C) $-2\, e^{\tan^2 x} \sec^2 x + c$ (D) $2\, e^{\tan^2 x} \cos^2 x + c$

***Solution:*** Let

$$I = \int (\sin 4x) e^{\tan^2 x}\, dx$$
$$= 4\int \sin x \cos x(\cos^2 x - \sin^2 x) e^{\tan^2 x} dx$$
$$= 4\int \sin x \cos^3 x\, (1 - \tan^2 x) e^{\tan^2 x}\, dx$$
$$= 4\int \tan x \cos^4 x(1 - \tan^2 x) e^{\tan^2 x} dx \qquad (4.13)$$

Put $t = \tan^2 x$. Therefore

$$dt = 2\tan x \sec^2 x\, dx$$

Hence

$$I = 2\int \frac{2\tan x \sec^2 x}{\sec^6 x}(1 - \tan^2 x) e^{\tan^2 x} dx$$
$$= 2\int \frac{1-t}{(1+t)^3} e^t\, dt$$
$$= -2\int \frac{t+1-2}{(1+t)^3} e^t\, dt$$
$$= -2\int \left(\frac{1}{(t+1)^2} - \frac{2}{(t+1)^3}\right) e^t dt$$
$$= -2\int [f(t) + f'(t)]\, e^t \quad \text{where } f(t) = \frac{1}{(t+1)^2}$$
$$= -\frac{2\, e^t}{(t+1)^2} + c$$
$$= -\frac{2\, e^{\tan^2 x}}{(1+\tan^2 x)^2} + c$$
$$= -2 e^{\tan^2 x} \cos^4 x + c$$

**Answer: (A)**

# 4.5 | Fundamental Classes of Integrable Functions

In this section we would discuss various classes of integrable function and give examples so that students understand how to evaluate each class.

### 4.5.1 Integration of Rational Functions of the Form $\frac{P(x)}{Q(x)}$ [P(x) and Q(x) are Polynomials]

***Method:*** Using partial fractions. For partial fractions, see Chapter 9 (Vol. 1).

**Example 4.101**

Evaluate $\int \frac{x^2}{(x-1)(x-2)(x-3)}\,dx$.

***Solution:*** Write

$$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

so that

$$x^2 = A(x-2)(x-3) + B(x-1)(sx-3) + C(x-1)(x-2)$$

Put $x = 1, 2, 3$ successively on both sides. Then

$$A = \frac{1}{2},\ B = -4,\ C = \frac{9}{2}$$

Therefore

$$\int \frac{x^2}{(x-1)(x-2)(x-3)}\,dx = \frac{1}{2}\int\frac{dx}{x-1} - 4\int\frac{dx}{x-2} + \frac{9}{2}\int\frac{dx}{x-3}$$

$$= \frac{1}{2}\log_e|x-1| - 4\log_e|x-2| + \frac{9}{2}\log_e|x-3| + c$$

**Example 4.102**

Evaluate $\int \frac{x^2}{(x-1)^2(x-2)}\,dx$.

***Solution:*** Write

$$\frac{x^2}{(x-1)^2(x-2)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$$

Therefore

$$x^2 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

Now,

$$x = 1 \Rightarrow B = -1$$
$$x = 2 \Rightarrow C = 4$$

Equating the coefficient of $x^2$ on both sides, we get

$$A + C = 1 \Rightarrow A = -3$$

Therefore

$$\int \frac{x^2}{(x-1)^2(x-2)}\,dx = -3\int\frac{dx}{x-1} - \int\frac{dx}{(x-1)^2} + 4\int\frac{dx}{x-2}$$

$$= -3\log_e|x-1| + \frac{1}{x-1} + 4\log_e|x-2| + c$$

### 4.5.2 To Evaluate Integrals of the Form $\int \frac{px+q}{ax^2+bx+c}\,dx$

**Case I:** If $ax^2 + bx + c = 0$ has real roots, use partial fractions.
**Case II:** If $ax^2 + bx + c = 0$ has no real roots. Then write

$$px+q = \lambda\frac{d}{dx}(ax^2+bx+c)+\mu$$

where $\lambda$ and $\mu$ are constants which can be determined by equating the coefficients on both sides.

**Case III:** To evaluate $\int \frac{P(x)}{ax^2+bx+c}\,dx$ where $P(x)$ is a polynomial of degree greater than or equal two. Write

$$P(x) = Q(x)(ax^2 + bx + c) + (px + q)$$

so that

$$\int \frac{P(x)}{ax^2+bx+c}\,dx = \int Q(x)\,dx + \int\frac{px+q}{ax^2+bx+c}$$

Now proceed as in Case II or Case I.

## Example 4.103

**(Based on Case II)**

Evaluate $\int \frac{3x+2}{x^2+2x+9}\,dx$.

***Solution:*** Let

$$3x+2 = \lambda \frac{d}{dx}(x^2+2x+9)+\mu$$
$$= \lambda(2x+2)+\mu$$

Therefore

$$3 = 2\lambda \quad \text{and} \quad 2\lambda+\mu = 2$$

$$\Rightarrow \lambda = \frac{3}{2} \quad \text{and} \quad \mu = -1$$

Now

$$\int \frac{3x+2}{x^2+2x+9}\,dx = \frac{3}{2}\int \frac{2x+2}{x^2+2x+9}\,dx - \int \frac{dx}{x^2+2x+9}$$
$$= \frac{3}{2}\log_e(x^2+2x+9) - \int \frac{dx}{(x+1)^2+8}$$
$$= \frac{3}{2}\log_e(x^2+2x+9) - \frac{1}{2\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{x+1}{2\sqrt{2}}\right)+c$$

## Example 4.104

**(Based on Case III)**

Evaluate $\int \frac{2x^3+x^2+3x-1}{2x^2+4x+9}\,dx$.

***Solution:*** Here

$$P(x) = 2x^3+x^2+3x-1$$

Divide $P(x)$ with $2x^2+4x+9$. Therefore

$$2x^3+x^2+3x-1 = \left(x-\frac{3}{2}\right)(2x^2+4x+9)+\frac{25}{2}$$

So

$$\int \frac{2x^3+x^2+3x-1}{2x^2+4x+9}\,dx = \int\left(x-\frac{3}{2}\right)dx + \int \frac{25/2}{2x^2+4x+9}\,dx$$
$$= \frac{x^2}{2}-\frac{3}{2}x+\frac{25}{4}\int \frac{dx}{x^2+2x+(9/2)}$$
$$= \frac{x^2}{2}-\frac{3}{2}x+\frac{25}{4}\int \frac{dx}{(x+1)^2+(7/2)}$$
$$= \frac{x^2}{2}-\frac{3}{2}x+\frac{25}{4}\times\frac{\sqrt{2}}{\sqrt{7}}\cdot\mathrm{Tan}^{-1}\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)+c$$

### 4.5.3 Integrals of the Form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ $(a \neq 0)$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

1. To evaluate $\int \frac{dx}{\sqrt{ax^2+bx+c}} (a \neq 0)$, make the coefficient of $x^2$ as $\pm 1$, complete the perfect square and use standard integrals.
2. To evaluate $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$, write

$$px+q = \lambda \frac{d}{dx}(ax^2+bx+c)+\mu$$

so that

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = \lambda\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \mu\int \frac{dx}{\sqrt{ax^2+bx+c}}$$
$$= 2\lambda\sqrt{ax^2+bx+c}+\mu\int \frac{dx}{ax^2+bx+c}$$

Now, use (1) for $\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

**Example 4.105**

Evaluate $\int \frac{dx}{\sqrt{3x^2+4x+5}}$.

***Solution:*** Let

$$I = \int \frac{dx}{\sqrt{3x^2+4x+5}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2 + \frac{4x}{3} + \frac{5}{3}}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(x+\frac{2}{3}\right)^2 + \frac{11}{9}}}$$

$$= \frac{1}{\sqrt{3}} \operatorname{Sinh}^{-1}\left(\frac{x+(2/3)}{(\sqrt{11}/3)}\right) + c$$

$$= \frac{1}{\sqrt{3}} \operatorname{Sinh}^{-1}\left(\frac{3x+2}{\sqrt{11}}\right) + c$$

**Example 4.106**

Evaluate $\int \frac{3-2x}{\sqrt{4+2x-x^2}}\,dx$.

***Solution:*** Let

$$I = \int \frac{3-2x}{\sqrt{4+2x-x^2}}\,dx$$

Let

$$3-2x = \lambda(-2x+2)+\mu$$

so that $-2\lambda = -2$, $2\lambda + \mu = 3$ and hence $\lambda = 1$ and $\mu = 1$.

Therefore

$$I = \int \frac{-2x+2}{\sqrt{4+2x-x^2}}\,dx + \int \frac{dx}{\sqrt{4+2x-x^2}}$$

$$= 2\sqrt{4+2x-x^2} + \int \frac{dx}{\sqrt{5-(x-1)^2}}$$

$$= 2\sqrt{4+2x-x^2} + \operatorname{Sin}^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + c$$

### 4.5.4 Integrals of the Form $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

To evaluate $\int (px+q)\sqrt{ax^2+bx+c}$, write

$$px+q = \lambda \frac{d}{dx}(ax^2+bx+c)+\mu$$

and then use standard integrals.

**Example 4.107**

Evaluate $\int (2x-5)\sqrt{2+3x-x^2}\,dx$.

***Solution:*** We have

$$2x-5 = \lambda(-2x+3)+\mu$$

so that $\lambda = -1$, $\mu = -2$. Therefore

$$\int (2x-5)\sqrt{2+3x-x^2} = -\int (3-2x)\sqrt{2+3x-x^2}\,dx - 2\int \sqrt{2+3x-x^2}\,dx$$

$$= -\frac{(2+3x-x^2)^{(1/2)+1}}{(1/2)+1} - 2\int \sqrt{\frac{17}{4} - \left(x-\frac{3}{2}\right)^2}\,dx$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - \left(\frac{2x-3}{2}\right)\sqrt{2+3x-x^2} - \frac{17}{4}\operatorname{Sin}^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

***Note:*** In the above integral $\int (px+q)\sqrt{ax^2+bx+c}\,dx$, if $p=0$ and $q=1$ then the integral can be reduced to the form $\int \sqrt{a^2 \pm x^2}\,dx$ or $\int \sqrt{x^2-a^2}\,dx$.

### 4.5.5 Integral of the Form $\int \frac{x^2 \pm 1}{x^4+kx^2+1}\,dx$

To evaluate $\int \frac{x^2 \pm 1}{x^4+kx^2+1}\,dx$ where $k = -1, 0,$ or $1$, divide numerator and denominator by $x^2$ and put the substitution $x + (1/x) = t$ or $x - (1/x) = t$ according as the numerator is $1 - (1/x^2)$ or $1 + (1/x^2)$.

**Example 4.108**

Evaluate $\int \frac{x^2+1}{x^4+x^2+1}\,dx$.

***Solution:*** We have

$$\int \frac{x^2+1}{x^4+x^2+1}\,dx = \int \frac{1+(1/x^2)}{x^2+(1/x^2)+1}\,dx$$

$$= \int \frac{dt}{t^2+3} \quad \text{where } t = x - \frac{1}{x}$$

$$= \frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\frac{t}{\sqrt{3}} + c$$

$$= \frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{x-(1/x)}{\sqrt{3}}\right) + c$$

### 4.5.6 Special Cases

In this section we discuss four special cases: To evaluate

$$\int \frac{dx}{f_1(x)\sqrt{f_2(x)}}$$

where

1. both $f_1$ and $f_2$ are linear expressions. In this case put $\sqrt{f_2(x)} = t$.
2. $f_1$ is quadratic and $f_2$ is linear. Then also put $\sqrt{f_2(x)} = t$.
3. $f_1$ is linear and $f_2$ is quadratic. In this case put $f_1(x) = 1/t$.
4. **(a)** To evaluate $\int \frac{x}{(ax^2+b)\sqrt{px^2+q}}\,dx$, put $\sqrt{px^2+q} = t$.

   **(b)** To evaluate $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}\,dx$, first put $x = \frac{1}{t}$, simplify the integral and then put $\sqrt{p+qt^2} = z$.

**Example 4.109**

Evaluate $\int \frac{dx}{(2x+3)\sqrt{x-1}}$.

***Solution:*** We have

$$I = \int \frac{dx}{(2x+3)\sqrt{x-1}}$$

$$= \int \frac{1}{[2(t^2+1)+3]t}(2t)\,dt \quad \text{where } t = \sqrt{x-1}$$

$$= 2\int \frac{dt}{2t^2+5}$$

$$= \int \frac{dt}{t^2+(5/2)}$$

$$= \frac{1}{\sqrt{5/2}}\mathrm{Tan}^{-1}\left(\frac{t}{\sqrt{5/2}}\right) + c$$

$$= \sqrt{\frac{2}{5}}\mathrm{Tan}^{-1}\left(\sqrt{\frac{2}{5}}(x-1)\right) + c$$

## Example 4.110

Evaluate $\int \frac{dx}{(x^2+1)\sqrt{x}}$.

***Solution:*** Let

$$I = \int \frac{dx}{(x^2+1)\sqrt{x}}$$

$$= \int \frac{1}{(t^4+1)t}(2t)\,dt \quad \text{where } t = \sqrt{x}$$

$$= \int \frac{2}{t^4+1}\,dt$$

$$= \int \frac{(t^2+1)-(t^2-1)}{t^4+1}\,dt$$

$$= \int \frac{t^2+1}{t^4+1}\,dt - \int \frac{t^2-1}{t^4+1}\,dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt - \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$= \int \frac{dz}{z^2+2} - \int \frac{du}{u^2-2} \qquad \left[\text{where } z = t - \frac{1}{t} \text{ and } u = t + \frac{1}{t}\right]$$

$$= \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{z}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log_e\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + c$$

$$= \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{t-(1/t)}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log_e\left|\frac{t+(1/t)-\sqrt{2}}{t+(1/t)+\sqrt{2}}\right| + c$$

$$= \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{x-1}{x\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log_e\left|\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right| + c$$

## Example 4.111

Evaluate $\int \frac{x}{(1-x^2)\sqrt{x^2+1}}\,dx$.

***Solution:*** We have

$$I = \int \frac{x}{(1-x^2)\sqrt{x^2+1}}\,dx$$

Put $\sqrt{x^2+1} = t$. Therefore $x^2 = t^2 - 1$ and $x\,dx = t\,dt$. So

$$I = \int \frac{1}{(2-t^2)}(t)\,dt$$

$$= \int \frac{dt}{2-t^2}$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}-t} + \frac{1}{\sqrt{2}+t}\right)dt$$

$$= \frac{1}{2\sqrt{2}}\log_e\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + c$$

$$= \frac{1}{2\sqrt{2}}\log_e\left|\frac{\sqrt{2}+\sqrt{x^2+1}}{\sqrt{2}-\sqrt{x^2+1}}\right| + c$$

## Example 4.112

Evaluate $\int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$.

***Solution:*** We have

$$I = \int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$$

Put $x = 1/t$ so that $dx = -(1/t^2)dt$. Therefore

$$I = \int \frac{(1/t)}{\left(\frac{1}{t^2}-1\right)\sqrt{\frac{1}{t^2}+1}}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int \frac{dt}{(1-t^2)\sqrt{1+t^2}}$$

Now proceed as in Example 4.111.

### 4.5.7 Integrals of the Form $\int R\left(x, x^{\frac{p_1}{q_1}}, x^{\frac{p_2}{q_2}}, \ldots, x^{\frac{p_k}{q_k}}\right) dx$

To evaluate $\int R\left(x, x^{\frac{p_1}{q_1}}, x^{\frac{p_2}{q_2}}, \ldots, x^{\frac{p_k}{q_k}}\right) dx$ where $R$ is a rational function of its variables $x, x^{p_1/q_1}, \ldots, x^{p_k/q_k}$, put $x = t^n$ where $n$ is the L.C.M. of the denominators of the fractions $p_1/q_1, p_2/q_2, \ldots, p_k/q_k$. If $R$ is a rational function of linear fractions of the form $\left(\dfrac{ax+b}{cx+d}\right)^{p/q}$, then put

$$\frac{ax+b}{cx+d} = n$$

where $n$ is the L.C.M. of the denominators of fractional powers of $(ax+b)/(cx+d)$.

**Example 4.113**

Evaluate $\displaystyle\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}\,dx$.

***Solution:*** We have

$$I = \int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}\,dx$$

Here the powers of $x$ are 1/2, 1/3, 5/4, 7/6 and the L.C.M. of the denominators is 12. Hence put $x = t^{12}$ so that $dx = 12\,t^{11}\,dt$. Therefore

$$I = \int \frac{t^6+t^4}{t^{15}-t^{14}}(12t^{11})dt$$

$$= 12\int \frac{t^3+t}{t-1}\,dt$$

$$= 12\int \left(t^2+t+2+\frac{2}{t-1}\right)dt$$

$$= 12\left[\frac{t^3}{3}+\frac{t^2}{2}+2t+2\log_e|t-1|\right]+c$$

where $t = x^{1/12}$.

**Example 4.114**

Evaluate $\displaystyle\int \frac{(2x+1)^{1/2}}{(2x+1)^{(1/3)}+1}\,dx$.

***Solution:*** We have

$$I = \int \frac{(2x+1)^{1/2}}{(2x+1)^{(1/3)}+1}\,dx$$

Here the powers of $2x+1$ are 1/2 and 1/3. Put $2x+1 = t^6$ so that $dx = 3t^5\,dt$. Therefore

$$I = \int \frac{t^3}{t^2+1}(3t^5)\,dt$$

$$= 3\int \frac{t^8}{t^2+1}\,dt$$

$$= 3\int \left(t^6-t^4+t^2-1+\frac{1}{t^2+1}\right)dt$$

$$= 3\left[\frac{t^7}{7}-\frac{t^5}{5}+\frac{t^3}{3}-t+\mathrm{Tan}^{-1}t\right]+c$$

where $t = (2x+1)^{1/6}$.

### 4.5.8 The Binomial Differential

The integral $\int x^m(a+bx^n)^p$ ($m, n, p$ are rational numbers) can be evaluated in the following four cases:

**Case I:** If $p$ is a positive integer, then use binomial expansion for positive integral index and then integrate.

**Case II:** If $p$ is a negative integer, then put the substitution $x = t^\alpha$ where $\alpha$ is the L.C.M. of the denominators of the fractions of $m$ and $n$.

**Case III:** If $\dfrac{m+1}{n}$ is an integer, then put the substitution $a + bx^n = t^k$ where $k$ is denominator of $p$.

**Case IV:** If $\dfrac{m+1}{n} + p$ is an integer, then put the substitution $a + bx^n = t^k x^n$ where $k$ is the denominator of $p$.

## Example 4.115

Evaluate $\displaystyle\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$.

***Solution:*** We have

$$I = \int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$$
$$= \int x^{-1/2}(1+x^{1/4})^{1/3}$$

Here

$$\frac{m+1}{n} = \frac{-(1/2)+1}{1/4} = 2 \text{ (Integer)}$$

So put

$$1 + x^{1/4} = t^3$$
$$\Rightarrow x = (t^3 - 1)^4$$
$$\Rightarrow dx = 12\, t^2(t^3 - 1)^3\, dt$$

Therefore

$$I = \int (t^3-1)^{-2}\, t[12t^2(t^3-1)^3]\, dt$$
$$= \int 12t^3(t^3-1)\, dt$$
$$= \frac{12}{7}t^7 - 3t^4 + c$$
$$= \frac{12}{7}(1+\sqrt[4]{x})^{7/3} - 3(1+\sqrt[4]{x})^{4/3} + c$$

## Example 4.116

Evaluate $\displaystyle\int \frac{dx}{\sqrt[3]{x^2}\,(1+\sqrt[3]{x^2})}$.

***Solution:*** We have

$$I = \int \frac{dx}{\sqrt[3]{x^2}\,(1+\sqrt[3]{x^2})}$$
$$= \int x^{-2/3}(1+x^{2/3})^{-1}\, dx$$

Here $p = -1$ is a negative integer and $m = -2/3, n = 2/3$. Put $x = t^3$. Then $dx = 3t^2 dt$. So

$$I = \int t^{-2}(1+t^2)^{-1} 3t^2 dt$$
$$= 3\int \frac{dt}{1+t^2}$$
$$= 3\,\mathrm{Tan}^{-1} t + c$$
$$= 3\,\mathrm{Tan}^{-1}(\sqrt[3]{x}) + c$$

### 4.5.9 Euler's Substitution

To evaluate $\int R(x, \sqrt{ax^2+bx+c})$, we use one of the following three substitutions called Euler's substitutions.

1. Put $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$.
2. Put $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$.
3. If $ax^2 + bx + c = a(x - \alpha)(x - \beta)$ where $\alpha, \beta$ are real roots of $ax^2 + bx + c = 0$, then put the substitution

$$\sqrt{ax^2+bx+c} = t(x-\alpha) \quad \text{or} \quad t(x-\beta)$$

and simplify the integral and perform integration.

***Note:*** Sometimes the integrals which can be evaluated by using Euler's substitution can also be evaluated by other substitution methods.

## Example 4.117

Evaluate $I = \int \frac{dx}{x+\sqrt{x^2+x+1}}$.

***Solution:*** Put

$$\sqrt{x^2+x+1} = t - x$$

$$\Rightarrow x^2+x+1 = t^2 - 2tx + x^2$$
$$\Rightarrow x\,(2t+1) = t^2 - 1$$
$$\Rightarrow x = \frac{t^2-1}{2t+1}$$

Therefore

$$dx = \frac{2t(2t+1)-2(t^2-1)}{(2t+1)^2}dt$$
$$= \frac{2t^2+2t+2}{(2t+1)^2}dt \qquad (4.14)$$

Again

$$x+\sqrt{x^2+x+1} = x+t-x = t$$

Therefore

$$I = \int \frac{1}{t}\frac{(2t^2+2t+2)}{(2t+1)^2}dt \quad \text{[from Eq. (4.14)]}$$
$$= 2\int \frac{t^2+t+1}{t(2t+1)^2}\,dt \quad \text{(use partial fractions)}$$

### 4.5.10 Integration of Trigonometric Functions

To evaluate $\int \sin^m x \cos^n x\,dx$ where $m$ and $n$ are rational numbers, use the substitution $t = \sin x$ or $\cos x$. The integral will be reduced to the form $\int t^m(1-t^2)^{(n-1)/2}dt$. Hence, it can be integrated in elementary functions only. See the following cases.

**Case I:** $n$ is odd positive integer so that $\frac{n-1}{2}$ is an integer.

**Case II:** If $m$ is an odd positive integer then $\left(\frac{m+1}{2}\right)$ is an integer.

**Case III:** $m+n$ is even and $\frac{m+1}{2}+\frac{n-1}{2}$ are integers, then

- **(a)** put $t = \sin x$ if $n$ is odd.
- **(b)** put $t = \cos x$ if $m$ is odd.
- **(c)** put $\tan x = t$ or $\cot x = t$ when $m+n$ is even.

## Example 4.118

Evaluate $I = \int \sin^5 x \cos^4 x\,dx$.

***Solution:*** Put $\cos x = t$ so that $\sin x\,dx = -dt$. Therefore

$$I = \int (1-t^2)^2 t^4\,(-dt)$$
$$= -\int (t^8 - 2t^6 + t^4)\,dt$$
$$= -\left[\frac{1}{9}t^9 - \frac{2}{7}t^7 + \frac{1}{5}t^5\right] + c$$

where $t = \cos x$.

**Example 4.119**

Evaluate $I = \int \sin^2 x \cos^3 x \, dx$.

***Solution:*** Put $\sin x = t$ so that $\cos x\, dx = dt$. Therefore

$$I = \int t^2 (1 - t^2)\, dt$$

$$= -\frac{1}{5}t^5 + \frac{1}{3}t^3 + c$$

$$= -\frac{1}{5}\sin^5 x + \frac{1}{3}\sin^3 x + c$$

**Example 4.120**

Evaluate $I = \int \frac{\sin^3 x}{\cos^4 x} dx$.

***Solution:*** Put $\cos x = t$ so that $\sin x\, dx = -dt$. Therefore

$$I = \int \left( \frac{1 - t^2}{t^4} \right)(-dt)$$

$$= \int \left( \frac{1}{t^2} - \frac{1}{t^4} \right) dt$$

$$= -\frac{1}{t} + \frac{1}{3}t^{-3} + c$$

$$= \frac{1}{3}\sec^3 x - \sec x + c$$

## 4.5.11 Reduction

At plus 2 level and in IIT-JEE, we have simple Reduction, viz. evaluation of integrals of the form $\int \sin^n x\, dx$, $\int \cos^n x\, dx$, $\int \tan^n x\, dx$, $\int \sec^n x\, dx$, $\int \mathrm{cosec}^n x\, dx$ and $\int \cot^n x\, dx$ where $n \geq 2$ integer.

**Examples**

**1.** Let

$$I_n = \int \sin^n x\, dx$$

$$= \int \sin^{n-1} x \sin x\, dx$$

Using by parts, we get

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\, I_{n-2} - (n-1)I_n$$

Therefore

$$nI_n = (n-1)I_{n-2} - \cos x \sin^{n-1} x$$

or

$$I_n = \left( \frac{n-1}{n} \right) I_{n-2} - \frac{1}{n}\cos x \sin^{n-1} x$$

Similarly, if $I_n = \int \cos^2 x\, dx$, then

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n}\sin x \cos^{n-1} x$$

**2.** Let

$$I_n = \int \tan^n x\, dx$$

$$= \int \tan^{n-2} x \tan^2 x\, dx$$

$$= \int \tan^{n-2} x\ (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x\ \sec^2 x\, dx - I_{n-2}$$

$$= \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$$

Therefore

$$I_n = \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$$

## WORKED-OUT PROBLEMS

### Single Correct Choice Type Questions

**1.** $\displaystyle\int \frac{x^m-1}{x^{m+1}\sqrt{1-2x^m+mx^{2m}}}dx = \frac{f(x)}{mx^m}+c$ where $f(x)$ is equal to

(A) $\sqrt{1-2x^m+mx^{2m}}$ (B) $\sqrt{1-x^m+x^{2m}}$

(C) $\sqrt{x^m+\dfrac{1}{x^m}}$ (D) $\sqrt{x^{2m}+\dfrac{1}{x^{2m}}}$

**Solution:** Let

$$I=\int \frac{x^m-1}{x^{m+1}\sqrt{1-2x^m+mx^{2m}}}dx$$

Dividing numerator and denominator with $x^{2m+1}$ we get

$$I=\int \frac{\dfrac{1}{x^{m+1}}-\dfrac{1}{x^{2m+1}}}{\sqrt{\dfrac{1}{x^{2m}}-\dfrac{2}{x^m}+m}}dx$$

Put

$$t^2=\frac{1}{x^{2m}}-\frac{2}{x^m}+m$$

Therefore

$$2t\ dt=\left(\frac{-2m}{x^{2m+1}}+\frac{2m}{x^{m+1}}\right)dx$$

$$\Rightarrow t\ dt = m\left(\frac{1}{x^{m+1}}-\frac{1}{x^{2m+1}}\right)dx$$

Hence

$$I=\int \frac{1}{t}\left(\frac{t}{m}\right)dt$$

$$=\frac{t}{m}+c$$

$$=\frac{1}{m}\sqrt{\frac{1}{x^{2m}}-\frac{2}{x^m}+m}+c$$

$$=\frac{1}{mx^m}\sqrt{1-2x^m+mx^{2m}}+c$$

**Answer: (A)**

**2.** If $\displaystyle\int \frac{dx}{(x^2-4)\sqrt{x+1}} = \frac{1}{k}\log_e\left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right|$ $-\dfrac{1}{2}\text{Tan}^{-1}\sqrt{x+1}+c$ then $k$ equals

(A) $2\sqrt{3}$ (B) $4\sqrt{3}$

(C) $\dfrac{1}{4\sqrt{3}}$ (D) $\dfrac{1}{3\sqrt{3}}$

**Solution:** Let

$$I=\int \frac{dx}{(x^2-4)\sqrt{x+1}}$$

Put $\sqrt{x+1}=t$ so that $dx = 2t\,dt$. Therefore

$$I=\int \frac{1}{\left[(t^2-1)^2-4\right]t}(2t)\ dt$$

$$=\int \frac{2}{(t^2-1-2)(t^2-1+2)}dt$$

$$=2\int \frac{dt}{(t^2-3)(t^2+1)}$$

$$=\frac{1}{2}\int\left(\frac{1}{t^2-3}-\frac{1}{t^2+1}\right)dt$$

$$=\frac{1}{2}\int\left(\frac{dt}{t^2-3}\right)-\frac{1}{2}\int\frac{dt}{t^2+1}$$

$$=\frac{1}{2}\times\frac{1}{2\sqrt{3}}\log_e\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|-\frac{1}{2}\text{Tan}^{-1}t+c$$

$$=\frac{1}{4\sqrt{3}}\log_e\left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right|-\frac{1}{2}\text{Tan}^{-1}\sqrt{x+1}+c$$

**Answer: (B)**

**3.** $\displaystyle\int \frac{dx}{(x-a)^{3/2}(x+a)^{1/2}}$ is equal to

(A) $\dfrac{1}{a}\sqrt{\dfrac{x+a}{x-a}}+c$ (B) $\dfrac{1}{2a}\sqrt{\dfrac{x+a}{x-a}}+c$

(C) $\dfrac{1}{a}\sqrt{\dfrac{x-a}{x+a}}+c$ (D) $-\dfrac{1}{a}\sqrt{\dfrac{x+a}{x-a}}+c$

**Solution:** Let

$$I=\int\frac{dx}{\sqrt{x^2-a^2}\,(x-a)}$$

Put $x-a=1/t$ so that $dx = (-1/t^2)dt$. Therefore

$$I=\int\frac{1}{\sqrt{\left(\dfrac{1}{t}+a\right)^2-a^2}\left(\dfrac{1}{t}\right)}\left(\frac{-1}{t^2}\right)dt$$

$$=-\int\frac{dt}{\sqrt{1+2at}}$$

$$=\frac{-1}{2a}\left[\frac{(1+2at)^{-(1/2)+1}}{-\dfrac{1}{2}+1}\right]+c$$

$$= -\frac{1}{a}\sqrt{1+2at}+c$$

$$= -\frac{1}{a}\sqrt{1+\frac{2a}{x-a}}+c$$

$$= -\frac{1}{a}\sqrt{\frac{x+a}{x-a}}+c$$

**Answer: (D)**

**4.** $\int \frac{dx}{(x-3)^2\sqrt{x^2-6x+4}}$ is equal to

(A) $\frac{\sqrt{x^2-6x+4}}{5|x-3|}+c$ (B) $\frac{\sqrt{x^2-6x-4}}{5|x-3|}+c$

(C) $\frac{\sqrt{x^2-6x-5}}{5|x-3|}+c$ (D) $\frac{\sqrt{x^2-6x+5}}{5|x-3|}+c$

**Solution:** Let

$$I = \int \frac{dx}{(x-3)^2\sqrt{x^2-6x+4}}$$

Put $x-3 = 1/t$ so that $dx = -(1/t^2)dt$. Therefore

$$I = \int \frac{t^2}{\sqrt{\left(\frac{1}{t}+3\right)^2-6\left(\frac{1}{t}+3\right)+4}}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int \frac{t}{\sqrt{(1+3t)^2-6t(1+3t)+4t^2}}dt$$

$$= -\int \frac{t}{\sqrt{1-5t^2}}dt$$

$$= \frac{1}{5}\sqrt{1-5t^2}+c \quad \text{where } t = \frac{1}{x-3}$$

**Answer: (A)**

**5.** $\int \frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}$ is equal to

(A) $\frac{1}{2}(\sin x-\cos x+x)+c$

(B) $\frac{1}{2}(\sin x-\cos x-\tan x+\cot x)+c$

(C) $\frac{1}{2}(\sin x-\cos x-x)+c$

(D) $\frac{1}{2}(\sin x+\cos x-\tan x-\cot x+x)+c$

**Solution:** Let $I$ be the given integral. Then

$$I = \int \frac{\sin x\cos x}{1+\cos x+\sin x}dx$$

$$= \int \frac{\sin x}{\sec x+1+\tan x}dx$$

$$= \int \frac{\sin x(1+\tan x-\sec x)}{(1+\tan x)^2-\sec^2 x}dx$$

$$= \int \frac{\sin x(1+\tan x-\sec x)}{2\tan x}dx$$

$$= \frac{1}{2}\int \cos x(1+\tan x-\sec x)\,dx$$

$$= \frac{1}{2}\int (\cos x+\sin x-1)\,dx$$

$$= \frac{1}{2}(\sin x-\cos x-x)+c$$

**Answer: (C)**

**6.** $\int \frac{a+b\sin x}{(b+a\sin x)^2}dx$ equals

(A) $-\left(\frac{a\cos x}{b+a\sin x}\right)+c$ (B) $-\left(\frac{\sin x}{a+b\cos x}\right)+c$

(C) $\frac{\cos x}{b+a\sin x}+c$ (D) $\frac{\sin x}{a+b\cos x}+c$

**Solution:** Let

$$I = \int \frac{a+b\sin x}{(b+a\sin x)^2}dx$$

$$= \frac{b}{a}\int \frac{\frac{a^2}{b}-b+(b+a\sin x)}{(b+a\sin x)^2}dx$$

$$= \frac{a^2-b^2}{a}\int \frac{dx}{(b+a\sin x)^2}+\frac{b}{a}\int \frac{dx}{b+a\sin x} \tag{4.15}$$

Let

$$f(x) = \frac{a\cos x}{b+a\sin x}$$

so that

$$f'(x) = \frac{-a-b\sin x}{(b+a\sin x)^2}$$

$$= -\frac{b}{a}\frac{\left(a\sin x+b+\frac{a^2}{b}-b\right)}{(b+a\sin x)^2}$$

$$= -\frac{b}{a}\left(\frac{1}{b+a\sin x}\right)-\frac{a^2-b^2}{a}\left(\frac{1}{(b+a\sin x)^2}\right) \tag{4.16}$$

Therefore

$$f(x) = \int f'(x)+c$$

$$= \frac{-b}{a}\int \frac{dx}{b+a\sin x}-\frac{a^2-b^2}{a}\int \frac{dx}{(b+a\sin x)^2}+c$$

[By Eq. (4.16)]

Hence from Eq. (4.15),

$$I = -f(x) + c = -\left(\frac{a\cos x}{b + a\sin x}\right) + c$$

**Answer: (A)**

**7.** If $I_n = \int (\sin x + \cos x)^n dx$ $(n \geq 2)$, then $nI_n - 2(n-1)\, I_{n-2}$ is equal to

(A) $(\sin x + \cos x)^n (\sin x - \cos x)$
(B) $(\sin x + \cos x)^{n-1} (\sin x - \cos x)$
(C) $(\sin x + \cos x)^{n+1} (\sin x - \cos x)$
(D) $(\sin x - \cos x)^{n-1} (\sin x + \cos x)$

**Solution:** Let

$$I_n = \int (\sin x + \cos x)^{n-1} (\sin x + \cos x)\, dx$$

Take

$$u = (\sin x + \cos x)^{n-1}$$

and

$$dv = (\sin x + \cos x) dx$$

so that

$$v = \sin x - \cos x$$

Therefore, using integration by parts we have

$$\begin{aligned} I_n &= (\sin x + \cos x)^{n-1} (\sin x - \cos x) \\ &\quad - \int (n-1)(\sin x + \cos x)^{n-2} \\ &\quad \times (\cos x - \sin x)(\sin x - \cos x) dx \\ &= (\sin x + \cos x)^{n-1}(\sin x - \cos x) \\ &\quad + (n-1)\int (\sin x + \cos x)^{n-2} (\cos x - \sin x)^2 dx \\ &= (\sin x + \cos x)^{n-1}(\sin x - \cos x) \\ &\quad + (n-1)\int (\sin x + \cos x)^{n-2} [2 - (\sin x + \cos x)^2] dx \\ &= (\sin x + \cos x)^{n-1}(\sin x - \cos x) \\ &\quad + 2(n-1) I_{n-2} - (n-1) I_n \end{aligned}$$

Therefore

$$I_n + (n-1)I_n - 2(n-1)I_{n-2} = (\sin x + \cos x)^{n-1}(\sin x - \cos x)$$

$$\Rightarrow nI_n - 2(n-1)I_{n-2} = (\sin x + \cos x)^{n-1}(\sin x - \cos x)$$

**Answer: (B)**

**8.** $\int \frac{x^2 + n(n-1)}{(x\sin x + n\cos x)^2} dx$ is equal to

(A) $\frac{x\cos x}{x\sin x + n\cos x} + \tan x + c$

(B) $\frac{-\sec x}{x\sin x + n\cos x} + \tan x + c$

(C) $\frac{x\sec x}{x\sin x + n\cos x} + \cot x + c$

(D) $\frac{-x^n \sec x}{x^n \sin x + nx^{n-1}\cos x} + \tan x + c$

**Solution:** Let $I$ be the given integral. Therefore

$$\begin{aligned} I &= \int \frac{[x^2 + n(n-1)]x^{2(n-1)}}{(x\sin x + n\cos x)^2 x^{2(n-1)}} dx \\ &= \int \frac{[x^2 + n(n-1)]x^{2n-1}}{(x^n \sin x + nx^{n-1}\cos x)^2} dx \end{aligned} \tag{4.17}$$

Put $t = x^n \sin x + nx^{n-1} \cos x$ so that

$$\begin{aligned} dt &= [nx^{n-1}\sin x + x^n \cos x + n(n-1)x^{n-2}\cos x \\ &\quad - nx^{n-1}\sin x] dx \\ &= [x^2 + n(n-1)]x^{n-2}\cos x\, dx \end{aligned} \tag{4.18}$$

So from Eq. (4.17),

$$I = \int \frac{[x^2 + n(n-1)]x^{n-2}\cos x \cdot x^n \sec x}{(x^n \sin x + nx^{n-1}\cos x)^2} dx$$

Take $u = x^n \sec x$ and

$$dv = \frac{[x^2 + n(n-1)]x^{n-2}\cos x}{(x^n \sin x + nx^{n-1}\cos x)^2} dx$$

So from Eq. (4.18),

$$\begin{aligned} I &= (x^n \sec x)\left(\frac{-1}{x^n \sin x + nx^{n-1}\cos x}\right) \\ &\quad + \int \frac{1}{x^n \sin x + nx^{n-1}\cos x}(nx^{n-1}\sec x + x^n \sec x \tan x) dx \\ &= \frac{-x^n \sec x}{x^n \sin x + nx^{n-1}\cos x} + \int \sec^2 x\, dx \\ &= \frac{-x^n \sec x}{x^n \sin x + nx^{n-1}\cos x} + \tan x + c \end{aligned}$$

**Answer: (D)**

**9.** $\int \frac{dx}{\cos^6 x + \sin^6 x}$ equals

(A) $\mathrm{Tan}^{-1}(\tan x + \cot x) + c$

(B) $\frac{1}{\sqrt{3}} \mathrm{Tan}^{-1} \frac{(\tan x - \cot x)}{\sqrt{3}} + c$

(C) $\frac{1}{\sqrt{3}} \mathrm{Tan}^{-1}(\sin x + \cos x) + c$

(D) $\frac{1}{\sqrt{3}} \mathrm{Tan}^{-1} \frac{(\sin x - \cos x)}{\sqrt{3}} + c$

**Solution:** We have

$$I = \int \frac{dx}{\cos^6 x + \sin^6 x}$$

$$= \int \frac{dx}{(\cos^2 x)^3 + (\sin^2 x)^3}$$

$$= \int \frac{dx}{\cos^4 x + \sin^2 x \cos^2 x + \sin^4 x}$$

$$= \int \frac{\sec^4 x}{1 + \tan^2 x + \tan^4 x} dx$$

$$= \int \frac{1+t^2}{1+t^2+t^4} dt \quad \text{where } t = \tan x$$

$$= \int \frac{1+(1/t^2)}{t^2+(1/t^2)+1} dt \quad \text{(see Sec. 4.5.5)}$$

$$= \int \frac{dz}{z^2+3} \quad \text{where } z = t - \frac{1}{t}$$

$$= \frac{1}{\sqrt{3}} \text{Tan}^{-1}\left(\frac{z}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \text{Tan}^{-1}\left(\frac{t-(1/t)}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \text{Tan}^{-1}\left(\frac{\tan x - \cot x}{\sqrt{3}}\right) + c$$

**Answer: (B)**

**10.** $\int \frac{x^2+2}{x^4+4} dx$ equals

(A) $\frac{1}{2}\text{Tan}^{-1}\left(\frac{x^2+2}{2x}\right)+c$ (B) $\frac{1}{2}\text{Tan}^{-1}\left(\frac{x^2+2}{x}\right)+c$

(C) $\frac{1}{2}\text{Tan}^{-1}\left(\frac{x^2-2}{2x}\right)+c$ (D) $\frac{1}{2}\text{Tan}^{-1}\left(\frac{x^2-2}{x}\right)+c$

**Solution:** Let

$$I = \int \frac{x^2+2}{x^4+4} dx$$

$$= \int \frac{1+(2/x^2)}{x^2+(4/x^2)} dx$$

Put $t = x-(2/x)$ (see Sec. 4.5.5). Therefore

$$dt = \left(1+\frac{2}{x^2}\right)dx$$

This gives

$$I = \int \frac{dt}{t^2+4}$$

$$= \frac{1}{2}\text{Tan}^{-1}\left(\frac{t}{2}\right)+c$$

$$= \frac{1}{2}\text{Tan}^{-1}\left(\frac{x-\frac{2}{x}}{2}\right)+c$$

$$= \frac{1}{2}\text{Tan}^{-1}\left(\frac{x^2-2}{2x}\right)+c$$

**Answer: (C)**

**11.** $\int \frac{x^2-2}{x^4+4} dx$ is equal to

(A) $\frac{1}{4}\log_e\left(\frac{x^2+2x+2}{x^2-2x+2}\right)+c$

(B) $\frac{1}{16}\text{Tan}^{-1}\left(\frac{x^2+2}{2x}\right)+c$

(C) $\frac{1}{4}\log_e\left(\frac{x^2-2x+2}{x^2+2x+2}\right)+c$

(D) $\frac{1}{16}\log_e\left(\frac{x^2+2x+2}{x^2-2x+2}\right)+c$

**Solution:** Let

$$I = \int \frac{x^2-2}{x^4+4} dx$$

$$= \int \frac{1-(2/x^2)}{x^2+(4/x^2)} dx$$

Put $x+(2/x) = t$. Therefore

$$I = \int \frac{dt}{t^2-4}$$

$$= \frac{1}{2(2)}\log_e\left|\frac{t-2}{t+2}\right|+c$$

$$= \frac{1}{4}\log_e\left|\frac{x+\frac{2}{x}-2}{x+\frac{2}{x}+2}\right|+c$$

$$= \frac{1}{4}\log_e\left(\frac{x^2-2x+2}{x^2+2x+2}\right)+c$$

($\because$ the expression within the logarithm is positive)

**Answer: (C)**

**Try it out** Solve

$$\int \frac{dx}{x^4+4}.$$

*Hint:* Using Problems 10 and 11, this integral can be evaluated because

$$\int \frac{dx}{x^4+4} = \frac{1}{4}\int \frac{(x^2+2)-(x^2-2)}{x^4+4} dx$$

$$= \frac{1}{4}\int \frac{x^2+2}{x^4+4} dx - \frac{1}{4}\int \frac{x^2-2}{x^4+4} dx$$

**12.** $\int \frac{x^2-1}{x^2+1}\cdot\frac{dx}{\sqrt{1+x^4}}$ is equal to

(A) $\text{Cos}^{-1}\frac{x}{x^2+1}+c$ (B) $\frac{1}{\sqrt{2}}\text{Cos}^{-1}\frac{x}{x^2+1}+c$

(C) $\frac{1}{\sqrt{2}}\text{Tan}^{-1}\frac{x\sqrt{2}}{x^2+1}+c$ (D) $\frac{1}{\sqrt{2}}\text{Cos}^{-1}\frac{x\sqrt{2}}{x^2+1}+c$

**Solution:** Let

$$I=\int \frac{x^2-1}{x^2+1}\cdot\frac{dx}{\sqrt{1+x^4}}$$

Dividing numerator and denominator with $x^2$ we get

$$I=\int \frac{1-(1/x^2)}{x+(1/x)}\cdot\frac{dx}{\sqrt{x^2+(1/x^2)}}$$

Put $x+(1/x)=t$. Therefore

$$I=\int \frac{dt}{t\sqrt{t^2-2}}$$
$$=\frac{1}{\sqrt{2}}\text{Sec}^{-1}\frac{t}{\sqrt{2}}+c$$
$$=\frac{1}{\sqrt{2}}\text{Cos}^{-1}\frac{\sqrt{2}}{t}+c$$

**Answer: (D)**

**13.** $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}dx$ is equal to

(A) $\frac{1}{2}\log_e\left(\frac{\sqrt{x^2+x+1}-\sqrt{x}}{\sqrt{x^2+x+1}+\sqrt{x}}\right)+c$

(B) $\frac{1}{2}\log_e\left(\frac{\sqrt{x^2+x+1}-\sqrt{x}}{\sqrt{x^2+x+1}+\sqrt{x}}\right)-\text{Tan}^{-1}\sqrt{\frac{x^2+x+1}{x}}+c$

(C) $\frac{1}{4}\log_e\left(\frac{x^2+x+1}{x^2-x+1}\right)+\text{Tan}^{-1}\left(\frac{x^2+x+1}{x}\right)+c$

(D) $\frac{1}{2}\log_e\left|\frac{x^2-1}{x^2+1}\right|-\text{Tan}^{-1}\left(\frac{x^2+1}{x^2-1}\right)+c$

**Solution:** Let

$$I=\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}dx$$
$$=\int \frac{x(x^2-1)}{(x^2+1)(x+1)^2\sqrt{x^3+x^2+x}}dx$$
$$=\int \frac{x(x^2-1)}{(x^2+1)(x^2+2x+1)}\frac{dx}{\sqrt{x^3+x^2+x}}$$

Dividing numerator and denominator with $x^3$ we get

$$I=\int \frac{1-(1/x^2)}{\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}+2\right)}\frac{dx}{\sqrt{x+\frac{1}{x}+1}}$$

Put $x+(1/x)+1=t^2$ so that

$$\left(1-\frac{1}{x^2}\right)dx=2t\,dt$$

Therefore

$$I=\int \frac{1}{(t^2-1)(t^2+1)}\cdot\frac{1}{t}(2t)dt$$
$$=\int \frac{2}{(t^2-1)(t^2+1)}dt$$
$$=\int\left(\frac{1}{t^2-1}-\frac{1}{t^2+1}\right)dt$$
$$=\frac{1}{2}\log_e\left|\frac{t-1}{t+1}\right|-\text{Tan}^{-1}t+c$$
$$=\frac{1}{2}\log_e\left|\frac{\sqrt{x^2+x+1}-\sqrt{x}}{\sqrt{x^2+x+1}+\sqrt{x}}\right|-\text{Tan}^{-1}\sqrt{\frac{x^2+x+1}{x}}+c$$

**Answer: (B)**

**14.** If $\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^{10}}=a(1+\sqrt[4]{x})^{-8}+b(1+\sqrt[4]{x})^{-9}$ then $a+b$ is

(A) $\frac{1}{8}$ (B) $-\frac{1}{18}$

(C) $\frac{1}{72}$ (D) $-\frac{1}{72}$

**Solution:** Let

$$I=\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^{10}}$$
$$=\int x^{-1/2}(1+\sqrt[4]{x})^{-10}dx$$

According to integration of Binomial differential (see Sec. 4.5.8) put $x=t^4$ so that $dx=4t^3dt$. Then

$$I=\int t^{-2}(1+t)^{-10}(4t^3)dt$$
$$=4\int\frac{t}{(1+t)^{10}}dt$$
$$=4\int\frac{t+1-1}{(1+t)^{10}}dt$$
$$=4\int\frac{dt}{(1+t)^9}-4\int\frac{dt}{(1+t)^{10}}$$
$$=\frac{-4}{8}\left(\frac{1}{(1+t)^8}\right)+\frac{4}{9}\left(\frac{1}{(1+t)^9}\right)+c$$

**Answer: (B)**

**15.** $\displaystyle\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$ equals

(A) $\dfrac{\sqrt{x^4+x^2+1}}{x}+c$ (B) $\dfrac{\sqrt{x^4-x^2+1}}{x}+c$

(C) $\operatorname{Tan}^{-1}\left(x^2+\dfrac{1}{x^2}\right)+c$ (D) $\log_e\left(\dfrac{x^2-2x+2}{x^2+2x+2}\right)+c$

**Solution:** Let

$$I=\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$$

$$=\int \frac{x-\dfrac{1}{x^3}}{\sqrt{x^2+\dfrac{1}{x^2}+1}}\,dx \quad \text{(See Sec. 4.5.5)}$$

Put $x^2+(1/x^2)+1=t^2$ so that

$$\left(x-\frac{1}{x^3}\right)dx=t\,dt$$

Therefore

$$I=\int \frac{1}{t}(t)dt$$
$$=t+c$$
$$=\sqrt{x^2+\frac{1}{x^2}+1}+c$$

**Answer: (A)**

**16.** $\displaystyle\int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}}\,dx=\log_e(f(x))+c$

where $f(x)$ is given by

(A) $\dfrac{x^2+1}{x}$ (B) $\dfrac{1}{x}\sqrt{x^4+3x^2+1}$

(C) $\dfrac{x^4+3x^2+1}{x^2}$ (D) $\dfrac{1}{x}(x^2+1+\sqrt{x^4+3x^2+1})$

**Solution:** Let

$$I=\int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}}\,dx$$

$$=\int \frac{1-(1/x^2)}{\sqrt{x^2+(1/x^2)+3}}\,dx$$

$$=\int \frac{1-(1/x^2)}{\sqrt{\left(x+\dfrac{1}{x}\right)^2+1}}\,dx$$

$$=\int \frac{dt}{\sqrt{t^2+1}} \quad \text{where } t=x+\frac{1}{x}$$

$$=\log_e(t+\sqrt{t^2+1})+c$$

$$=\log_e\left(x+\frac{1}{x}+\sqrt{\left(x+\frac{1}{x}\right)^2+1}\right)+c$$

**Answer: (D)**

**17.** If $\displaystyle\int \frac{dx}{1-\sin^4 x}=\frac{1}{2}\tan x+f(x)+c$ then $f(x)$ is

(A) $\operatorname{Tan}^{-1}(\sqrt{2}\tan x)$

(B) $\dfrac{1}{\sqrt{2}}\operatorname{Tan}^{-1}(\sqrt{2}\tan x)$

(C) $\dfrac{1}{2\sqrt{2}}\operatorname{Tan}^{-1}(\sqrt{2}\tan x)$

(D) $\dfrac{1}{2\sqrt{2}}\operatorname{Tan}^{-1}(\sqrt{2}\sin x)$

**Solution:** Let

$$I=\int \frac{dx}{1-\sin^4 x}$$

$$=\int \frac{\sec^2 x}{1+\sin^2 x}\,dx$$

$$=\int \frac{\sec^2 x\cdot\sec^2 x}{1+2\tan^2 x}\,dx$$

$$=\int \frac{1+t^2}{1+2t^2}\,dt \quad \text{where } t=\tan x$$

$$=\frac{1}{2}\int \frac{1+1+2t^2}{1+2t^2}\,dt$$

$$=\frac{1}{2}\int dt+\frac{1}{2}\int \frac{dt}{1+2t^2}\,dt$$

$$=\frac{t}{2}+\frac{1}{2\sqrt{2}}\operatorname{Tan}^{-1}(\sqrt{2}t)+c$$

$$=\frac{1}{2}\tan x+\frac{1}{2\sqrt{2}}\operatorname{Tan}^{-1}(\sqrt{2}\tan x)+c$$

**Answer: (C)**

**18.** If $\displaystyle\int \frac{\cos 2x\ \sin 4x}{\cos^4 x(1+\cos^2 2x)}\,dx = 2\log_e(1+\cos 2x)-$ $\log_e(1+\cos^2 2x)+f(x)+c$ then $f(x)$ is

(A) $\sec^2 x$ (B) $\tan x$

(C) $\operatorname{cosec}^2 x$ (D) $\cot x$

**Solution:** Let

$$I=\int\frac{\cos 2x\ \sin 4x}{\cos^4 x(1+\cos^2 2x)}dx$$
$$=8\int\frac{\cos^2 2x\ \sin 2x}{(1+\cos 2x)^2(1+\cos^2 2x)}dx$$

Put $\cos 2x = t$ so that $-2\sin 2x\,dx = dt$. Therefore

$$I=-4\int\frac{t^2}{(1+t)^2(1+t^2)}dt$$

Using partial fractions we have

$$\frac{t^2}{(1+t)^2(1+t^2)}=\frac{1}{2(1+t)^2}-\frac{1}{2(1+t)}+\frac{t}{2(1+t^2)}$$

Therefore

$$I=-4\int\left(\frac{1}{2(1+t)^2}-\frac{1}{2(1+t)}+\frac{t}{2(1+t^2)}\right)dt$$
$$=\frac{2}{1+t}+2\log_e(1+t)-\int\frac{2t}{1+t^2}dt$$
$$=\frac{2}{1+\cos 2x}+2\log_e(1+\cos 2x)-\log_e(1+\cos^2 2x)+c$$
$$=\sec^2 x+2\log_e(1+\cos 2x)-\log_e(1+\cos^2 2x)+c$$

Therefore $f(x) = \sec^2 x$.

**Answer: (A)**

**19.** If $\int\sqrt[3]{x}(1+\sqrt[3]{x^4})^{1/7}dx=\alpha(1+\sqrt[3]{x^4})^{\beta}+c$, then $\alpha+\beta^{-1}$ is

(A) $\frac{7}{6}$ (B) $\frac{32}{49}$

(C) $\frac{7}{9}$ (D) $\frac{49}{32}$

**Solution:** Let

$$I=\int\sqrt[3]{x}(1+\sqrt[3]{x^4})^{1/7}dx$$

This is Binomial differential (see Sec. 4.5.8) in which $m = 1/3$, $n = 4/3$ and $p = 1/7$. Now,

$$\frac{m+1}{n}=\frac{(1/3)+1}{4/3}=1 \quad \text{(integer)}$$

Hence put $1+\sqrt[3]{x^4}=t^7$ so that $x=(t^7-1)^{3/4}$. Therefore

$$dx=\frac{3}{4}(t^7-1)^{-1/4}(7t^6)dt$$

Hence

$$I=\int(t^7-1)^{1/4}(t)\frac{21}{4}(t^7-1)^{-1/4}(t^6)\,dt$$
$$=\frac{21}{4}\int t^7dt$$
$$=\frac{21}{4}\left(\frac{t^8}{8}\right)+c$$
$$=\frac{21}{32}(1+\sqrt[3]{x^4})^{8/7}+c$$

So

$$\alpha+\beta^{-1}=\frac{21}{32}+\frac{7}{8}=\frac{49}{32}$$

**Answer: (D)**

**20.** $\int x^{-6}(1+2x^3)^{2/3}dx$ equals

(A) $\frac{-1}{5}(2+x^{-3})^{5/3}+c$ (B) $\frac{1}{5}(x^{-3}+2)^{5/3}+c$

(C) $\frac{3}{5}(2+x^{-3})^{5/3}+c$ (D) $\frac{-3}{5}(x^{-3}+2)^{-5/3}+c$

**Solution:** Let $I$ be the given integral which is Binomial differential (See Sec. 4.5). In this case $m = -6$, $n = 3$ and $p = 2/3$ so that

$$\frac{m+1}{n}+p=\frac{-5}{3}+\frac{2}{3}=-1 \quad \text{(integer)}$$

Put $1 + 2x^3 = x^3t^3$ so that $x^3(t^3-2) = 1$. Therefore

$$x=(t^3-2)^{-1/3}$$
$$\Rightarrow dx=\frac{-1}{3}(t^3-2)^{-4/3}(3t^2)dt$$
$$=-t^2(t^3-2)^{-4/3}dt$$

Hence

$$I=\int(t^3-2)^2\left[(t^3-2)^{-1}t^3\right]^{2/3}(-t^2)(t^3-2)^{-4/3}dt$$
$$=-\int(t^3-2)^2(t^3-2)^{-2/3}t^4(t^3-2)^{-4/3}dt$$
$$=-\int t^4(t^3-2)^{2-2/3-4/3}dt$$
$$=-\int t^4dt$$
$$=-\frac{1}{5}t^5+c$$
$$=-\frac{1}{5}\left(\frac{1+2x^3}{x^3}\right)^{5/3}+c$$
$$=-\frac{1}{5}(2+x^{-3})^{5/3}+c$$

**Answer: (A)**

**21.** $\int x^{-2/3}(1+x^{1/3})^{1/2}dx$ is equal to

(A) $2(1+x^{1/3})^{2/3}+c$ (B) $2(1+x^{1/3})^{3/2}+c$

(C) $3(1+x^{1/3})^{2/3}+c$ (D) $3(1+x^{1/3})^{3/2}+c$

**Solution:** Let

$$I = \int x^{-2/3}(1+x^{1/3})^{1/2}\,dx$$

This is a case of Binomial differential (see Sec. 4.5.8) in which $m = -2/3$, $n = 1/3$ and $p = 1/2$ so that

$$\frac{m+1}{n} = \frac{-(2/3)+1}{1/3} = 1 \qquad \text{(integer)}$$

Now put $1 + x^{1/3} = t^2$ so that

$$x = (t^2-1)^3$$

and $$dx = 6t(t^2-1)^2\,dt$$

Therefore

$$I = \int (t^2-1)^{-2} t(6t)(t^2-1)^2\,dt$$
$$= 6\int t^2 dt$$
$$= 2t^3 + c$$
$$= 2(1+x^{1/3})^{3/2} + c$$

**Answer: (B)**

**22.** If

$$\int x^{-11}(1+x^4)^{-1/2}\,dx = -\frac{1}{10}x^{-a}(1+x^4)^{5/2} + \frac{1}{3}x^{-b}(1+x^4)^{3/2} + \frac{-1}{2}x^{-c}(1+x^4)^{1/2} + k$$

then $a + b + c$ equals

(A) 10 (B) 14
(C) 16 (D) 18

**Solution:** Let

$$I = \int x^{-11}(1+x^4)^{-1/2}\,dx$$

This integral is again integration of Binomial differential (see Sec. 4.5) in which $m = -11$, $n = 4$ and $p = -1/2$ so that

$$\frac{m+1}{n} + p = \frac{-11+1}{4} - \frac{1}{2} = -\frac{5}{2} - \frac{1}{2} = -3 \ \text{(integer)}$$

Therefore put $1 + x^4 = x^4 t^2$ so that

$$x^4 = \frac{1}{t^2-1}$$
$$\Rightarrow x = (t^2-1)^{-1/4}$$

Thus

$$dx = -\frac{1}{4}(t^2-1)^{-5/4}(2t)dt = -\frac{t}{2}(t^2-1)^{-5/4}\,dt$$

Therefore

$$x^{-11} = (t^2-1)^{11/4}$$
$$1 + x^4 = x^4 t^2 = (t^2-1)^{-1} t^2$$

Hence

$$I = \int (t^2-1)^{11/4}(t^2-1)^{1/2}\frac{1}{t}\left[-\frac{t}{2}(t^2-1)^{-5/4}\right]dt$$
$$= -\frac{1}{2}\int (t^2-1)^2\,dt$$
$$= -\frac{1}{2}\left[\frac{t^5}{5} - \frac{2}{3}t^3 + t\right] + k$$
$$= -\frac{1}{10}\left(\frac{1+x^4}{x^4}\right)^{5/2} + \frac{1}{3}\left(\frac{1+x^4}{x^4}\right)^{3/2} - \frac{1}{2}\left(\frac{1+x^4}{x^4}\right)^{1/2} + k$$
$$= -\frac{1}{10}x^{-10}(1+x^4)^{5/2} + \frac{1}{3}x^{-6}(1+x^4)^{3/2}$$
$$-\frac{1}{2}x^{-2}(1+x^4)^{1/2} + k$$

Therefore

$$a + b + c = 10 + 6 + 2 = 18$$

**Answer: (D)**

**23.** If

$$\int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx = (x^2+2x+5)^{1/5} + f(x) + c$$

then $f(x)$ is equal to

(A) $\frac{1}{2}\log_e(x+1+\sqrt{x^2+2x+5})$

(B) $-\log_e(x+1+\sqrt{x^2+2x+5})$

(C) $-\frac{1}{2}\log_e(x+1+\sqrt{x^2+2x+5})$

(D) $\frac{(x+1)}{2}\sqrt{x^2+2x+5}$

**Solution:** Let

$$I = \int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx$$

Refer to Sec. 4.5.3. Let

$$5x + 4 = \lambda\frac{d}{dx}(x^2+2x+5) + \mu$$
$$= \lambda(2x+2) + \mu$$

Therefore $\lambda = 5/2$, $\mu = -1$. Hence

$$I = \int \frac{2x+2}{\sqrt{x^2+2x+5}}\,dx - \int \frac{dx}{\sqrt{x^2+2x+5}}$$

$$= 2\sqrt{x^2+2x+5} - \int \frac{dx}{\sqrt{(x+1)^2+4}}$$

$$= 2\sqrt{x^2+2x+5} - \mathrm{Sinh}^{-1}\frac{x+1}{2} + c$$

$$= 2\sqrt{x^2+2x+5} - \log_e(x+1+\sqrt{x^2+2x+5}) + c$$

So

$$f(x) = -\log_e(x+1+\sqrt{x^2+2x+5})$$

**Answer: (B)**

**24.** If

$$\int \frac{2x+5}{\sqrt{9x^2+6x+2}}\,dx = a\sqrt{9x^2+6x+2} + b\log_e(3x+1+\sqrt{9x^2+6x+2}) + c$$

then $9(a+b)$ is equal to

(A) 9 (B) 4

(C) 41 (D) 2

**Solution:** Refer Sec. 4.5.3. Let

$$2x+5 = \lambda\frac{d}{dx}(9x^2+6x+2) + \mu$$
$$= \lambda(18x+6) + \mu$$

Therefore

$$\lambda = \frac{2}{18} = \frac{1}{9}$$

and

$$\mu = 5 - 6\lambda = 5 - \frac{6}{9} = \frac{13}{3}$$

Hence

$$\int \frac{2x+5}{\sqrt{9x^2+6x+2}}\,dx = \frac{1}{9}\int \frac{18x+6}{\sqrt{9x^2+6x+2}} + \frac{13}{3}\int \frac{dx}{\sqrt{(3x+1)^2+1}}$$

$$= \frac{2}{9}\sqrt{9x^2+6x+2} + \frac{13}{3}\log_e(3x+1+\sqrt{9x^2+6x+2}) + c$$

Therefore

$$9(a+b) = 9\left(\frac{2}{9} + \frac{13}{3}\right) = 41$$

**Answer: (C)**

**25.** If

$$\int \sqrt{4x^2-4x+3}\,dx = \frac{1}{2}\log_e(2x-1+\sqrt{4x^2-4x+3}) + f(x) + c$$

then $f(x)$ equals

(A) $\dfrac{2x-1}{4}\sqrt{4x^2-4x+3}$ (B) $\dfrac{x-1}{4}\sqrt{4x^2-4x+3}$

(C) $\dfrac{x}{4}\sqrt{4x^2-4x+3}$ (D) $\dfrac{2x+1}{4}\sqrt{4x^2-4x+3}$

**Solution:** Let

$$I = \int \sqrt{4x^2-4x+3}\,dx$$

$$= 2\int \sqrt{x^2 - x + \frac{3}{4}}\,dx$$

$$= 2\int \sqrt{\left(x-\frac{1}{2}\right)^2 + \frac{1}{2}}\,dx$$

According to Formula 11

$$I = 2\left[\frac{x-(1/2)}{2}\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{1}{2}} + \frac{1}{2}\cdot\frac{1}{4}\mathrm{Sinh}^{-1}\left(\left(x-\frac{1}{2}\right)\sqrt{2}\right)\right] + c$$

$$= \frac{2x-1}{4}\sqrt{4x^2-4x+3} + \frac{1}{2}\log_e\left(\frac{2x-1}{\sqrt{2}} + \frac{\sqrt{4x^2-4x+3}}{\sqrt{2}}\right) + c$$

$$= \frac{2x-1}{4}\sqrt{4x^2-4x+3} + \frac{1}{2}\log_e(2x-1+\sqrt{4x^2-4x+3}) + c$$

Therefore

$$f(x) = \frac{2x-1}{4}\sqrt{4x^2-4x+3}$$

**Answer: (A)**

**26.** $\displaystyle\int \frac{2}{(2-x)^2}\left(\frac{2-x}{2+x}\right)^{1/3} dx =$

(A) $\left(\dfrac{2+x}{2-x}\right)^{2/3} + c$ (B) $\left(\dfrac{2-x}{2+x}\right)^{2/3} + c$

(C) $\dfrac{3}{4}\left(\dfrac{2+x}{2-x}\right)^{2/3} + c$ (D) $\dfrac{3}{4}\left(\dfrac{2+x}{2-x}\right)^{3/2} + c$

**Solution:** Let $I$ be the given integral. Put

$$\frac{2-x}{2+x} = t^3$$

so that

$$x = \frac{2-2t^3}{1+t^3}$$

$$2 - x = 2 - \frac{2-2t^3}{1+t^3} = \frac{4t^3}{1+t^3}$$

and

$$dx = \frac{-6t^2(1+t^3) - 3t^2(2-2t^3)}{(1+t^3)^2}dt = \frac{-12t^2}{(1+t^3)^2}dt$$

Therefore

$$I = -\int \frac{2(1+t^3)^2}{16t^6}\cdot(t)\frac{12t^2}{(1+t^3)^2}dt$$
$$= \frac{-3}{2}\int\frac{dt}{t^3}$$
$$= \frac{-3}{2}\left(\frac{t^{-3+1}}{-3+1}\right)+c$$
$$= \frac{3}{4}\left(\frac{1}{t^2}\right)+c$$
$$= \frac{3}{4}(t^{-2})+c$$
$$= \frac{3}{4}\left(\frac{2-x}{2+x}\right)^{-2/3}+c$$
$$= \frac{3}{4}\left(\frac{2+x}{2-x}\right)^{2/3}+c$$

**Answer: (C)**

**27.** $\displaystyle\int \frac{\sin x}{\sin x - \cos x}dx$ equals

(A) $\frac{1}{2}\log_e|\sin x - \cos x| + \frac{x}{2}+c$

(B) $\frac{1}{2}\log_e|\sin x - \cos x| - \frac{x}{2}+c$

(C) $\log_e\left|\sqrt{\tan x}+\sqrt{\cot x}\right| - \frac{x}{2}+c$

(D) $\frac{1}{2}\log_e|\operatorname{cosec} x - \sec x| + \frac{x}{2}+c$

**Solution:** We have

$$I = \int \frac{\sin x}{\sin x - \cos x}dx$$
$$= \frac{1}{2}\int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x - \cos x}dx$$
$$= \frac{1}{2}\int\left(1+\frac{\sin x + \cos x}{\sin x - \cos x}\right)dx$$
$$= \frac{x}{2}+\int \frac{\frac{1}{2}\frac{d}{dx}(\sin x - \cos x)}{\sin x - \cos x}dx + c$$
$$= \frac{x}{2}+\frac{1}{2}\log_e|\sin x - \cos x| + c$$

**Answer: (A)**

**28.** $\displaystyle\int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}} =$

(A) $4\int\frac{du}{8-3u^2}$ where $u=\sqrt{1-z^2}$, $z=\frac{1}{t}$ and $x+\frac{1}{2}=\frac{t\sqrt{5}}{2}$

(B) $4\int\frac{du}{8+3u^2}$ where $u=\sqrt{1-z^2}$, $z=\frac{1}{t}$ and $x+\frac{1}{2}=\frac{t\sqrt{5}}{2}$

(C) $5\int\frac{du}{5+3u^2}$ where $u=\sqrt{1-z^2}$, $z=\frac{1}{t}$ and $x+\frac{1}{2}=\frac{t\sqrt{5}}{2}$

(D) $5\int\frac{du}{8u^2-3}$ where $u=\sqrt{1-z^2}$, $z=\frac{1}{t}$ and $x+\frac{1}{2}=\frac{t\sqrt{5}}{2}$

**Solution:** Let

$$I = \int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}}$$
$$= \int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{5}{4}}}$$

Put

$$x+\frac{1}{2}=\frac{t\sqrt{5}}{2}$$

Then

$$dx = \frac{\sqrt{5}}{2}dt$$

Therefore

$$I = \int \frac{1}{\left(\frac{5t^2}{4}+\frac{3}{4}\right)\frac{\sqrt{5}}{2}\sqrt{t^2-1}}\left(\frac{\sqrt{5}}{2}\right)dt$$
$$= 4\int\frac{dt}{(5t^2+3)\sqrt{t^2-1}}$$

Put $t = 1/z$ so that $dt = (-1/z^2)dz$. Then

$$I = -4\int\frac{z}{(5+3z^2)\sqrt{1-z^2}}dz$$

Now, put $1-z^2=u^2$. Then

$$I = -4\int\frac{1}{5+3(1-u^2)u}(-u)du$$
$$= 4\int\frac{du}{8-3u^2}$$

**Answer: (A)**

**29.** $\int \frac{dx}{(x+1)^{2/3}(x-1)^{4/3}}$ equals

(A) $\frac{3}{2}\left(\frac{1+x}{1-x}\right)^{1/3}+c$  (B) $\frac{2}{3}\left(\frac{1+x}{1-x}\right)^{1/3}+c$

(C) $\frac{3}{2}\left(\frac{1-x}{1+x}\right)^{1/3}+c$  (D) $\frac{2}{3}\left(\frac{1-x}{1+x}\right)^{1/3}+c$

**Solution:** Let

$$I=\int \frac{dx}{(x+1)^{2/3}(x-1)^{4/3}}$$
$$=\int \frac{dx}{\left(\frac{x+1}{x-1}\right)^{2/3}(x-1)^2}$$

Put $\frac{x+1}{x-1}=t^3$ so that

$$x=\frac{t^3+1}{t^3-1}\Rightarrow x-1=\frac{2}{t^3-1}$$

and $$dx=\frac{-6t^2}{(t^3-1)^2}dt$$

Therefore

$$I=\int \frac{1}{t^2\left(\frac{4}{(t^3-1)^2}\right)}\left(\frac{-6t^2}{(t^3-1)^2}\right)dt$$
$$=\frac{-3}{2}\int dt$$
$$=\frac{-3}{2}\left(\frac{x+1}{x-1}\right)^{1/3}+c$$
$$=\frac{3}{2}\left(\frac{1+x}{1-x}\right)^{1/3}+c$$

**Answer: (A)**

**30.** $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$ equals

(A) $-2\tan\left(\frac{\sqrt{1+x-x^2}+1}{x}+1\right)+c$

(B) $-2\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{1+x-x^2}+1}{x}+1\right)+c$

(C) $-2\log_e\left(\frac{\sqrt{1+x-x^2}+1}{x}+1\right)$

(D) $-2\left(\frac{\sqrt{1+x-x^2}+1}{x}+1\right)+c$

**Solution:** Let

$$I=\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$$

Put $\sqrt{1+x-x^2}=tx-1$ (Euler's substitution). Therefore

$$1+x-x^2=t^2x^2-2tx+1$$
$$\Rightarrow 1-x=t^2x-2t$$
$$\Rightarrow x=\frac{1+2t}{1+t^2}$$

Hence

$$dx=\frac{2(1+t^2)-2t(1+2t)}{(1+t^2)^2}dt=\frac{2(1-t-t^2)}{(1+t^2)^2}dt$$

Now

$$\sqrt{1+x-x^2}=tx-1$$
$$=\frac{t(1+2t)}{1+t^2}-1=\frac{t^2+t-1}{1+t^2}$$

Therefore

$$I=\int \frac{1}{\left(1+\frac{1+2t}{1+t^2}\right)}\cdot\frac{1}{\left(\frac{t^2+t-1}{1+t^2}\right)}\cdot\frac{2(1-t-t^2)}{(1+t^2)^2}dt$$
$$=-2\int \frac{dt}{2+2t+t^2}$$
$$=-2\int \frac{dt}{(1+t)^2+1}$$
$$=-2\,\mathrm{Tan}^{-1}(1+t)+c$$
$$=-2\,\mathrm{Tan}^{-1}\left(1+\frac{\sqrt{1+x-x^2}+1}{x}\right)+c$$

**Answer: (B)**

**31.** $\int \frac{x}{(x^2-3x+2)\sqrt{x^2-4x+3}}dx$ is equal to

(A) $-2\,\mathrm{Sin}^{-1}\frac{1}{x-2}+\frac{\sqrt{x^2-4x+3}}{x-1}+c$

(B) $2\,\mathrm{Sin}^{-1}\frac{1}{x-2}+\frac{\sqrt{x^2-4x+3}}{x-1}+c$

(C) $-2\ \text{Sin}^{-1}\dfrac{1}{x-2} - \dfrac{\sqrt{x^2-4x+3}}{x-1} + c$

(D) $2\ \text{Sin}^{-1}\dfrac{1}{x-2} - \dfrac{\sqrt{x^2-4x+3}}{x-1} + c$

**Solution:** Let

$$I = \int \frac{x}{(x^2-3x+2)\sqrt{x^2-4x+3}}dx$$
$$= \int \frac{x}{(x-2)(x-1)\sqrt{x^2-4x+3}}dx$$
$$= \int\left(\frac{2}{x-2} - \frac{1}{x-1}\right)\frac{dx}{\sqrt{x^2-4x+3}}$$

Therefore

$$I = 2\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} - \int \frac{dx}{(x-1)\sqrt{x^2-4x+3}} \qquad (4.19)$$

Let

$$I_1 = \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$$
$$= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$$
$$= \int \frac{\dfrac{1}{(x-2)^2}}{\sqrt{1-\left(\dfrac{1}{x-2}\right)^2}}dx$$
$$= -\text{Sin}^{-1}\frac{1}{x-2}$$

Now, let

$$I_2 = \int \frac{dx}{(x-1)\sqrt{x^2-4x+3}}$$
$$= \int \frac{dx}{(x-1)^{3/2}(x-3)^{1/2}}$$
$$= \int \frac{dx}{\left(\dfrac{x-1}{x-3}\right)^{3/2}(x-3)^2}$$

Put

$$\frac{x-1}{x-3} = t^2$$

so that

$$x = \frac{1-3t^2}{1-t^2} = 3 - \frac{2}{1-t^2}$$

$$\Rightarrow x-3 = \frac{-2}{1-t^2}$$

Therefore

$$dx = \frac{-4t}{(1-t^2)^2}dt$$

This gives

$$I_2 = \int \frac{1}{t^3\left(\dfrac{4}{(1-t^2)^2}\right)} \cdot \frac{-4t}{(1-t^2)^2}dt$$
$$= -\int \frac{dt}{t^2}$$
$$= \frac{1}{t}$$
$$= \sqrt{\frac{x-3}{x-1}}$$
$$= \sqrt{\frac{(x-3)(x-1)}{(x-1)^2}}$$
$$= \frac{\sqrt{x^2-4x+3}}{x-1}$$

Substituting the values of $I_1$ and $I_2$ in Eq. (4.19), we have

$$I = -2\ \text{Sin}^{-1}\frac{1}{x-2} - \frac{\sqrt{x^2-4x+3}}{x-1} + c$$

**Answer: (C)**

**32.** $\displaystyle\int \frac{dx}{\sin x + \sec x}$ is equal to

$$\frac{1}{2\sqrt{3}}\log_e\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + \text{Tan}^{-1}z + c$$

where

(A) $t = \sin x + \cos x,\ z = \sin x - \cos x$
(B) $t = \sin x - \cos x,\ z = \sin x + \cos x$
(C) $t = \tan x - \cot x,\ z = \tan x + \cot x$
(D) $t = \sin x + \text{cosec}\ x,\ z = \cos x - \sin x$

**Solution:** Let

$$I = \int \frac{dx}{\sin x + \sec x}$$
$$= \int \frac{\cos x}{\sin x\cos x + 1}dx$$
$$= \int \frac{2\cos x}{2 + 2\sin x\cos x}dx$$
$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + 2\sin x\cos x}dx$$

$$= \int \frac{\cos x + \sin x}{2 + 2\sin x \cos x} dx + \int \frac{\cos x - \sin x}{2 + 2\sin x \cos x} dx$$

$$= \int \frac{\cos x + \sin x}{3 - (\sin x - \cos x)^2} dx + \int \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} dx$$

$$= \int \frac{dt}{3 - t^2} + \int \frac{dz}{1 + z^2} \quad \text{where } t = \sin x - \cos x \text{ and}$$

$$z = \sin x + \cos x$$

$$= \frac{1}{2\sqrt{3}} \log_e \left| \frac{\sqrt{3} + t}{\sqrt{3} - t} \right| + \mathrm{Tan}^{-1} z + c$$

**Answer: (B)**

**33.** $\int \frac{x^2 - 2}{x^3 \sqrt{x^2 - 1}} dx$ equals

(A) $\frac{-\sqrt{x^2 - 1}}{x^2} + c$ (B) $\frac{\sqrt{x^2 - 1}}{x^2} + c$

(C) $\frac{(x^2 - 1)^{3/2}}{x^2} + c$ (D) $\frac{-\sqrt{x^2 - 1}}{x} + c$

**Solution:** Let

$$I = \int \frac{x^2 - 2}{x^3 \sqrt{x^2 - 1}} dx$$

Put $x^2 - 1 = t^2$ so that $xdx = tdt$. Therefore

$$I = \int \frac{t^2 - 1}{(t^2 + 1)^2} dt$$

$$= \int \frac{t^2 + 1 - 2}{(t^2 + 1)^2} dt$$

$$= \int \frac{dt}{1 + t^2} - 2\int \frac{dt}{(1 + t^2)^2}$$

Hence

$$I = \mathrm{Tan}^{-1} t - 2\int \frac{dt}{(1 + t^2)^2} \tag{4.20}$$

Now let

$$I_1 = \int \frac{dt}{(1 + t^2)^2}$$

Put $t = \tan\theta$. Therefore

$$I = \int \frac{1}{\sec^4 \theta} (\sec^2 \theta) d\theta$$

$$= \int \cos^2 \theta \, d\theta$$

$$= \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4}$$

$$= \frac{1}{2} \mathrm{Tan}^{-1} t + \frac{t}{2(1 + t^2)}$$

Substituting the value of $I_1$ in Eq. (4.20), we get

$$I = \mathrm{Tan}^{-1} t - 2\left( \frac{1}{2} \mathrm{Tan}^{-1} t + \frac{t}{2(1 + t^2)} \right) + c$$

$$= \frac{-t}{1 + t^2} + c$$

$$= \frac{-\sqrt{x^2 - 1}}{x^2} + c$$

**Answer: (A)**

**34.** $\int \frac{dx}{\sqrt[4]{1 + x^4}}$ is

(A) $\frac{1}{4} \log_e \left| \frac{(1 + x^4)^{1/4} - x}{(1 + x^4)^{1/4} + x} \right| + \frac{1}{2} \mathrm{Tan}^{-1} \left( \frac{(1 + x^4)^{1/4}}{x} \right) + c$

(B) $\frac{1}{4} \log_e \left| \frac{(1 + x)^{1/4} - x}{(1 + x)^{1/4} + x} \right| - \frac{1}{2} \mathrm{Tan}^{-1} \left( \frac{(1 + x)^{1/4}}{x} \right) + c$

(C) $\frac{-1}{4} \log_e \left| \frac{(1 + x^4)^{1/4} - x}{(1 + x^4)^{1/4} + x} \right| - \frac{1}{2} \mathrm{Tan}^{-1} \left( \frac{(1 + x^4)^{1/4}}{x} \right) + c$

(D) $\frac{-1}{4} \log \left| \frac{(1 + x^4)^{1/4} - x}{(1 + x^4)^{1/4} + x} \right| + \frac{1}{2} \mathrm{Tan}^{-1} \left( \frac{(1 + x^4)^{1/4}}{x} \right) + c$

**Solution:** Let

$$I = \int \frac{dx}{\sqrt[4]{1 + x^4}}$$

Put $1 + x^4 = t^4 x^4$ (see Binomial differential). Then

$$x = (t^4 - 1)^{-1/4}$$

and $$dx = -t^3 (t^4 - 1)^{-5/4} dt$$

Also

$$(1 + x^4)^{1/4} = tx = t(t^4 - 1)^{-1/4}$$

Therefore

$$I = \int \frac{(t^4 - 1)^{1/4}}{t} \cdot (-t^3)(t^4 - 1)^{-5/4} dt$$

$$= -\int \frac{t^2}{t^4 - 1} dt$$

$$= -\frac{1}{2} \int \left( \frac{1}{t^2 - 1} + \frac{1}{t^2 + 1} \right) dt$$

$$= -\frac{1}{2} \cdot \frac{1}{2} \log_e \left| \frac{t - 1}{t + 1} \right| - \frac{1}{2} \mathrm{Tan}^{-1} t + c$$

$$= \frac{-1}{4}\log_e\left|\frac{(1+x^4)^{1/4}-x}{(1+x^4)^{1/4}+x}\right| - \frac{1}{2}\text{Tan}^{-1}\left(\frac{(1+x^4)^{1/4}}{x}\right)+c$$

**Answer: (C)**

**35.** If $F_n(x)=\int(\log x)^n dx$ and $\phi_n = F_n + nF_{n-1}$, then $\phi_n(e)-\phi_n(1)$ is equal to

(A) 0 (B) 1

(C) $ne$ (D) $e$

**Solution:** Using integration by parts, we get

$$F_n(x) = x(\log x)^n - \int x\, n\frac{(\log x)^{n-1}}{x}dx$$
$$= x(\log x)^n - nF_{n-1}(x)+c$$

Therefore

$$F_n(x)+nF_{n-1}(x) = x(\log x)^n + c$$

This implies

$$\phi_n(x) = x(\log x)^n + c$$
$$\Rightarrow \phi_n(e)-\phi_n(1) = e$$

**Answer: (D)**

**36.** If $f:\mathbb{R}\to\mathbb{R}$ is a function satisfying the following:

(i) $f(-x)=-f(x)$

(ii) $f(x+1)=f(x)+1$

(iii) $f\left(\frac{1}{x}\right)=\frac{f(x)}{x^2}\ \forall\ x\neq 0$

then $\int e^x f(x)dx$ is equal to

(A) $e^x(x-1)+c$ (B) $e^x\log x + c$

(C) $\frac{e^x}{x}+c$ (D) $\frac{e^x}{x+1}+c$

**Solution:** First, we determine $f(x)$.

$$f(0)=f(-0)=-f(0)\Rightarrow f(0)=0$$

and $\quad 0=f(0)=f(1+(-1))=f(-1)+1\Rightarrow f(-1)=-1$

Therefore $f(0)=0$ and $f(-1)=-1$. Let $x\neq 0$ and $-1$. From (ii), we have

$$f\left(\frac{1}{x}+1\right)=f\left(\frac{1}{x}\right)+1=\frac{f(x)}{x^2}+1 \quad \text{[By (iii)]}$$

Therefore

$$f\left(\frac{1+x}{x}\right)=\frac{f(x)}{x^2}+1 \tag{4.21}$$

Again

$$f\left(\frac{x}{1+x}\right)=f\left(\frac{1}{\left(\frac{1+x}{x}\right)}\right)$$
$$=\frac{f\left(\frac{1+x}{x}\right)}{\left(\frac{1+x}{x}\right)^2}$$
$$=\left(\frac{x}{1+x}\right)^2\left(\frac{f(x)}{x^2}+1\right) \quad \text{[By Eq. (4.21)]}$$

Therefore

$$(1+x)^2 f\left(\frac{x}{1+x}\right)=f(x)+x^2 \tag{4.22}$$

Also

$$(1+x)^2 f\left(\frac{x}{1+x}\right)=(1+x)^2 f\left(1-\frac{1}{x+1}\right)$$
$$=(1+x)^2 f\left(\frac{-1}{x+1}+1\right)$$
$$=(1+x)^2\left[-f\left(\frac{1}{x+1}\right)+1\right]$$
$$[\because f(-x)=-f(x)]$$
$$=(1+x)^2\left[\frac{-f(x+1)}{(x+1)^2}+1\right]$$
$$\left(\because f\left(\frac{1}{x}\right)=\frac{f(x)}{x^2}\right)$$
$$=-f(x+1)+(x+1)^2$$
$$=-(f(x)+1)+(x+1)^2$$
$$=-f(x)+x^2+2x \tag{4.23}$$

Therefore from Eqs. (4.22) and (4.23), we have

$$f(x)+x^2=-f(x)+x^2+2x$$
$$\Rightarrow 2f(x)=2x$$
$$\Rightarrow f(x)=x$$

Thus $f(0)=0, f(-1)=-1$ and $f(x)=x\ \forall\ x\neq 0,-1$. So, $f(x)=x\ \forall\ x\in\mathbb{R}$. Hence

$$\int e^x f(x)dx=\int e^x x dx$$
$$=xe^x-\int e^x dx$$
$$=xe^x-e^x+c$$
$$=e^x(x-1)+c$$

**Answer: (A)**

**37.** $\displaystyle\int \frac{\sin x - \cos x}{(\sin x + \cos x)\sqrt{\sin x \cos x + \sin^2 x \cos^2 x}}\,dx =$

(A) $-\sin(\sin 2x + 1) + c$ (B) $\operatorname{cosec}(\sin 2x + 1)$

(C) $-\text{Sec}^{-1}(\sin 2x + 1) + c$ (D) $\text{Tan}^{-1}(\sin 2x + 1) + c$

**Solution:** Let $I$ be the given integral. Then

$$\begin{aligned}
I &= \int \frac{\sin^2 x - \cos^2 x}{(\sin x + \cos x)^2\sqrt{\sin x \cos x + \sin^2 x \cos^2 x}}\,dx \\
&= -\int \frac{\cos 2x}{(1+\sin 2x)\sqrt{\left(\sin x \cos x + \frac{1}{2}\right)^2 - \frac{1}{4}}}\,dx \\
&= -\int \frac{2\cos 2x}{(1+\sin 2x)\sqrt{(\sin 2x + 1)^2 - 1}} \\
&= -\int \frac{dt}{t\sqrt{t^2-1}} \quad \text{where } t = 1 + \sin 2x \\
&= -\text{Sec}^{-1} t + c \\
&= -\text{Sec}^{-1}(1+\sin 2x) + c
\end{aligned}$$

**Answer: (C)**

**38.** If $f(x)$ is a quadratic expression such that $f(0) = f(1) = 3f(2) = -3$, then $\displaystyle\int \frac{f(x)}{x^3-1}\,dx =$

(A) $\log_e|x-1| + \log_e(x^2+x+1) + \dfrac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + c$

(B) $\log_e|x-1| + \log_e(x^2+x+1) + \dfrac{1}{\sqrt{3}}\text{Tan}^{-1}\left(\dfrac{x+1}{\sqrt{3}}\right) + c$

(C) $\log_e(x^2+x+1) + \log_e|x| + \dfrac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\dfrac{2x+3}{\sqrt{3}}\right) + c$

(D) $\log_e|x^3+x^2+x| + \dfrac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + c$

**Solution:** Let $f(x) = ax^2 + bx + c$. Then

$$f(0) = -3 \Rightarrow c = -3$$

and

$$\begin{aligned}
& f(1) = -3 \\
& \Rightarrow a + b + c = -3 \\
& \Rightarrow a + b = 0 \qquad (4.24)
\end{aligned}$$

and

$$\begin{aligned}
& f(2) = -1 \\
& \Rightarrow 4a + 2b + c = -1 \\
& \Rightarrow 4a + 2b = 2 \\
& \Rightarrow 2a + b = 1 \qquad (4.25)
\end{aligned}$$

From Eqs. (4.24) and (4.25), $a = 1, b = -1$. Therefore

$$f(x) = x^2 - x - 3$$

So

$$\begin{aligned}
\int \frac{f(x)}{x^3-1}\,dx &= \int \frac{x^2-x-3}{x^3-1}\,dx \\
&= \int \left(\frac{-1}{x-1} + \frac{2x+2}{x^2+x+1}\right) dx \\
&= \int \frac{1}{1-x}\,dx + \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \log_e|x-1| + \log_e(x^2+x+1) + \frac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c
\end{aligned}$$

**Answer: (A)**

**39.** $\displaystyle\int \cos 2\theta \log_e\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta = \frac{\sin 2\theta}{2}$
$\times \log_e\left(\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \dfrac{1}{2}f(x) + c$

where $f(x)$ is

(A) $\sec 2\theta$ (B) $\log_e(\sec 2\theta)$

(C) $2\,\text{Tan}^{-1}\theta$ (D) $\tan 2\theta$

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**Solution:** Let $I$ be the given integral. We can see that the derivative of $\log_e\left(\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$ is $2\sec 2\theta$, so that using integration by parts we have

$$\begin{aligned}
I &= \frac{\sin 2\theta}{2}\log_e\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \int \frac{\sin 2\theta}{2}\cdot(2\sec 2\theta)\,d\theta \\
&= \frac{\sin 2\theta}{2}\log_e\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \int \tan 2\theta\,d\theta \\
&= \frac{\sin 2\theta}{2}\log_e\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \frac{1}{2}\log_e(\sec 2\theta) + c
\end{aligned}$$

Therefore

$$f(x) = \log_e(\sec 2\theta)$$

**Answer: (B)**

**40.** $\displaystyle\int \operatorname{cosec}^2 x \log_e(\sin x)\,dx$ equals

(A) $-\cot x[1 + \log_e(\sin x)] + c$

(B) $-\cot x[1 + \log_e(\sin x)] - x + c$

(C) $-\cot x \log_e(\sin x) - x + c$

(D) $\operatorname{cosec} x \log_e(\sin x) - x + c$

**Solution:** Using integration by parts, we get

$$\int \operatorname{cosec}^2 x \log_e(\sin x)dx = -\cot x \log_e(\sin x) - \int(-\cot x)\cot x dx$$

$$= -\cot x \log_e(\sin x) + \int(\operatorname{cosec}^2 x - 1)dx$$

$$= -\cot x \log_e(\sin x) - \cot x - x + c$$

$$= -\cot x[1 + \log_e(\sin x)] - x + c$$

**Answer: (B)**

**41.** $\int \log(x+\sqrt{x^2-1})\,dx =$

(A) $x\log_e(x+\sqrt{x^2-1}) - \sqrt{x^2-1} + c$

(B) $x\log_e(x+\sqrt{x^2-1}) + \sqrt{x^2-1} + c$

(C) $\dfrac{x\sqrt{x^2-1}}{2} + \log_e(x+\sqrt{x^2-1}) + c$

(D) $\dfrac{\sqrt{x^2-1}}{2} - \log_e(x+\sqrt{x^2-1}) + c$

**Solution:** Using integration by parts we have

$$\int \log_e(x+\sqrt{x^2-1})\,dx = x\log_e(x+\sqrt{x^2-1})$$

$$-\int x\left(\frac{1}{x+\sqrt{x^2-1}}\right)\left(1+\frac{x}{\sqrt{x^2-1}}\right)dx$$

$$= x\log_e(x+\sqrt{x^2-1}) - \int \frac{x}{\sqrt{x^2-1}}dx$$

$$= x\log_e(x+\sqrt{x^2-1}) - \sqrt{x^2-1} + c$$

**Answer: (A)**

**Try it out** Using Problems 40 and 41, evaluate

$$\int \frac{\log_e(\cos x+\sqrt{\cos 2x})}{1-\cos^2 x}\,dx$$

**42.** If $\int \dfrac{dx}{1+\sin x} = \tan\left(\dfrac{x}{2}+a\right)+c$, then $a$ is equal to

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{2}$

(C) $-\dfrac{\pi}{2}$ (D) $-\dfrac{\pi}{4}$

**Solution:** We have

$$\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{\cos^2 x}dx$$

$$= \int \sec^2 x dx - \int \sec x \tan x dx$$

$$= \tan x - \sec x + c$$

$$= \frac{\sin x - 1}{\cos x} + c$$

$$= \frac{\dfrac{2\tan(x/2)}{1+\tan^2(x/2)} - 1}{\left(\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)} + c$$

$$= -\frac{(1-\tan(x/2))^2}{1-\tan^2(x/2)}$$

$$= \frac{\tan(x/2)-1}{1+\tan(x/2)}$$

$$= \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+c$$

Therefore $a = -\pi/4$.

**Answer: (D)**

**43.** $\int x\tan x\sec^2 x dx =$

(A) $\dfrac{x\sec^2 x}{2} - \dfrac{x\tan x}{2} + c$

(B) $\dfrac{x\sec^2 x}{2} - \dfrac{1}{2}\tan x + c$

(C) $\dfrac{x\sec x}{2} - \dfrac{\tan x}{2} + c$

(D) $\dfrac{x\sec^2 x}{2} + \dfrac{\tan x}{2} + c$

**Solution:** Take $u = x$ and $dv = \tan x \sec^2 x dx$ so that

$$v = \int \tan x \sec^2 x dx = \frac{1}{2}\sec^2 x$$

Therefore

$$\int x\tan x\sec^2 x dx = x\left(\frac{1}{2}\sec^2 x\right) - \int \frac{1}{2}\sec^2 x dx$$

$$= \frac{x\sec^2 x}{2} - \frac{1}{2}\tan x + c$$

**Answer: (B)**

**44.** $\int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx =$

(A) $-2\sqrt{1-x} + \operatorname{Cos}^{-1}\sqrt{x} + \sqrt{x-x^2} + c$

(B) $-2\sqrt{1-x} - \operatorname{Cos}^{-1}\sqrt{x} + \sqrt{x-x^2} + c$

(C) $-2\sqrt{1-x}+\text{Cos}^{-1}\sqrt{x}-\sqrt{x-x^2}+c$

(D) $2\ \text{Cos}^{-1}x-\sqrt{x-x^2}+c$

**Solution:** Let

$$I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

Put $x=\cos^2\theta$ so that

$$dx=-2\cos\theta\sin\theta d\theta=-\sin 2\theta d\theta$$

Therefore

$$\begin{aligned}I&=\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin 2\theta)d\theta\\&=-\int\tan\frac{\theta}{2}\sin 2\theta d\theta\\&=-\int\frac{\sin(\theta/2)}{\cos(\theta/2)}(2\sin\theta\cos\theta)d\theta\\&=-4\int\frac{\sin(\theta/2)}{\cos(\theta/2)}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\right)d\theta\\&=-4\int\sin^2\left(\frac{\theta}{2}\right)\cos\theta d\theta\\&=-2\int(1-\cos\theta)\cos\theta d\theta\\&=-2\int\cos\theta+\int 2\cos^2\theta d\theta\\&=-2\int\cos\theta+\int(1+\cos 2\theta)d\theta\\&=-2\sin\theta+\theta+\frac{1}{2}\sin 2\theta+c\\&=-2\sqrt{1-\cos^2\theta}+\text{Cos}^{-1}\sqrt{x}+\sin\theta\cos\theta+c\\&=-2\sqrt{1-x}+\text{Cos}^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+c\\&=-2\sqrt{1-x}+\text{Cos}^{-1}\sqrt{x}+\sqrt{x-x^2}+c\end{aligned}$$

**Answer: (A)**

**45.** $\displaystyle\int\frac{x^4(1-x)^4}{1+x^2}\,dx=f(x)-4\ \text{Tan}^{-1}x+c$ where $f(x)$ is

(A) $\dfrac{x^7}{7}-\dfrac{2}{3}x^6+x^5-\dfrac{4x^3}{3}+4x$

(B) $\dfrac{x^7}{7}+\dfrac{2}{3}x^6-x^5-\dfrac{4}{3}x^3+4x$

(C) $\dfrac{x^7}{7}-\dfrac{2}{3}x^6+x^5-\dfrac{4x^3}{3}-4x$

(D) $-\dfrac{x^7}{7}+\dfrac{2}{3}x^6-x^5+\dfrac{4x^3}{3}+x$

**Solution:** Dividing $x^4(1-x)^4$ with $1+x^2$, we have

$$\begin{aligned}\int\frac{x^4(1-x)^4}{1+x^2}\,dx&=\int\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx\\&=\frac{x^7}{7}-\frac{2}{3}x^6+x^5-\frac{4}{3}x^3+4x-4\ \text{Tan}^{-1}x+c\end{aligned}$$

Therefore

$$f(x)=\frac{x^7}{7}-\frac{2}{3}x^6+x^5-\frac{4}{3}x^3+4x$$

**Answer: (A)**

**46.** $\displaystyle\int\frac{\text{Sin}^{-1}\sqrt{x}-\text{Cos}^{-1}\sqrt{x}}{\text{Sin}^{-1}\sqrt{x}+\text{Cos}^{-1}\sqrt{x}}dx=$

(A) $\dfrac{4}{\pi}\left[\dfrac{\sqrt{x-x^2}}{2}+x\ \text{Sin}^{-1}\sqrt{x}-\dfrac{1}{2}\text{Sin}^{-1}\sqrt{1-x}\right]-x+c$

(B) $\dfrac{2}{\pi}\left[\sqrt{x-x^2}+x\ \text{Sin}^{-1}x-\text{Sin}^{-1}\sqrt{1-x}\right]-x+c$

(C) $\dfrac{2}{\pi}\left[\sqrt{x-x^2}+x\ \text{Sin}^{-1}x-\dfrac{1}{2}\text{Sin}^{-1}\sqrt{1-x}\right]-x+c$

(D) $\dfrac{2}{\pi}\left[\sqrt{x-x^2}+x\ \text{Sin}^{-1}x+\dfrac{1}{2}\text{Sin}^{-1}\sqrt{1-x}-x\right]+c$

**Solution:** Let $I$ be the given integral. Since

$$\text{Sin}^{-1}x+\text{Cos}^{-1}x=\frac{\pi}{2}$$

we have

$$\begin{aligned}I&=\frac{2}{\pi}\int\left(2\ \text{Sin}^{-1}x-\frac{\pi}{2}\right)dx\\&=\frac{4}{\pi}\int\text{Sin}^{-1}\sqrt{x}dx-x\end{aligned}\tag{4.26}$$

Now, let $I_1=\int\text{Sin}^{-1}\sqrt{x}\,dx$. Using integration by parts, we get

$$\begin{aligned}I_1&=x\ \text{Sin}^{-1}\sqrt{x}-\int\frac{x}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}dx\\&=x\ \text{Sin}^{-1}\sqrt{x}-\frac{1}{2}\int x^{1/2}(1-x)^{-1/2}dx\end{aligned}$$

Let $1-x=t^2$ so that $dx=-2t\,dt$. Then

$$\begin{aligned}I_1&=x\ \text{Sin}^{-1}\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{1-t^2}}{t}(-2t)dt\\&=x\ \text{Sin}^{-1}\sqrt{x}+\int\sqrt{1-t^2}\,dt\\&=x\text{Sin}^{-1}\sqrt{x}+\left[\frac{t\sqrt{1-t^2}}{2}-\frac{1}{2}\text{Sin}^{-1}t\right]\\&=x\ \text{Sin}^{-1}\sqrt{x}+\left[\frac{\sqrt{x-x^2}}{2}-\frac{1}{2}\ \text{Sin}^{-1}\sqrt{1-x}\right]\end{aligned}$$

Substituting the value of $I_1$ in Eq. (4.26), we have

$$I = \frac{4}{\pi}\left[x\ \text{Sin}^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2} - \frac{1}{2}\ \text{Sin}^{-1}\sqrt{1-x}\right] - x + c$$

**Answer: (A)**

**47.** $\displaystyle\int \frac{dx}{(x^6+1)^2} - \frac{5}{6}\int \frac{dx}{x^6+1}$ is equal to

(A) $\dfrac{x}{6(x^6+1)}$ (B) $\dfrac{x^5}{6(x^6+1)}$

(C) $\dfrac{6x^5}{x^6+1}$ (D) $\dfrac{6x}{x^6+1}$

**Solution:** Let

$$I = \int \frac{dx}{(x^6+1)^2} = \frac{1}{6}\int \frac{6x^5}{x^5(x^6+1)^2}\,dx$$

Take

$$u = \frac{1}{x^5} \text{ and } dv = \frac{6x^5}{(x^6+1)^2}$$

so that

$$v = \frac{-1}{x^6+1}$$

Therefore using integration by parts, we get

$$I = \frac{1}{6}\left[\frac{-1}{x^5(x^6+1)} - \int \frac{-1}{x^6+1}(-5x^{-6})\,dx\right]$$
$$= \frac{-1}{6x^5(x^6+1)} - \frac{5}{6}\int \frac{dx}{x^6(x^6+1)}$$
$$= \frac{-1}{6x^5(x^6+1)} - \frac{5}{6}\int\left(\frac{1}{x^6} - \frac{1}{x^6+1}\right)dx$$
$$= \frac{-1}{6x^5(x^6+1)} + \frac{1}{6x^5} + \frac{5}{6}\int \frac{dx}{x^6+1}$$

So,

$$\int \frac{dx}{(x^6+1)^2} - \frac{5}{6}\int \frac{dx}{x^6+1} = -\frac{1}{6x^5(x^6+1)} + \frac{1}{6x^5}$$
$$= \frac{x}{6(x^6+1)}$$

**Answer: (A)**

**48.** If $f:\mathbb{R}\to\mathbb{R}$ is a function such that

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x\ \forall\ x \neq 0$$

then $\displaystyle\int xf(x)\,dx$ is equal to

(A) $2x - \dfrac{1}{3}x^3 + c$ (B) $2x + \dfrac{1}{3}x^3 + c$

(C) $\log_e|x| - \dfrac{1}{2}x^2 + c$ (D) $\log_e|x| + \dfrac{1}{2}x^2 + c$

**Solution:** Given that

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \quad (4.27)$$

Replacing $x$ with $1/x$ we get

$$2f(x) + f\left(\frac{1}{x}\right) = \frac{3}{x} \quad (4.28)$$

Solving for $f(x)$, we have

$$f(x) = \frac{2-x^2}{x}$$

Therefore

$$\int xf(x)dx = \int (2-x^2)dx = 2x - \frac{1}{3}x^3 + c$$

**Answer: (A)**

**49.** $\displaystyle\int \frac{e^{x\sqrt{2}}(1-x^2)}{(1-x\sqrt{2})\sqrt{1-2x^2}}\,dx$ is equal to

(A) $\dfrac{1}{2\sqrt{2}}e^{x\sqrt{2}}\left(\dfrac{1+x\sqrt{2}}{1-x\sqrt{2}}\right) + c$

(B) $\dfrac{1}{2\sqrt{2}}e^{x\sqrt{2}}\left(\dfrac{1-x\sqrt{2}}{1+x\sqrt{2}}\right) + c$

(C) $\dfrac{1}{2\sqrt{2}}e^{x\sqrt{2}}\sqrt{\dfrac{1+x\sqrt{2}}{1-x\sqrt{2}}} + c$

(D) $\dfrac{1}{2\sqrt{2}}e^{x\sqrt{2}}\sqrt{\dfrac{1-x\sqrt{2}}{1+x\sqrt{2}}} + c$

**Solution:** Let

$$I = \int \frac{e^{x\sqrt{2}}(1-x^2)}{(1-x\sqrt{2})\sqrt{1-2x^2}}\,dx$$

Put $x\sqrt{2} = t$. Then

$$I = \frac{1}{\sqrt{2}}\int \frac{e^t[1-(t^2/2)]}{(1-t)\sqrt{1-t^2}}\,dt$$
$$= \frac{1}{2\sqrt{2}}\int \frac{e^t(1-t^2+1)}{(1-t)\sqrt{1-t^2}}\,dt$$
$$= \frac{1}{2\sqrt{2}}\int e^t\left[\sqrt{\frac{1+t}{1-t}} + \frac{1}{(1-t)\sqrt{1-t^2}}\right]dt$$
$$= \frac{1}{2\sqrt{2}}e^t\sqrt{\frac{1+t}{1-t}} + c \quad \text{where } t = x\sqrt{2}$$

**Answer: (C)**

**50.** $\displaystyle\int \frac{(\sin x+\cos x)(2-\sin 2x)}{\sin^2 2x}dx =$

(A) $\dfrac{\sin x+\cos x}{\sin 2x}+c$ (B) $\dfrac{\sin x-\cos x}{\sin 2x}+c$

(C) $\dfrac{\sin x}{\sin x+\cos x}+c$ (D) $\dfrac{\sin x}{\sin x-\cos x}+c$

**Solution:** Put $t=\sin x-\cos x$ so that

$$dt=(\cos x+\sin x)dx$$

and $$1-\sin 2x=t^2$$

Therefore

$$I=\int\frac{(1+t^2)}{(1-t^2)^2}dt=\int\frac{(1/t^2)+1}{[(1/t)-t]^2}dt=-\int\frac{dz}{z^2}$$

where $z=(1/t)-t$. So

$$I=\frac{1}{z}+c$$
$$=\frac{1}{(1/t)-t}+c$$
$$=\frac{t}{1-t^2}+c$$
$$=\frac{\sin x-\cos x}{\sin 2x}+c$$

**Answer: (B)**

**51.** $\displaystyle\int\frac{3x^2+2x}{\sqrt{x^6+2x^5+x^4-2x^3-2x^2+5}}dx$ equals

(A) $\sinh^{-1}\left(\dfrac{x^3+x^2-1}{2}\right)+c$

(B) $2\cosh^{-1}\left(\dfrac{x^3+x^2-1}{2}\right)$

(C) $\sin^{-1}\left(\dfrac{x^3+x^2-1}{2}\right)+c$

(D) $2\cos^{-1}\left(\dfrac{x^3+x^2-1}{2}\right)+c$

**Solution:** The expression under the square root is

$$(x^3+x^2)^2-2(x^3+x^2)+5$$

Let $I$ be the given integral. Put $x^3+x^2=t$. Then

$$I=\int\frac{dt}{\sqrt{t^2-2t+5}}$$
$$=\int\frac{dt}{\sqrt{(t-1)^2+4}}$$
$$=\sinh^{-1}\frac{t-1}{2}+c$$
$$=\sinh^{-1}\left(\frac{x^3+x^2-1}{2}\right)+c$$

**Answer: (A)**

**52.** $\displaystyle\int(\sin x)^{-3/2}(\cos x)^{-5/2}dx=$

(A) $\dfrac{2}{3}\sqrt{\cot x}-2\sqrt{\tan^3 x}+c$

(B) $2\sqrt{\tan^3 x}+\dfrac{2}{3}\sqrt{\cot x}+c$

(C) $-2\sqrt{\cot x}+\dfrac{2}{3}\sqrt{\tan^3 x}+c$

(D) $-2\cot x+\dfrac{2}{3}\tan^3 x+c$

**Solution:** We have

$$I=\int(\sin x)^{-3/2}(\cos x)^{-5/2}dx$$
$$=\int\frac{dx}{(\sin x)^{3/2}(\cos x)^{5/2}}$$
$$=\int\frac{dx}{(\sin x/\cos x)^{3/2}\cos^4 x}$$
$$=\int\frac{\sec^4 x}{(\tan x)^{3/2}}dx$$
$$=\int\frac{1+t^2}{t^{3/2}}dt \quad \text{where } t=\tan x$$
$$=\int(t^{-3/2}+t^{1/2})dt$$
$$=\frac{t^{-3/2+1}}{-(3/2)+1}+\frac{t^{3/2}}{3/2}+c$$
$$=-2t^{-1/2}+\frac{2}{3}t^{3/2}+c$$
$$=-2\sqrt{\cot x}+\frac{2}{3}(\tan x)^{3/2}+c$$

**Answer: (C)**

**53.** If

$$\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx=px+q\log_e|c\sin x+d\cos x|+c$$

where $p+q$ is equal to

(A) $\dfrac{c(a+b)-d(a-b)}{c^2+d^2}$ (B) $\dfrac{c(a+b)+d(a-b)}{c^2+d^2}$

(C) $\dfrac{a(d+c)-b(d-c)}{a^2+b^2}$ (D) $\dfrac{a(d+c)+b(d-c)}{a^2+b^2}$

**Solution:** Write

$$\begin{aligned} a\sin x+b\cos x &= \lambda(c\sin x+d\cos x) \\ &\quad +\mu\frac{d}{dx}(c\sin x+d\cos x) \\ &= \lambda(c\sin x+d\cos x)+\mu(-d\sin x+c\cos x) \end{aligned}$$

Equating the corresponding coefficients of $\sin x$ and $\cos x$ on both sides we get

$$\lambda c-\mu d=a \quad (4.29)$$

$$\mu c+\lambda d=b \quad (4.30)$$

Solving Eqs. (4.29) and (4.30) for $\lambda$ and $\mu$, we have

$$\lambda=\frac{ac+bd}{c^2+d^2} \quad \text{and} \quad \mu=-\frac{(ad-bc)}{c^2+d^2}$$

Therefore

$$\begin{aligned} I &= \int\left(\lambda+\mu\frac{c\cos x-d\sin x}{c\sin x+d\cos x}\right)dx \\ &= \lambda\int dx+\mu\int\frac{c\cos x-d\sin x}{c\sin x+d\cos x}\,dx \\ &= \lambda x+\mu\log_e|c\sin x+d\cos x|+c \\ &= \left(\frac{ac+bd}{c^2+d^2}\right)x-\frac{(ad-bc)}{c^2+d^2}\log_e|c\sin x+d\cos x|+c \end{aligned}$$

**Answer: (A)**

**QUICK LOOK**

**Important Formula:**

$$\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}\,dx=\left(\frac{ac+bd}{c^2+d^2}\right)x-\frac{(ad-bc)}{c^2+d^2}\times\log_e|c\sin x+d\cos x|+c$$

**54.** If

$$\int\frac{3\sin x+2\cos x}{2\sin x+3\cos x}\,dx=lx+m\log_e|2\sin x+3\cos x|+c$$

then

(A) $l=\dfrac{-5}{13}, m=\dfrac{12}{13}$ (B) $l=\dfrac{12}{13}, m=-\dfrac{5}{13}$

(C) $l=\dfrac{-5}{13}, m=\dfrac{-12}{13}$ (D) $l=\dfrac{-12}{13}, m=\dfrac{-5}{3}$

**Solution:** In the above formula, take $a=3, b=2, c=2, d=3$, so that

$$l=\frac{ac+bd}{c^2+d^2}=\frac{12}{13}$$

$$m=\frac{-(ad-bc)}{c^2+d^2}=\frac{-5}{13}$$

**Answer: (B)**

**55.** If

$$\int\frac{dx}{1+\tan x}=px+q\log_e|\cos x+\sin x|+c$$

then $p+q$ equals

(A) $\dfrac{3}{2}$ (B) 2

(C) $\dfrac{1}{2}$ (D) 1

**Solution:** We have

$$\int\frac{dx}{1+\tan x}=\int\frac{\cos x}{\sin x+\cos x}\,dx$$

$$=\int\frac{0(\sin x)+\cos x}{\sin x+\cos x}$$

Now $a=0, b=1, c=1, d=1$. Therefore

$$p=\frac{ac+bd}{c^2+d^2}=\frac{1}{2}$$

$$q=\frac{-(ad-bc)}{c^2+d^2}=\frac{-(0-1)}{2}=\frac{1}{2}$$

So

$$p+q=\frac{1}{2}+\frac{1}{2}=1$$

**Answer: (D)**

**56.** $\displaystyle\int\frac{x^4(x^{10}-1)}{x^{20}+3x^{10}+1}\,dx$ equals

(A) $\mathrm{Tan}^{-1}\left(x^5+\dfrac{1}{x^5}\right)+c$

(B) $\dfrac{1}{5}\mathrm{Tan}^{-1}\left(x^5+\dfrac{1}{x^5}\right)+c$

(C) $5\,\mathrm{Tan}^{-1}\left(x^5+\dfrac{1}{x^5}\right)+c$

(D) $\dfrac{1}{10}\mathrm{Tan}^{-1}\left(x^{10}+\dfrac{1}{x^{10}}\right)+c$

**Solution:** Let

$$I=\int \frac{x^4(x^{10}-1)}{x^{20}+3x^{10}+1}dx$$

Put $t=x^5$ so that $(1/5)dt = x^4 dx$. Therefore

$$I=\int \frac{t^2-1}{t^4+3t^2+1}\left(\frac{1}{5}\right)dt$$

$$=\frac{1}{5}\int \frac{1-(1/t^2)}{t^2+(1/t^2)+3}dt$$

$$=\frac{1}{5}\int \frac{1-(1/t^2)}{[t+(1/t)]^2+1}dt$$

$$=\frac{1}{5}\text{Tan}^{-1}\left(t+\frac{1}{t}\right)+c$$

$$=\frac{1}{5}\text{Tan}^{-1}\left(x^5+\frac{1}{x^5}\right)+c$$

**Answer: (B)**

**57.** $\int \frac{x^2}{(x\sin x+\cos x)^2}dx$ equals

(A) $\frac{\cos x - x\sin x}{x\sin x+\cos x}+c$ (B) $\frac{\cos x + x\sin x}{x\cos x-\sin x}+c$

(C) $\frac{\sin x - x\cos x}{x\sin x+\cos x}+c$ (D) $\frac{\sin x + x\cos x}{x\sin x-\cos x}+c$

**Solution:** Let

$$I=\int \frac{x^2}{(x\sin x+\cos x)^2}dx$$

We know that

$$\frac{d}{dx}(x\sin x+\cos x)=x\cos x$$

Therefore

$$I=\int \frac{x\cos x}{(x\sin x+\cos x)^2}\left(\frac{x}{\cos x}\right)dx$$

Take

$$u=\frac{x}{\cos x} \quad \text{and} \quad dv=\frac{x\cos x}{(x\sin x+\cos x)^2}dx$$

so that

$$v=\frac{-1}{x\sin x+\cos x}$$

Hence

$$I=\frac{x}{\cos x}\left(\frac{-1}{x\sin x+\cos x}\right)-\int \frac{-1}{x\sin x+\cos x}\left[\frac{\cos x - x(-\sin x)}{\cos^2 x}\right]dx$$

$$=\frac{-x}{\cos x(x\sin x+\cos x)}+\int \frac{1}{x\sin x+\cos x}\left(\frac{\cos x+x\sin x}{\cos^2 x}\right)dx$$

$$=\frac{-x}{\cos x(x\sin x+\cos x)}+\int \sec^2 x\,dx$$

$$=\frac{-x}{\cos x(x\sin x+\cos x)}+\tan x+c$$

$$=\frac{-x+\sin x(x\sin x+\cos x)}{\cos x(x\sin x+\cos x)}+c$$

$$=\frac{-x(1-\sin^2 x)+\sin x\cos x}{\cos x(x\sin x+\cos x)}+c$$

$$=\frac{\cos x(\sin x - x\cos x)}{\cos x(x\sin x+\cos x)}+c$$

$$=\frac{\sin x - x\cos x}{x\sin x+\cos x}+c$$

**Answer: (C)**

**58.** $\int [x+\sqrt{x^2+1}]^n dx\ (n\neq \pm 1)$ is

(A) $\frac{1}{2}\left[\frac{(x+\sqrt{x^2+1})^{n+1}}{n+1}+\frac{(x+\sqrt{x^2+1})^{n-1}}{n-1}\right]+c$

(B) $\frac{(x+\sqrt{x^2+1})^{2n}}{n(n-1)}+c$

(C) $\frac{1}{2}(x+\sqrt{x^2+1})^n\left[\frac{x+\sqrt{x^2+1}}{n+1}+\frac{1}{n}\right]+c$

(D) $\frac{(x+\sqrt{x^2+1})^{n+1}}{2(n+1)}-\frac{(x+\sqrt{x^2+1})^{n-1}}{2(n-1)}+c$

**Solution:** Let

$$I=\int [x+\sqrt{x^2+1}]^n dx$$

Put $t=x+\sqrt{x^2+1}$ so that

$$\frac{1}{t}=\sqrt{x^2+1}-x$$

Therefore

$$t-\frac{1}{t}=2x$$

and

$$t+\frac{1}{t}=2\sqrt{x^2+1}$$

Now,

$$dt=\left(1+\frac{x}{\sqrt{x^2+1}}\right)dx$$

$$=\frac{t}{(1/2)[t+(1/t)]}dx$$

or $$dx = \frac{t+(1/t)}{2t}dt = \frac{t^2+1}{2t^2}dt$$

Therefore

$$I = \int t^n \left(\frac{t^2+1}{2t^2}\right)dt$$
$$= \frac{1}{2}\int (t^n + t^{n-2})\,dt$$
$$= \frac{1}{2}\left[\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1}\right] + c$$
$$= \frac{1}{2}\left[\frac{(x+\sqrt{x^2+1})^{n+1}}{n+1} + \frac{(x+\sqrt{x^2+1})^{n-1}}{n-1}\right] + c$$

**Answer: (A)**

**Try it out** Try the case when $n = \pm 1$.

**59.** $\displaystyle\int \left(\frac{\cos x - \cos^3 x}{1-\cos^3 x}\right)^{1/2} dx =$

(A) $-\frac{2}{3}\mathrm{Sin}^{-1}\sqrt{\cos x} + c$ (B) $-\frac{2}{3}\mathrm{Sin}^{-1}(\cos^{3/2} x) + c$

(C) $\frac{2}{3}\mathrm{Sin}^{-1}(\sin^{3/2} x) + c$ (D) $\frac{2}{3}\mathrm{Cos}^{-1}\sqrt{\cos x} + c$

**Solution:** Let the given integral be $I$. Then

$$I = \int \frac{\sqrt{\cos x}(\sin x)}{\sqrt{1-\cos^3 x}}dx$$

Put $\cos^{3/2} x = t$ so that

$$\frac{3}{2}\sqrt{\cos x}(-\sin x)dx = dt \quad \text{and} \quad \cos^3 x = t^2$$

Then

$$I = \int \frac{1}{\sqrt{1-t^2}}\left(-\frac{2}{3}\right)dt$$
$$= -\frac{2}{3}\mathrm{Sin}^{-1} t + c$$
$$= -\frac{2}{3}\mathrm{Sin}^{-1}(\cos^{3/2} x) + c$$

**Answer: (B)**

**60.** $\displaystyle\int (\sin x)^{-11/3}(\cos x)^{-1/3}dx = -\frac{3(1+4\tan^2 x)}{8H(x)} + c,$

where $H(x)$ is

(A) $\tan^3 x$ (B) $(\tan x)^{5/3}$

(C) $(\tan x)^{2/3}$ (D) $(\tan x)^{8/3}$

**Solution:** Since sum of the indices of $\sin x$ and $\cos x$ is $-4$ (even), put $t = \tan x$ [see Sec. 4.5.10, Case III, part (c)]. Now

$$I = \int \left(\frac{\sin x}{\cos x}\right)^{-11/3} \cos^{-4} x dx$$
$$= \int \frac{1}{(\tan x)^{11/3}} \sec^4 x dx$$
$$= \int \frac{1}{t^{11/3}} \cdot (1+t^2)\, dt$$
$$= \int (t^{-11/3} + t^{-5/3})\, dt$$
$$= \frac{t^{(-11/3)+1}}{(-11/3)+1} + \frac{t^{(-5/3)+1}}{(-5/3)+1} + c$$
$$= -\frac{3}{8}t^{-8/3} - \frac{3}{2}t^{-2/3}$$
$$= -\frac{3}{8}\left(\frac{1}{t^{8/3}} + \frac{4}{t^{2/3}}\right) + c$$
$$= -\frac{3}{8}\left(\frac{1}{\tan^2 x(\tan x)^{2/3}} + \frac{4}{(\tan x)^{2/3}}\right) + c$$
$$= -\frac{3}{8}\left[\frac{1+4\tan^2 x}{\tan^2 x(\tan x)^{2/3}}\right] + c$$

Therefore $H(x) = (\tan x)^{8/3}$.

**Answer: (D)**

**61.** $\displaystyle\int \cos(\log_e x)\,dx$ is equal to

(A) $x[\cos(\log_e x) + \sin(\log_e x)] + c$

(B) $\frac{x}{2}[\cos(\log_e x) - \sin(\log_e x)] + c$

(C) $\frac{x}{2}[\cos(\log_e x) + \sin(\log_e x)] + c$

(D) $\frac{x}{2}[\sin(\log_e x) - \cos(\log_e x)] + c$

**Solution:** Let

$$I = \int \cos(\log_e x)dx$$
$$= x\cos(\log_e x) - \int x\frac{[-\sin(\log_e x)]}{x}\,dx$$
$$= x\cos(\log_e x) + \int \sin(\log_e x)dx$$
$$= x\cos(\log_e x) + \left[x\sin(\log_e x) - \int x\frac{\cos(\log_e x)}{x}dx\right]$$
$$= x[\cos(\log_e x) + \sin(\log_e x)] - I$$

Therefore

$$2I = x[\cos(\log_e x)+\sin(\log_e x)]$$
$$\Rightarrow I = \frac{x}{2}[\cos(\log_e x)+\sin(\log_e x)]$$

**Answer: (C)**

**62.** $\int \frac{\sin^4 x}{\cos x}dx = \frac{1}{2}\log_e\left(\frac{1+\sin x}{1-\sin x}\right) - g(x)+c$ where $g(x)$ equals

(A) $\frac{1}{3}\sin^3 x+\sin x$ (B) $\frac{1}{3}\cos^3 x+\cos x$

(C) $\frac{1}{3}\sin^3 x-\sin x$ (D) $\frac{1}{3}\cos^3 x-\cos x$

**Solution:** Let

$$I = \int \frac{\sin^4 x}{\cos x}dx$$
$$= \int \frac{\sin^4 x\cos x}{\cos^2 x}dx$$
$$= \int \frac{t^4}{1-t^2}dt \quad \text{where } t=\sin x$$
$$= \int \frac{t^4-1+1}{1-t^2}\,dt$$
$$= -\int (t^2+1)dt + \int \frac{dt}{1-t^2}$$
$$= \frac{-t^3}{3} - t + \frac{1}{2}\log_e\left|\frac{1+t}{1-t}\right| + c$$
$$= \frac{-1}{3}\sin^3 x - \sin x + \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)+c$$

**Answer: (A)**

**63.** $\int \frac{\log x}{x^2}dx$ is equal to

(A) $\log x + \frac{1}{x}+c$ (B) $-\log x - \frac{1}{x}+c$

(C) $-\frac{\log x}{x}+\frac{1}{x}+c$ (D) $-\frac{(1+\log x)}{x}+c$

**Solution:** Let

$$I = \int \frac{\log x}{x^2}dx$$
$$= \frac{-1}{x}\log x - \int \frac{-1}{x}\left(\frac{1}{x}\right)dx$$
$$= -\frac{\log x}{x} + \int \frac{1}{x^2}dx$$
$$= \frac{-\log x}{x} - \frac{1}{x}+c$$

**Answer: (D)**

**64.** $\int \frac{e^x}{(1+e^{2x})^2}dx =$

(A) $\frac{e^x}{1+e^{2x}} + \mathrm{Tan}^{-1}e^x + c$

(B) $\frac{1}{2}\left(\frac{e^x}{1+e^{2x}}\right) - \mathrm{Tan}^{-1}x + c$

(C) $\frac{1}{2}\left(\frac{e^x}{1+e^{2x}}\right) + \frac{1}{2}\mathrm{Tan}^{-1}e^x + c$

(D) $\mathrm{Tan}^{-1}\left(\frac{e^x}{2}\right) + \frac{1}{2}\left(\frac{e^x}{1+e^{2x}}\right)+c$

**Solution:** Let

$$I = \int \frac{e^x}{(1+e^{2x})^2}dx$$
$$= \int \frac{dt}{(1+t^2)^2} \quad \text{where } t=e^x$$

Put $t=\tan\theta$. Then

$$I = \int \frac{\sec^2\theta}{\sec^4\theta}d\theta$$
$$= \int \frac{1+\cos 2\theta}{2}d\theta$$
$$= \frac{\theta}{2}+\frac{1}{4}\sin 2\theta + c$$
$$= \frac{1}{2}\mathrm{Tan}^{-1}e^x + \frac{1}{4}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)c$$
$$= \frac{1}{2}\mathrm{Tan}^{-1}e^x + \frac{1}{2}\left(\frac{e^x}{1+e^{2x}}\right)+c$$

**Answer: (C)**

**65.** $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx =$

(A) $\mathrm{Cos}^{-1}\left(\frac{x}{a}\right)^{3/2}+c$ (B) $\frac{2}{3}\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)^{3/2}+c$

(C) $\frac{2}{3}\mathrm{Cos}^{-1}\left(\frac{x}{a}\right)^{3/2}+c$ (D) $\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)^{3/2}+c$

**Solution:** Let

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$$
$$= \int x^{1/2}(a^3-x^3)^{-1/2}dx$$

Put $x = a\sin^{2/3}\theta$ so that

$$dx = \frac{2a}{3}\sin^{-1/3}\theta\cos\theta d\theta$$

Therefore

$$I=\int \frac{\sqrt{a}\sin^{1/3}\theta}{a^{3/2}\cos\theta}\left(\frac{2}{3}a\sin^{-1/3}\theta\cos\theta\right)d\theta$$
$$=\frac{2}{3}\int d\theta$$
$$=\frac{2}{3}\theta+c$$
$$=\frac{2}{3}\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)^{3/2}+c$$

**Answer: (B)**

**66.** Let $f$ be a positive differentiable function defined on $(0,\infty)$ and

$$\phi(x)=\lim_{n\to\infty}\left(\frac{f\left(x+\frac{1}{n}\right)}{f(x)}\right)^n$$

Then $\int \log_e(\phi(x))\,dx$ is equal to

(A) $\frac{1}{2}(f(x))^2+c$ (B) $f(x)f'(x)$

(C) $\log_e(f(x))+c$ (D) $\log_e\left(\frac{f'(x)}{f(x)}\right)+c$

**Solution:** We have

$$\phi(x)=\exp\left[\lim_{n\to\infty}\left(\frac{f(x+(1/n))}{f(x)}-1\right)^n\right]$$
$$=\exp\left[\lim_{n\to\infty}\left(\frac{f(x+(1/n))-f(x)}{(x+(1/n))-x}\right)\cdot\frac{1}{f(x)}\right]$$
$$=e^{f'(x)/f(x)}$$

Therefore

$$\log_e\phi(x)=\frac{f'(x)}{f(x)}$$

Integrating both sides we get

$$\int \log_e(\phi(x))\,dx=\int\frac{f'(x)}{f(x)}dx$$
$$=\log_e(f(x))+c$$

**Answer: (C)**

## Multiple Correct Choice Type Questions

**1.** $\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$ is equal to

(A) $\sqrt{2}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

(B) $\sqrt{2}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{2}}\right)+c$

(C) $\sqrt{2}\,\mathrm{Sin}^{-1}(\sin x-\cos x)+c$

(D) $\sqrt{2}\,\mathrm{Tan}^{-1}(\sin x-\cos x)+c$

**Solution:** Let

$$I=\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

Put $\tan x=t^2$ so that $\sec^2 x dx=2t\,dt$. Then

$$dx=\frac{2t}{1+t^4}dt$$

Now

$$I=\int\left(t+\frac{1}{t}\right)\frac{2t}{1+t^4}dt$$
$$=\int\frac{2(t^2+1)}{t^4+1}dt$$
$$=2\int\frac{1+(1/t^2)}{t^2+(1/t^2)}dt$$
$$=2\int\frac{dz}{z^2+2}\quad\text{where } z=t-\frac{1}{t}$$
$$=\frac{2}{\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{z}{\sqrt{2}}\right)+c$$
$$=\sqrt{2}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$$

Hence (A) is correct.
Also

$$I=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$
$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$
$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$
$$=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}\quad\text{where } t=\sin x-\cos x$$
$$=\sqrt{2}\,\mathrm{Sin}^{-1}t+c$$
$$=\sqrt{2}\,\mathrm{Sin}^{-1}(\sin x-\cos x)+c$$

Hence (C) is correct.

**Answers: (A), (C)**

**2.** $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$ is equal to

(A) $-2\text{Tan}^{-1}\left(\frac{\sqrt{1+x-x^2}+1}{x}+1\right)+c$

(B) $2\text{Tan}^{-1}\left(\frac{\sqrt{1+x-x^2}+1}{x}\right)+c$

(C) $\text{Sin}^{-1}\left(\frac{3x+1}{5(x+1)}\right)+c$

(D) $\text{Sin}^{-1}\left(\frac{3x+1}{\sqrt{5}(x+1)}\right)+c$

**Solution:** First, we use Euler's substitution. Put

$$\sqrt{1+x-x^2} = tx-1 \quad (\because c>0)$$

This gives

$$1+x-x^2 = t^2x^2-2tx+1$$
$$\Rightarrow 1-x = t^2x-2t$$
$$\Rightarrow x = \frac{1+2t}{1+t^2}$$

So

$$dx = \frac{2(1-t-t^2)}{(1+t^2)^2}dt$$

Also

$$\sqrt{1+x-x^2} = tx-1 = \frac{t(1+2t)}{1+t^2}-1 = \frac{t^2+t-1}{1+t^2}$$

Therefore

$$\int \frac{1}{(1+x)\sqrt{1+x-x^2}}dx = \int \frac{1}{\left(1+\frac{1+2t}{1+t^2}\right)\left(\frac{t^2+t-1}{1+t^2}\right)} \cdot \frac{2(1-t-t^2)}{(1+t^2)^2}dt$$
$$= -2\int \frac{dt}{t^2+2t+2}$$
$$= -2\int \frac{dt}{(t+1)^2+1}$$
$$= -2\,\text{Tan}^{-1}(t+1)+c$$
$$= -2\,\text{Tan}^{-1}\left(\frac{\sqrt{1+x-x^2}+1}{x}+1\right)+c$$

So (A) is correct. Now, we put

$$t = \frac{1}{x+1} \quad \text{or} \quad x+1 = \frac{1}{t}$$

Therefore

$$dx = -\frac{dt}{t^2}$$

So

$$\int \frac{dx}{(1+x)\sqrt{1+x-x^2}} = \int \frac{1}{\frac{1}{t}\sqrt{\frac{1}{t}-\left(\frac{1}{t}-1\right)^2}}\left(-\frac{dt}{t^2}\right)$$
$$= -\int \frac{dt}{\sqrt{3t-t^2-1}}$$
$$= -\int \frac{dt}{\sqrt{\frac{5}{4}-\left(t-\frac{3}{2}\right)^2}}$$
$$= -\text{Sin}^{-1}\left(\frac{t-\frac{3}{2}}{\sqrt{5}/2}\right)+c$$
$$= -\text{Sin}^{-1}\left(\frac{2t-3}{\sqrt{5}}\right)+c$$
$$= -\text{Sin}^{-1}\left(\frac{\frac{2}{x+1}-3}{\sqrt{5}}\right)+c$$
$$= -\text{Sin}^{-1}\left(\frac{-3x-1}{(x+1)\sqrt{5}}\right)+c$$
$$= \text{Sin}^{-1}\left(\frac{3x+1}{\sqrt{5}(x+1)}\right)+c$$

Hence (D) is correct.

**Answers: (A), (D)**

**3.** By a suitable substitution, the indefinite integral

$$\int \frac{dx}{\sin x+\sec x}$$

can be expressed as sum of two of the following integrals. Identify them.

(A) $\int \frac{dt}{1-t^2}$ (B) $\int \frac{dt}{1+t^2}$

(C) $\int \frac{dt}{3-t^2}$ (D) $\int \frac{dt}{3+t^2}$

**Solution:** Let

$$I = \int \frac{dx}{\sin x+\sec x} = \int \frac{\cos x}{\sin x\cos x+1}dx$$

$$= \int \frac{2\cos x}{2+2\sin x\cos x}dx$$
$$= \int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{2+2\sin x\cos x}dx$$
$$= \int \frac{\cos x+\sin x}{2+2\sin x\cos x}dx + \int \frac{\cos x-\sin x}{2+2\sin x\cos x}dx$$
$$= \int \frac{\cos x+\sin x}{3-(\sin x-\cos x)^2}dx + \int \frac{\cos x-\sin x}{1+(\sin x+\cos x)^2}dx$$
$$= \int \frac{dt}{3-t^2} + \int \frac{dz}{1+z^2}$$

where

$$t = \sin x - \cos x$$

and $$z = \sin x + \cos x$$

**Answers: (B), (C)**

**4.** $\displaystyle\int \frac{x^3+3}{(x+1)(x^2+1)}dx$ equals

(A) $x+\log_e|x+1|-\log_e(x^2+1)-\mathrm{Cot}^{-1}x+c$

(B) $x-\log_e|x+1|+\log_e(x^2+1)+\mathrm{Tan}^{-1}x+c$

(C) $x+\log_e|x+1|-\log_e(x^2+1)+\mathrm{Tan}^{-1}x+c$

(D) $x-\log_e|x+1|-\log_e(x^2+1)-\mathrm{Tan}^{-1}x+c$

**Solution:** Let

$$I = \int \frac{x^3+3}{(x+1)(x^2+1)}dx$$
$$= \int \frac{x^3+1+2}{(x+1)(x^2+1)}dx$$
$$= \int \frac{x^2-x+1}{x^2+1}dx + 2\int \frac{dx}{(x+1)(x^2+1)}$$
$$= \int \left(1-\frac{x}{x^2+1}\right)dx + \int \left(\frac{1}{x+1}-\frac{x-1}{x^2+1}\right)dx$$
$$= x-\frac{1}{2}\log_e(x^2+1)+\log_e|x+1|-\frac{1}{2}\log_e(x^2+1)$$
$$+\mathrm{Tan}^{-1}x+c$$
$$= x+\log_e|x+1|-\log_e(x^2+1)+\mathrm{Tan}^{-1}x+c$$

Hence (C) is correct. Also

$$\mathrm{Tan}^{-1}x = \frac{\pi}{2}-\mathrm{Cot}^{-1}x$$

Therefore (A) is also correct.

**Answers: (A), (C)**

**5.** $\displaystyle\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$ equals

(A) $-\frac{19}{36}x+\frac{35}{36}\log(9e^x-4e^{-x})+c$

(B) $-\frac{3}{2}x+\frac{35}{36}\log(9e^{2x}-4)+c$

(C) $\frac{4}{9}x+\frac{70}{9}\int\frac{dx}{9e^{2x}-4}+c$

(D) $\frac{4}{9}x-\frac{70}{9}\int\frac{dx}{9e^{2x}-4}+c$

**Solution:** Let

$$I = \int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx$$

Write

$$4e^x+6e^{-x} = \lambda(9e^x-4e^{-x})+\mu\frac{d}{dx}(9e^x-4e^{-x})$$
$$= \lambda(9e^x-4e^{-x})+\mu(9e^x+4e^{-x})$$

Equating the coefficients of $e^x$ and $e^{-x}$ on both sides we get

$$9\lambda+9\mu=4$$
$$-4\lambda+4\mu=6$$

Solving these equations we get

$$\lambda=-\frac{19}{36},\quad \mu=\frac{35}{36}$$

Therefore

$$I = \int \lambda dx + \mu\int \frac{9e^x+4e^{-x}}{9e^x-4e^{-x}}dx$$
$$= -\frac{19}{36}x+\frac{35}{36}\log_e(9e^x-4e^{-x})+c$$

So (A) is correct.

In the above answer, if we write $e^{-x}=1/e^x$ then we get that (B) is also correct.

Also

$$I = \int \frac{4e^{2x}+6}{9e^{2x}-4}dx \tag{4.31}$$

That is

$$\frac{4e^{2x}+6}{9e^{2x}-4} = \frac{4}{9}+\frac{70}{9(9e^{2x}-4)}$$

Therefore from Eq. (4.31), we have

$$I = \frac{4}{9}x+\frac{70}{9}\int\frac{dx}{9e^{2x}-4}$$

Hence (C) is also correct.

**Answers: (A), (B), (C)**

**6.** If $\int \sin x(\sec x \tan x)\, dx = f(x) + g(x) + c$, then

(A) $f(x) = \tan x$ (B) $f(x) = \sec x$

(C) $g(x) = -2x$ (D) $g(x) = -x$

**Solution:** We have

$$\int \sin x \sec x \tan x dx = \int \tan^2 x dx$$
$$= \int (\sec^2 x - 1)\, dx$$
$$= \tan x - x + c$$

**Answers: (A), (D)**

**7.** If

$$\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = \lambda f(x) + c$$

then

(A) $\lambda = -\frac{1}{8}, f(x) = \cos 4x$

(B) $\lambda = \frac{1}{8}, f(x) = \sin 4x$

(C) Least period of $f(x)$ is $\frac{\pi}{2}$

(D) $f(x)$ is an even function

**Solution:** We have

$$\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = \int \frac{2\cos^2 2x \sin x \cos x}{\cos 2x} dx$$
$$= \int \cos 2x \sin 2x dx$$
$$= \frac{1}{2}\int \sin 4x dx$$
$$= \frac{-1}{8}\cos 4x + c$$

**Answers: (A), (C), (D)**

**8.** Let $f(x) = x^3 + ax^2 + bx + c$ where $a, b, c$ are real numbers. If $f(x)$ has local minimum at $x = 1$ and a local maximum at $x = -1/3$ and $f(2) = 0$, then

(A) $f(x) = x^3 - x^2 - x + 2$

(B) $f(x) = x^3 - x^2 - x - 2$

(C) $f(x) = x^3 + x^2 + x - 2$

(D) If $F(x) = \int f(x) dx$, then $F(1) - F(-1) = \frac{-14}{3}$

**Solution:** Since $f(x)$ has local maximum and local minimum at $x = -1/3$ and $x = 1$, respectively, we have

$$f'\left(\frac{-1}{3}\right) = 0 \quad \text{and} \quad f'(1) = 0$$

Now,

$$f'(x) = 3x^2 + 2ax + b$$

Therefore

$$f'\left(\frac{-1}{3}\right) = 0 \Rightarrow -2a + 3b = -1 \tag{4.32}$$

and $$f'(1) = 0 \Rightarrow 2a + b = -3 \tag{4.33}$$

From Eqs. (4.32) and (4.33), we obtain $a = -1, b = -1$. Further

$$f(2) = 0 \Rightarrow 8 + 4a + 2b + c = 0$$
$$\Rightarrow c = -2 \quad (\because a = -1, b = -1)$$

Therefore

$$f(x) = x^3 - x^2 - x - 2$$

Hence (B) is correct. Also

$$F(x) = \int f(x) dx + c$$
$$= \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + c$$

so that

$$F(1) - F(-1) = \frac{1}{4}(1-1) - \frac{1}{3}(1+1) - \frac{1}{2}(1-1) - 2(1+1)$$
$$= \frac{-2}{3} - 4 = \frac{-14}{3}$$

Hence (D) is correct.

**Answers: (B), (D)**

**9.** $\int \frac{1 - \cos x}{\cos x(1 + \cos x)} dx$ is equal to

(A) $\log_e |\sec x + \tan x| - 2\tan\frac{x}{2} + c$

(B) $\log_e \left|\tan\frac{x}{2}\right| - 2(\sec x + \tan x) + c$

(C) $-2\tan\frac{x}{2} + \log_e \left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

(D) $2\tan\frac{x}{2} + \log_e \left|\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c$

**Solution:** Let

$$I = \int \frac{1 - \cos x}{\cos x(1 + \cos x)} dx$$

$$= \int \left(\tan^2 \frac{x}{2}\right) \frac{\left(1+\tan^2 \frac{x}{2}\right)}{1-\tan^2 \frac{x}{2}} dx$$

Put $\tan(x/2) = t$ so that $\sec^2(x/2)dx = 2dt$. Therefore

$$I = \int \frac{2t^2}{1-t^2} dt$$

$$= -2\int \left(1 - \frac{1}{1-t^2}\right) dt$$

$$= -2t + 2\left(\frac{1}{2}\log_e \left|\frac{1+t}{1-t}\right|\right) + c$$

$$= -2\tan\frac{x}{2} + \log_e \left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + c$$

$$= -2\tan\frac{x}{2} + \log_e \left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$$

Also

$$I = \int \left(\frac{1}{\cos x} - \frac{2}{1+\cos x}\right) dx$$

$$= \int \left(\sec x - \sec^2 \frac{x}{2}\right) dx$$

$$= \log_e |\sec x + \tan x| - 2\tan\frac{x}{2} + c$$

***Note:***

$$\log_e \left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| = \log_e \left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right|$$

$$= \log_e \left|\frac{1+\sin x}{\cos x}\right|$$

$$= \log_e |(\sec x + \tan x)|$$

**Answers: (A), (C)**

**10.** $\int \sqrt{\frac{a+x}{a-x}}\, dx$ equals

(A) $a\,\mathrm{Sin}^{-1}\frac{x}{a} + \frac{\sqrt{a^2-x^2}}{2} + c$

(B) $a\,\mathrm{Sin}^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$

(C) $a\,\mathrm{Sin}^{-1}\frac{x}{a} + \sqrt{a^2-x^2} + c$

(D) $-a\,\mathrm{Cos}^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$

**(IIT-JEE 2004)**

**Solution:** We have

$$I = \int \sqrt{\frac{a+x}{a-x}}\, dx$$

$$= \int \frac{a+x}{\sqrt{a^2-x^2}} dx$$

$$= a\int \frac{dx}{\sqrt{a^2-x^2}} + \int \frac{x}{\sqrt{a^2-x^2}} dx$$

$$= a\,\mathrm{Sin}^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$$

Hence (B) is correct.
Also

$$I = \int \sqrt{\frac{a+x}{a-x}}\, dx$$

Note that $-a < x < a$. Put $x = a\cos\theta$ so that $dx = (-a\sin\theta)\,d\theta$. Therefore

$$I = \int \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} (-a\sin\theta)\, d\theta$$

$$= -a\int \frac{\cos(\theta/2)}{\sin(\theta/2)} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) d\theta$$

$$= -a\int 2\cos^2\frac{\theta}{2} d\theta$$

$$= -a\int (1+\cos\theta)\, d\theta$$

$$= -a\theta - a\sin\theta + c$$

$$= -a\,\mathrm{Cos}^{-1}\frac{x}{a} - a\sqrt{1-\cos^2\theta} + c$$

$$= -a\,\mathrm{Cos}^{-1}\frac{x}{a} - a\sqrt{1-\frac{x^2}{a^2}} + c$$

$$= -a\,\mathrm{Cos}^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$$

So (D) is correct.

***Note:*** In (B), if we replace $\mathrm{Sin}^{-1}(x/a)$ by $(\pi/2) - \mathrm{Cos}^{-1}(x/a)$ we obtain (D). But the procedure is different.

**Answers: (B), (D)**

**11.** $\int \frac{x^3}{\sqrt{x^8+1}} dx$ can be evaluated by using one (or more) substitutions.

(A) $x^4 = t$ (B) $x^4 t^2 = \sqrt{1+x^8}$

(C) $t^2x^8 = \sqrt{1+x^8}$ (D) $t^4x^8 = \sqrt{1+x^8}$

**Solution:** Let

$$I = \int \frac{x^3}{\sqrt{x^8+1}}dx$$

Put $x^4 = t$ so that $x^3dx = (1/4)dt$. Therefore

$$\begin{aligned} I &= \frac{1}{4}\int \frac{dt}{\sqrt{t^2+1}} \\ &= \frac{1}{4}\mathrm{Sinh}^{-1}t \\ &= \frac{1}{4}\log(t+\sqrt{1+t^2}) \\ &= \frac{1}{4}\log(x^4+\sqrt{1+x}\ ) \end{aligned}$$

Hence $x^4 = t$ is useful. Also

$$I = \int x^3(1+x^8)^{-1/2}dx$$

According to integration of Binomial differential, $m=3$, $n=8$, $p=-1/2$ so that

$$\frac{m+1}{n}+p = \frac{4}{8}-\frac{1}{2} = 0 \quad \text{(integer)}$$

Hence the substitution $x^8t^2 = \sqrt{1+x^8}$ can be used.

**Answers: (A), (C)**

**12.** Which of the following methods are handy to evaluate the indefinite integral $\int \frac{x}{\sqrt{9+8x-x^2}}dx$?

(A) The substitution $x-4=t$
(B) Writing $x = \lambda(8-2x)+\mu$
(C) Substitution $9+8x-x^2 = t^2$
(D) Both (A) and (B)

**Solution:** We have

$$\begin{aligned} I &= \int \frac{x}{\sqrt{9+8x-x^2}}dx \\ &= \int \frac{x}{\sqrt{25-(x-4)^2}}dx \\ &= \int \frac{t+4}{\sqrt{25-t^2}}dt \quad \text{where } t = x-4 \\ &= \int \frac{t}{\sqrt{25-t^2}}dt + 4\int \frac{dt}{\sqrt{25-t^2}} \\ &= -\sqrt{25-t^2}+4\mathrm{Sin}^{-1}\left(\frac{t}{5}\right)+c \\ &= -\sqrt{9+8x-x^2}+4\mathrm{Sin}^{-1}\left(\frac{x-4}{5}\right)+c \end{aligned}$$

Also,

$$x = \lambda(8-2x)+\mu$$

$$\Rightarrow \lambda = \frac{-1}{2} \quad \text{and} \quad \mu = 4$$

Therefore

$$\begin{aligned} I &= -\frac{1}{2}\int \frac{8-2x}{\sqrt{9+8x-x^2}}dx + 4\int \frac{dx}{\sqrt{9+8x-x^2}} \\ &= -\frac{1}{2}\times 2\sqrt{9+8x-x^2}+4\int \frac{dx}{\sqrt{25-(x-4)^2}} \\ &= -\sqrt{9+8x-x^2}+4\mathrm{Sin}^{-1}\left(\frac{x-4}{5}\right)+c \end{aligned}$$

**Answers: (A), (B)**

**13.** The indefinite integral

$$\int \frac{\sin x\cos x}{a^2\cos^2 x+b^2\sin^2 x}dx$$

can be evaluated by means of the following substitutions. Identify them. Given that $ab \neq 0$ and $b^2-a^2>0$.

(A) $\tan = t$ (B) $a^2\cos^2 x+b^2\sin^2 x = t$
(C) $t = \cos^2 x$ (D) $t = \sin^2 x$

**Solution:** We have

$$\begin{aligned} I &= \int \frac{\sin x\cos x}{a^2\cos^2 x+b^2\sin^2 x}dx \\ &= \int \frac{\tan x}{a^2+b^2\tan^2 x}dx \end{aligned}$$

**(i)** Put $t = \tan x$ so that $dt = \sec^2 x\,dx$ and hence

$$dx = \frac{1}{1+t^2}dt$$

Thus

$$I = \int \frac{t}{(a^2+b^2t^2)(1+t^2)}dt$$

Now, use partial fractions. Therefore $t = \tan x$ is workable.

**(ii)** We have

$$t = a^2\cos^2 x+b^2\sin^2 x$$

So

$$dt = 2(b^2-a^2)\sin x\cos x dx$$

Therefore

$$\begin{aligned} I &= \frac{1}{2(b^2-a^2)}\int \frac{dt}{t} \\ &= \frac{1}{2(b^2-a^2)}\log_e(a^2\cos^2 x+b^2\sin^2 x)+c \end{aligned}$$

Hence $t = a^2\cos^2 x + b^2\sin^2 x$ is useful.

**(iii)** Let $t = \cos^2 x$ so that $dt = (-2\sin x\cos x)dx$. Therefore

$$I = -\frac{1}{2}\int \frac{dt}{a^2 t + b^2(1-t)}$$

$$= -\frac{1}{2}\int \frac{dt}{b^2 - (b^2 - a^2)\,t}$$

Therefore $t = \cos^2 x$ is also useful. Similarly, $t = \sin^2 x$ is also workable.

**Answers: (A), (B), (C), (D)**

## Matrix-Match Type Questions

**1.** Match the integrals of Column I with their values in Column II.

| Column I | Column II |
|---|---|
| (A) $\int \frac{dx}{\sqrt{1+\sin x}}$ | (p) $\log_e\left(\tan x \tan\frac{x}{2}\right) + c$ |
| (B) $\int \frac{dx}{\sin x + \cos x}$ | (q) $\sqrt{2}\log\tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$ |
| (C) $\int \frac{dx}{\sin x + \sqrt{3}\cos x}$ | (r) $\frac{1}{2}\log\left(\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)\right) + c$ |
| (D) $\int \frac{1+\cos x}{\sin x\cos x}\,dx$ | (s) $\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$ |

**Solution:**

(A) Let

$$I = \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= \int \frac{dx}{\sin\frac{x}{2} + \cos\frac{x}{2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin\frac{x}{2} + \frac{1}{\sqrt{2}}\cos\frac{x}{2}}$$

$$= \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2} + \frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}(\sqrt{2})^2\log\left(\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)\right) + c$$

$$\left[\because \operatorname{cosec} x\,dx = \log\tan\left(\frac{x}{2}\right) + c\right]$$

**Answer: (A) → (q)**

(B) Let

$$I = \int \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log_e\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)$$

**Answer: (B) → (s)**

(C) Let

$$I = \int \frac{dx}{\sin x + \sqrt{3}\cos x}$$

$$= \frac{1}{2}\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

$$= \frac{1}{2}\int \operatorname{cosec}\left(x + \frac{\pi}{3}\right)dx$$

$$= \frac{1}{2}\log_e\tan\left(\frac{x}{2} + \frac{\pi}{6}\right) + c$$

**Answer: (C) → (r)**

(D) Let

$$I = \int \frac{1+\cos x}{\sin x\cos x}\,dx$$

$$= 2\int \operatorname{cosec} 2x\,dx + \int \operatorname{cosec} x\,dx$$

$$= 2\left(\frac{1}{2}\right)\log(\tan x) + \log_e\left(\tan\frac{x}{2}\right) + c$$

$$= \log_e\left(\tan x\tan\frac{x}{2}\right) + c$$

**Answer: (D) → (p)**

**2.** Match the integrals of Column I with their corresponding values in Column II.

| Column I | Column II |
|---|---|
| (A) $\int x\cos^3(x^2)\sin(x^2)\,dx$ | (p) $-\tan\left(\frac{1}{x}\right) + c$ |
| (B) $\int \frac{1}{2\sqrt{x}}\tan\sqrt{x}\,\sec\sqrt{x}\,dx$ | (q) $-2\cos\sqrt{x} + c$ |

*(Continued)*

| *Column I* | *Column II* |
|---|---|
| (C) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ | (r) $-\frac{1}{8}\cos^4\left(x^2\right)+c$ |
| (D) $\int \frac{1}{x^2\cos^2\left(\frac{1}{x}\right)}dx$ | (s) $\sec\sqrt{x}+c$ |

**Solution:**

(A) Let

$$I=\int x\cos^3(x^2)\sin(x^2)\,dx$$

Put $\cos(x^2)=t$. Then

$$-2x\sin(x^2)\,dx=dt$$

Therefore

$$I=\frac{-1}{2}\int t^3dt$$
$$=-\frac{1}{8}t^4+c$$
$$=-\frac{1}{8}\cos^4(x^2)+c$$

**Answer: (A) → (r)**

(B) Let

$$I=\int\frac{1}{2\sqrt{x}}\tan\sqrt{x}\sec\sqrt{x}\,dx$$

Put $\sec\sqrt{x}=t$. Then

$$\frac{1}{2\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}=dt$$

Therefore

$$I=\int dt=t+c=\sec\sqrt{x}+c$$

**Answer: (B) → (s)**

(C) Let

$$I=\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx$$

Put $t=\sqrt{x}$ so that $dt=(1/2\sqrt{x})dx$. Therefore

$$I=\int 2\sin t\,dt=-2\cos t+c=-2\cos\sqrt{x}+c$$

**Answer: (C) → (q)**

(D) Let

$$I=\int\frac{dx}{x^2\cos^2 x}$$

Put $1/x=t$ so that

$$-\frac{1}{x^2}dx=dt$$

Therefore

$$I=-\int\sec^2t\,dt$$
$$=-\tan t$$
$$=-\tan\frac{1}{x}$$

**Answer: (D) → (p)**

**3.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int\frac{1+\log_e x}{1+x\log_e x}dx$ equals | (p) $\frac{1}{6}\log_e\left\lvert\frac{x-1}{x+5}\right\rvert$ |
| (B) $\int\frac{dx}{x^2+4x-5}$ is | (q) $\log_e\lvert 1+\log_e x\rvert+c$ |
| (C) $\int\left[1+2\tan x(\tan x+\sec x)\right]^{1/2}dx$ is | (r) $\log_e\lvert\sec x(\sec x+\tan x)\rvert$ |
| (D) $\int\frac{dx}{x(1+\log_e x)}$ equals | (s) $\log_e\lvert 1+x\log_e x\rvert+c$ |

**Solution:**

(A) We have

$$\int\frac{1+\log_e x}{1+x\log_e x}dx=\int\frac{\frac{d}{dx}(1+x\log_e x)}{1+x\log_e x}dx$$
$$=\log_e(1+x\log_e x)+c$$

**Answer: (A) → (s)**

(B) We have

$$\int\frac{dx}{x^2+4x-5}=\int\frac{dx}{(x+5)(x-1)}$$
$$=\frac{1}{6}\int\left(\frac{1}{x-1}-\frac{1}{x+5}\right)dx$$
$$=\frac{1}{6}\log\left\lvert\frac{x-1}{x+5}\right\rvert+c$$

**Answer: (B) → (p)**

(C) We have

$$\int\left[1+2\tan x(\tan x+\sec x)\right]^{1/2}dx$$
$$=\int(1+\tan^2x+\tan^2x+2\tan x\sec x)^{1/2}dx$$

$$= \int (\sec^2 x + 2\tan x \sec x + \tan^2 x)^{1/2} dx$$
$$= \int (\sec x + \tan x) dx$$
$$= \log_e |\sec x + \tan x| + \log_e |\sec x| + c$$
$$= \log_e |\sec x(\sec x + \tan x)| + c$$

**Answer: (C) → (r)**

(D) We have

$$\int \frac{dx}{x(1+\log_e x)} = \int \frac{(1/x)dx}{(1+\log_e x)}$$
$$= \log_e |1 + \log_e x| + c$$

**Answer: (D) → (q)**

**4.** Match the entries of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int e^x \frac{(2+\sin 2x)}{1+\cos 2x} dx$ | (p) $e^x \tan\frac{x}{2} + c$ |
| (B) $\int e^x \frac{(2-\sin 2x)}{1-\cos 2x} dx$ | (q) $-e^{-x/2} \sec\frac{x}{2} + c$ |
| (C) $\int e^x \frac{(1+\sin x)}{1+\cos x} dx$ | (r) $e^x \tan x + c$ |
| (D) $\int e^{-x/2} \frac{\sqrt{1-\sin x}}{1+\cos x} dx$ | (s) $-e^x \cot x + c$ |

**Solution:**

(A) We have

$$I = \int e^x \frac{2(1+\sin x \cos x)}{2\cos^2 x} dx$$
$$= \int e^x (\tan x + \sec^2 x) dx = e^x \tan x + c$$

**Answer: (A) → (r)**

(B) We have

$$I = \int e^x \frac{2(1-\sin x \cos x)}{2\sin^2 x} dx$$
$$= \int e^x (\mathrm{cosec}^2 x - \cot x) dx = e^x (-\cot x) + c$$

**Answer: (B) → (s)**

(C) We have

$$I = \int e^x \frac{(1+\sin x)}{1+\cos x} dx$$
$$= \int e^x \frac{\left(1 + 2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{2\cos^2\frac{x}{2}} dx$$
$$= \int e^x \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx$$
$$= e^x \tan\frac{x}{2} + c$$

**Answer: (C) → (p)**

(D) We have

$$I = \int e^{-x/2} \frac{\sqrt{1-\sin x}}{1+\cos x} dx$$

Put $-x/2 = t$ so that

$$I = -2\int e^t \frac{\sqrt{1+\sin 2t}}{1+\cos 2t} dt$$
$$= -2\int e^t \frac{(\sin t + \cos t)}{2\cos^2 t} dt$$
$$= -\int e^t (\sec t + \sec t \tan t) dt$$
$$= -e^t \sec t + c$$
$$= -e^{-x/2} \sec(x/2) + c$$

**Answer: (D) → (q)**

**5.** Match the entries of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int e^x \frac{(x^3+x+1)}{(1+x^2)^{3/2}} dx$ | (p) $e^x \left(\frac{x+1}{x+2}\right) + c$ |
| (B) $\int \frac{e^x (x^3 - x + 2)}{(x^2+1)^2} dx$ | (q) $\frac{e^x}{\sqrt{1+x^2}} + c$ |
| (C) $\int e^x \left[\frac{1-x}{1+x^2}\right]^2 dx$ | (r) $\frac{e^x x}{\sqrt{1+x^2}} + c$ |
| (D) $\int e^x \frac{(x^2+3x+3)}{(x+2)^2} dx$ | (s) $\frac{e^x (x+1)}{x^2+1} + c$ |

**Solution:**

(A) We have

$$\int \frac{e^x (x^3+x+1)}{(1+x^2)^{3/2}} dx = \int \frac{e^x [x(x^2+1)+1]}{(1+x^2)^{3/2}} dx$$
$$= \int e^x \left[\frac{x}{\sqrt{1+x^2}} + \frac{1}{(1+x^2)^{3/2}}\right] dx$$
$$= \int e^x [f(x) + f'(x)]\, dx$$

where $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$= e^x f(x) + c$$

$$= \frac{e^x x}{\sqrt{x^2+1}} + c$$

**Answer: (A) → (r)**

(B) We have

$$\int e^x \frac{(x^3 - x + 2)}{(x^2+1)^2} dx = \int e^x \frac{[(x+1)(x^2+1) + 1 - 2x - x^2]}{(x^2+1)^2} dx$$

$$= \int e^x \left[\frac{x+1}{x^2+1} + \frac{1-2x-x^2}{(x^2+1)^2}\right] dx$$

$$= \frac{e^x(x+1)}{x^2+1} + c$$

**Answer: (B) → (s)**

(C) We have

$$\int e^x \left[\frac{1-x}{1+x^2}\right]^2 dx = \int e^x \left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right] dx$$

$$= \frac{e^x}{1+x^2} + c$$

**Answer: (C) → (q)**

(D) We have

$$\int e^x \frac{(x^2+3x+3)}{(x+2)^2} dx = \int e^x \frac{(x+1)(x+2)+1}{(x+2)^2} dx$$

$$= \int e^x \left[\frac{x+1}{x+2} + \frac{1}{(x+2)^2}\right] dx$$

$$= \frac{e^x(x+1)}{x+2} + c$$

**Answer: (D) → (p)**

**6.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$ | (p) $\frac{1}{6}(\sec^3 2x - 3\sec 2x)$ |
| (B) $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x} dx$ | (q) $\frac{2}{9}(2 + \sin 3x)^{3/2}$ |
| (C) $\int \tan^3 2x \sec 2x\, dx$ | (r) $\frac{-1}{2}(2\sin x + \sin 2x)$ |
| (D) $\int \sqrt{2 + \sin 3x}\cos 3x dx$ | (s) $\frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + c$ |

**Solution:**

(A) We have

$$I = \int \frac{(\cos 5x + \cos 4x)}{1 - 2\cos 3x} dx$$

$$= \int \frac{(\cos 5x + \cos 4x)\sin 3x}{\sin 3x - \sin 6x} dx$$

$$= \int \frac{\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)\sin 3x}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}} dx$$

$$= \int \frac{-\cos\frac{x}{2}\left(2\sin\frac{3x}{2}\cos\frac{3x}{2}\right)}{\sin\frac{3x}{2}} dx$$

$$= -\int (\cos 2x + \cos x) dx$$

$$= -\left(\frac{1}{2}\sin 2x + \sin x\right) + c$$

$$= -\frac{1}{2}(2\sin x + \sin 2x) + c$$

**Answer: (A) → (r)**

(B) We have

$$I = \int \frac{(\cos 7x - \cos 8x)\sin 5x}{\sin 5x + \sin 10x} dx$$

$$= \int \frac{\left(2\sin\frac{15x}{2}\sin\frac{x}{2}\right)\left(2\sin\frac{5x}{2}\cos\frac{5x}{2}\right)}{2\sin\frac{15x}{2}\cdot\cos\frac{5x}{2}} dx$$

$$= \int 2\sin\frac{x}{2}\sin\frac{5x}{2} dx$$

$$= \int (\cos 2x - \cos 3x) dx$$

$$= \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + c$$

**Answer: (B) → (s)**

(C) We have

$$I = \int \tan^3 2x \sec 2x\ dx$$

$$= \int \tan^2 2x(\sec 2x \tan 2x)\ dx$$

$$= \int (\sec^2 2x - 1)\sec 2x \tan 2x\, dx$$

Put $t = \sec 2x$, so that

$$\frac{dt}{2} = (\sec 2x \tan 2x) dx$$

Therefore

$$I = \frac{1}{2}\int (t^2 - 1)\ dt$$

$$= \frac{1}{2}\left(\frac{1}{3}t^3 - t\right) + c$$

$$=\frac{1}{6}\sec^3 2x-\frac{1}{2}\sec 2x+c$$

**Answer: (C) → (p)**

(D) We have

$$I=\int\sqrt{2+\sin 3x}\cos 3x\,dx$$

Put $2+\sin 3x=t$. Therefore $3\cos 3x dx=dt$. Now

$$I=\int\sqrt{t}\cdot\frac{1}{3}dt$$
$$=\frac{2}{9}(t)^{3/2}+c$$
$$=\frac{2}{9}(2+\sin 3x)^{3/2}+c$$

**Answer: (D) → (q)**

**7.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) If $\int\frac{\sqrt{x^2+1}}{x^4}dx=\frac{-1}{3}\frac{(1+x^2)^{3/2}}{x^k}$, then $k$ equals | (p) 3 |
| (B) If $\int\frac{2+\sqrt{x}}{(x+\sqrt{x}+1)^2}dx=\frac{\lambda x}{x+\sqrt{x}+1}$, then $\lambda$ is | (q) 2 |
| (C) If $\int\frac{\sqrt{x^{10}+2}}{x^{16}}dx=\frac{-1}{30}\frac{(x^{10}+2)^{3/2}}{x^m}+c$, then $m$ is | (r) 15 |
| (D) If $\int\frac{dx}{x+\sqrt{x}}=n\log_e(1+\sqrt{x})$, then $n$ is equal to | (s) 5 |

**Solution:**

(A) Let

$$I=\int\frac{\sqrt{x^2+1}}{x^4}dx$$

Put $x=\tan\theta$. Then

$$I=\int\frac{\sec\theta\sec^2\theta}{\tan^4\theta}d\theta$$
$$=\int\frac{\cos\theta}{\sin^4\theta}d\theta$$
$$=\int t^{-4}dt \quad \text{where } t=\sin\theta$$
$$=\frac{t^{-4+1}}{-4+1}+c=\frac{-1}{3}t^{-3}+c$$
$$=\frac{-1}{3}\cdot\frac{1}{\sin^3\theta}+c$$
$$=\frac{-1}{3}\frac{1}{(\sin^2\theta)^{3/2}}+c$$
$$=\frac{-1}{3}(\mathrm{cosec}^2\theta)^{3/2}$$
$$=\frac{-1}{3}(1+\cot^2\theta)^{3/2}$$
$$=\frac{-1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}$$
$$=\frac{-1}{3}\frac{(x^2+1)^{3/2}}{x^3}$$

Therefore $k=3$.

**Answer: (A) → (p)**

(B) We have

$$I=\int\frac{2+\sqrt{x}}{(x+\sqrt{x}+1)^2}$$
$$=\int\frac{2+\sqrt{x}}{x^2\left(1+\frac{1}{\sqrt{x}}+\frac{1}{x}\right)^2}dx$$
$$=\int\frac{\frac{2}{x^2}+\frac{1}{x\sqrt{x}}}{\left(1+\frac{1}{\sqrt{x}}+\frac{1}{x}\right)^2}dx$$

Put

$$1+\frac{1}{\sqrt{x}}+\frac{1}{x}=t$$

so that

$$\left(\frac{-1}{2x\sqrt{x}}-\frac{1}{x^2}\right)dx=dt$$
$$\Rightarrow\left(\frac{2}{x^2}+\frac{1}{x\sqrt{x}}\right)dx=-2dt$$

Therefore

$$I=\int\frac{1}{t^2}(-2)dt$$
$$=\frac{2}{t}+c$$
$$=\frac{2}{\frac{1}{x}+\frac{1}{\sqrt{x}}+1}+c$$
$$=\frac{2x}{1+\sqrt{x}+x}+c$$

Therefore $\lambda = 2$.

**Answer: (B) → (q)**

(C) We have

$$I = \int \frac{\sqrt{x^{10}+2}}{x^{16}}\,dx$$
$$= \int \frac{\sqrt{x^{10}+2}}{x^{11}\cdot x^5}\,dx$$
$$= \int \frac{\sqrt{1+\frac{2}{x^{10}}}}{x^{11}}\,dx$$

Put

$$1+\frac{2}{x^{10}} = t^2$$

so that

$$\frac{-20}{x^{11}}dx = 2t\,dt$$
$$\Rightarrow \frac{dx}{x^{11}} = \frac{-1}{10}t\,dt$$

Therefore

$$I = \int t\left(\frac{-1}{10}\right) t\; dt$$
$$= -\frac{t^3}{30}+c$$
$$= \frac{-1}{30}\left(1+\frac{2}{x^{10}}\right)^{3/2}+c$$
$$= -\frac{1}{30}\frac{(x^{10}+2)^{3/2}}{x^{15}}+c$$

So $m = 15$.

**Answer: (C) → (r)**

(D) We have

$$I = \int \frac{dx}{x+\sqrt{x}}$$
$$= \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

Put $\sqrt{x} = t$ so that

$$\frac{1}{\sqrt{x}}dx = 2dt$$

Therefore

$$I = \int \frac{2}{t+1}dt$$
$$= 2\log(t+1)$$
$$= 2\log(\sqrt{x}+1)+c$$

So $n = 2$.

**Answer: (D) → (q)**

## *Comprehension-Type Questions*

**1. Passage:** Using the formula $\int u dv = uv - \int v du$, answer the following questions.

**(i)** $\int \sec^3 x dx$ is equal to

(A) $\frac{1}{2}(\sec x \tan x + \log_e|\sec x + \tan x|)+c$

(B) $\frac{1}{2}(\sec x + \log_e|\sec x + \tan x|)+c$

(C) $\frac{1}{2}(\sec x + \tan x + \log_e|\sec x \tan x|)+c$

(D) $\sec x \tan x + 2\log_e|\sec x + \tan x|+c$

**(ii)** $\int \log_e(1+\cos x)dx - \int x\tan\frac{x}{2}dx$ equals

(A) $-x\log_e(1+\cos x)+c$

(B) $x\log_e(1+\cos x)+c$

(C) $x\log_e(1+\tan x)+c$

(D) $x\log_e(1+\sin x)+c$

**(iii)** $\int e^x \sin x dx$ is equal to

(A) $\frac{e^x}{2}(\sin x - \cos x)+c$

(B) $\frac{-e^x}{2}(\sin x - \cos x)+c$

(C) $\frac{e^x}{4}(\sin x - \cos x)+c$

(D) $\frac{e^x}{4}(\cos x - \sin x)+c$

**Solution:**

**(i)** We have

$$I=\int \sec^3 x dx$$
$$=\int \sec x \sec^2 x dx$$
$$=\sec x \tan x-\int \sec x \tan^2 x\, dx$$
$$=\sec x \tan x-\int \sec x(\sec^2 x-1)dx$$
$$=\sec x \tan x-I+\int \sec x\ dx$$

Therefore

$$2I=\sec \tan x+\log|\sec x+\tan x|$$

**Answer: (A)**

**(ii)** We have

$$I=\int \log_e(1+\cos x)\ dx$$
$$=x\log_e(1+\cos x)-\int x\left(\frac{-\sin x}{1+\cos x}\right)dx$$
$$=x\log_e(1+\cos x)+\int x\tan\frac{x}{2}\ dx$$

Therefore

$$I-\int x\tan\frac{x}{2}dx=x\log_e(1+\cos x)$$

**Answer: (B)**

**(iii)** We have

$$I=\int e^x \sin x$$
$$=e^x \sin x-\int e^x \cos x\ dx$$
$$=e^x \sin x-\left[e^x \cos x-\int e^x(-\sin x)dx\right]$$
$$=e^x \sin x-e^x \cos x-I$$

Therefore

$$2I=e^x(\sin x-\cos x)$$
$$\Rightarrow I=\frac{e^x}{2}(\sin x-\cos x)$$

**Answer: (A)**

2. **Passage:** If $m, n, p$ are rational numbers, then the indefinite integral $\int x^m(a+bx^n)^p dx$ can be evaluated in the following cases with the help of the substitution suggested.

***Case 1:*** If $\dfrac{m+1}{n}$ is an integer, put $a+bx^n=t^\alpha$, where $\alpha$ is the denominator of $p$.

***Case 2:*** If $\dfrac{m+1}{n}+p$ is an integer, put $a+bx^n=t^\alpha x^n$, where $\alpha$ is the denominator of $p$.

Answer the following questions:

**(i)** If

$$\int x^{1/3}(2+x^{2/3})^{1/4}dx=\frac{2}{3}(2+x^{2/3})^{\alpha_1}-\frac{12}{5}(2+x^{2/3})^{\alpha_2}+c$$

then $\alpha_1-\alpha_2$ is equal to

(A) 4 (B) 3 (C) 2 (D) 1

**(ii)** If

$$\int \frac{dx}{x^3\sqrt[5]{1+(1/x)}}=\frac{5}{4}\left(1+\frac{1}{x}\right)^\alpha-\frac{5}{9}\left(1+\frac{1}{x}\right)^\beta+c$$

then $\beta-\alpha$ equals

(A) 1 (B) 2 (C) –1 (D) 3

**Solution:**

**(i)** Let

$$I=\int x^{1/3}(2+x^{2/3})^{1/4}dx$$

Now $m=1/3,\ n=2/3,\ p=1/4$. Therefore

$$\frac{m+1}{n}=\frac{(1/3)+1}{2/3}=2 \quad \text{(integer)}$$

So put

$$2+x^{2/3}=t^4$$

so that

$$x=(t^4-2)^{3/2}$$

Therefore

$$dx=\frac{3}{2}(4t^3)(t^4-2)^{1/2}dt$$
$$=6t^3(t^4-2)^{1/2}dt$$

So

$$I=\int (t^4-2)^{1/2}(t)(6t^3)(t^4-2)^{1/2}dt$$
$$=6\int t^4(t^4-2)\ dt$$
$$=6\left[\frac{t^9}{9}-\frac{2}{5}t^5\right]+c$$
$$=\frac{2}{3}(2+x^{2/3})^{9/4}-\frac{12}{5}(2+x^{2/3})^{5/4}+c$$

Hence

$$\alpha_1-\alpha_2=\frac{9}{4}-\frac{5}{4}=1$$

**Answer: (D)**

**(ii)** Let

$$I = \int x^{-14/5}(1+x)^{-1/5}\,dx$$

Now $m = -14/5$, $n = 1$, $p = -1/5$. Therefore

$$\frac{m+1}{n} + p = \frac{-14}{5} + 1 - \frac{1}{5} = -2 \quad \text{(integer)}$$

Put $(1+x) = xt^5$ so that

$$x = \frac{1}{t^5 - 1} = (t^5 - 1)^{-1}$$

So

$$dx = (-1)\,(5t^4)\,(t^5-1)^{-2}\,dt$$

Hence

$$I = \int (t^5-1)^{14/5}[t^5(t^5-1)^{-1}]^{-1/5}(-5t^4)(t^5-1)^{-2}\,dt$$

$$= -5\int (t^5-1)^{(14/5)+(1/5)-2}(t^3)dt$$

$$= -5\int t^3(t^5-1)\,dt$$

$$= -5\left[\frac{t^9}{9} - \frac{t^4}{4}\right] + c$$

$$= \frac{5}{4}\left(\frac{1+x}{x}\right)^{4/5} - \frac{5}{9}\left(\frac{1+x}{x}\right)^{9/5} + c$$

Therefore

$$\beta - \alpha = \frac{9}{5} - \frac{4}{5} = 1$$

**Answer: (A)**

**3. Passage:** To evaluate

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$$

where $c^2 + d^2 \neq 0$, express the numerator as $a\sin x + b\cos x = \lambda$ (Denominator) $+\,\mu$ (Derivative of the denominator)

Answer the following questions

**(i)** $\displaystyle\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$ equals

(A) $\dfrac{18}{25}x + \dfrac{1}{25}\log_e|3\sin x + 4\cos x| + c$

(B) $\dfrac{-18}{25}x + \dfrac{1}{25}\log_e|3\sin x + 4\cos x| + c$

(C) $\dfrac{x}{25} + \dfrac{18}{25}\log_e|2\sin x + 3\cos x| + c$

(D) $\dfrac{-x}{25} + \dfrac{18}{25}\log_e|3\sin x + 4\cos x| + c$

**(ii)** If $\displaystyle\int \frac{\sin x + 2\cos x}{2\sin x + \cos x}\,dx$ is equal to $px + q\log_e|2\sin x + \cos x| + c$, then $p+q$ equals

(A) $\dfrac{8}{5}$ (B) $\dfrac{6}{5}$ (C) $\dfrac{7}{5}$ (D) 1

**Solution:**

**(i)** Let

$$I = \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$$

Let

$$2\sin x + 3\cos x = \lambda(3\sin x + 4\cos x) + \mu(3\cos x - 4\sin x)$$

Now equating the coefficients $\sin x$ and $\cos x$ on both sides we get

$$3\lambda - 4\mu = 2$$

$$4\lambda + 3\mu = 3$$

Solving these equations, we get that $\lambda = 18/25$ and $\mu = 1/25$. Therefore

$$I = \int \frac{18}{25}dx + \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}dx$$

$$= \frac{18}{25}x + \frac{1}{25}\log_e|3\sin x + 4\cos x| + c$$

**Answer: (A)**

**(ii)** Let

$$I = \int \frac{\sin x + 2\cos x}{2\sin x + \cos x}\,dx$$

Let

$$\sin x + 2\cos x = \lambda(2\sin x + \cos x) + \mu(2\cos x - \sin x)$$

Therefore

$$2\lambda - \mu = 1$$
$$\lambda + 2\mu = 2$$

so that $\lambda = 4/5$ and $\mu = 3/5$. Hence

$$p = \frac{4}{5} \quad \text{and} \quad q = \frac{3}{5}$$

and so

$$p + q = \frac{7}{5}$$

**Answer: (C)**

**4. Passage:** To evaluate

$$\int \frac{1}{(\text{Linear})\sqrt{\text{Linear}}}$$

and $$\int \frac{1}{(\text{Quadratic})\sqrt{\text{Linear}}}$$

put $\sqrt{\text{Linear}} = t$. Answer the following questions.

**(i)** If $\int \frac{dx}{(x+2)\sqrt{3x+4}} = \sqrt{2} f(x) + c$, then $f(x)$ equals

(A) $\text{Tan}^{-1}\left(\frac{3x+4}{2}\right)$ (B) $\text{Tan}^{-1}\left(\left(\frac{3x+4}{2}\right)^{1/2}\right)$

(C) $\text{Tan}^{-1}\left(\frac{x+2}{2}\right)$ (D) $\text{Tan}^{-1}\sqrt{\frac{x+2}{2}} + c$

**(ii)** If

$$\int \frac{dx}{(x^2-4)\sqrt{x+1}} = p \log_e \left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right| - q\,\text{Tan}^{-1}\sqrt{x+1} + c$$

then $pq$ is equal to

(A) $\frac{1}{4\sqrt{3}}$ (B) $\frac{1}{8}$

(C) $\frac{1}{3\sqrt{3}}$ (D) $\frac{1}{8\sqrt{3}}$

**(iii)** If $\int \frac{dx}{(x-3)\sqrt{x+2}}$ is equal to

(A) $\frac{1}{\sqrt{5}} \log_e \left|\frac{x+2-\sqrt{5}}{x+2+\sqrt{5}}\right| + c$

(B) $\frac{1}{\sqrt{5}} \log_e \left|\frac{x+2+\sqrt{5}}{x+2-\sqrt{5}}\right| + c$

(C) $\frac{1}{\sqrt{5}} \log_e \left|\frac{\sqrt{x+2}-\sqrt{5}}{\sqrt{x+2}+\sqrt{5}}\right| + c$

(D) $\frac{1}{\sqrt{5}} \log_e \left|\frac{\sqrt{x+2}+\sqrt{5}}{\sqrt{x+2}-\sqrt{5}}\right| + c$

**Solution:**

**(i)** We have

$$I = \int \frac{dx}{(x+2)\sqrt{3x+4}}$$

Put $3x+4 = t^2$ so that

$$x = \frac{t^2-4}{3}$$

and $$dx = \frac{2t}{3} dt$$

Therefore

$$I = \int \frac{1}{\left(\frac{t^2-4}{3}+2\right)t}\left(\frac{2t}{3}\right) dt$$

$$= 2\int \frac{dt}{t^2+2}$$

$$= \frac{2}{\sqrt{2}} \text{Tan}^{-1} \frac{t}{\sqrt{2}}$$

$$= \sqrt{2}\,\text{Tan}^{-1}\left(\frac{\sqrt{3x+4}}{\sqrt{2}}\right) + c$$

So

$$f(x) = \text{Tan}^{-1}\sqrt{\frac{3x+4}{2}}$$

**Answer: (B)**

**(ii)** We have

$$I = \int \frac{dx}{(x^2-4)\sqrt{x+1}}$$

Put $x+1 = t^2$. So $dx = 2t\,dt$. Therefore

$$I = \int \frac{1}{[(t^2-1)^2-4]t}(2t)\, dt$$

$$= 2\int \frac{dt}{(t^2-1+2)(t^2-1-2)}$$

$$= 2\int \frac{dt}{(t^2+1)(t^2-3)}$$

$$= \frac{2}{4}\int\left(\frac{1}{t^2-3} - \frac{1}{t^2+1}\right) dt$$

$$= \frac{1}{2}\left[\frac{1}{2\sqrt{3}} \log_e \left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \text{Tan}^{-1} t\right] + c$$

$$= \frac{1}{4\sqrt{3}} \log_e \left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right| - \frac{1}{2}\text{Tan}^{-1}\sqrt{x+1} + c$$

Therefore

$$pq = \frac{1}{4\sqrt{3}} \times \frac{1}{2} = \frac{1}{8\sqrt{3}}$$

**Answer: (D)**

**(iii)** Let

$$I = \int \frac{dx}{(x-3)\sqrt{x+2}}$$

Put $x+2 = t^2$. Then

$$I = \int \frac{1}{(t^2-5)t}(2t)\, dt$$

$$= 2\int \frac{dt}{t^2-5}$$

$$= \frac{2}{2\sqrt{5}} \log_e \left| \frac{t-\sqrt{5}}{t+\sqrt{5}} \right| + c \quad \text{where } t = \sqrt{x+2}$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{x+2}-\sqrt{5}}{\sqrt{x+2}+\sqrt{5}} \right| + c$$

**Answer: (C)**

5. **Passage:** $\int f(g(x))\, g'(x)\, dx = \int f(t) dt$ where $t = g(x)$.
This rule is called substitution rule. Answer the following questions:

**(i)** If

$$\int \sec x \tan x \sqrt{\sec^2 x + 1}\, dx = \frac{1}{2} \text{Sinh}^{-1}(\sec x) + \frac{1}{2} f(x) + c$$

then $f(x)$ is

(A) $\tan x \sqrt{\sec^2 x + 1}$ (B) $\sec x \sqrt{\sec^2 x + 1}$

(C) $\sec x \sqrt{\tan x}$ (D) $\sec^2 x \sqrt{\tan x}$

**(ii)** $\int \frac{x^2-1}{x^2+1} \frac{dx}{\sqrt{1+x^4}} =$

(A) $\frac{1}{\sqrt{2}} \text{Cosh}^{-1} \frac{x\sqrt{2}}{1+x^2} + c$

(B) $\frac{1}{\sqrt{2}} \text{Sinh}^{-1} \frac{x\sqrt{2}}{1+x^2} + c$

(C) $\frac{1}{\sqrt{2}} \text{Tan}^{-1} \left( \frac{x\sqrt{2}}{1+x^2} \right) + c$

(D) $\frac{1}{\sqrt{2}} \text{Cos}^{-1} \frac{x\sqrt{2}}{1+x^2}$

**(iii)** If

$$\int \frac{dx}{(x+1)^3 \sqrt{x^2+2x-3}} = \frac{\sqrt{x^2+2x-3}}{8(x+1)^2} - \frac{1}{16} f(x) + c$$

then $f(x)$ is equal to

(A) $\text{Sin}^{-1}\left(\frac{2}{x+1}\right)$ (B) $\text{Cos}^{-1}\left(\frac{x+1}{2}\right)$

(C) $\text{Sin}^{-1}\left(\frac{x+1}{2}\right)$ (D) $\text{Sec}^{-1}\left(\frac{x+1}{2}\right)$

**Solution:**

**(i)** Let

$$I = \int \sec x \tan x \sqrt{\sec^2 x + 1}\, dx$$

Put $\sec x = t$ so that $\sec x \tan x\, dx = dt$. Therefore

$$I = \int \sqrt{t^2+1}\, dt$$

$$= \frac{t\sqrt{t^2+1}}{2} + \frac{1}{2} \text{Sinh}^{-1} t + c$$

$$= \frac{\sec x \sqrt{\sec^2 x + 1}}{2} + \frac{1}{2} \text{Sinh}^{-1}(\sec x) + c$$

So

$$f(x) = \sec x \sqrt{\sec^2 x + 1}$$

**Answer: (B)**

**(ii)** Let

$$I = \int \frac{x^2-1}{x^2+1} \frac{dx}{\sqrt{x^4+1}}$$

Dividing numerator and denominator with $x^2$ we get

$$I = \int \frac{1-(1/x^2)}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}} dx$$

$$= \int \frac{dt}{t\sqrt{t^2-2}} \quad \text{where } t = x + \frac{1}{x}$$

$$= \int \frac{t}{t^2\sqrt{t^2-2}} dt$$

Put $t^2 - 2 = z^2$ so that $tdt = zdz$. Therefore

$$I = \int \frac{1}{(z^2+2)\, z} (z)\, dz$$

$$= \frac{1}{\sqrt{2}} \text{Tan}^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \text{Tan}^{-1}\left(\frac{\sqrt{t^2-2}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \text{Tan}^{-1} \sqrt{\frac{x^2+(1/x^2)}{2}} + c$$

If

$$\text{Tan}^{-1} \sqrt{\frac{x^2+(1/x^2)}{2}} = \theta$$

then

$$\tan^2\theta = \frac{x^2+(1/x^2)}{2}$$

or $$\sec^2\theta = \frac{[x+(1/x)]^2}{2}$$

so that

$$\cos\theta = \frac{\sqrt{2}}{x+(1/x)} = \frac{x\sqrt{2}}{x^2+1}$$

Hence

$$\theta = \mathrm{Cos}^{-1}\frac{x\sqrt{2}}{x^2+1}$$

Therefore

$$I = \frac{1}{\sqrt{2}}\mathrm{Cos}^{-1}\left(\frac{x\sqrt{2}}{x^2+1}\right)+c$$

**Answer: (D)**

**(iii)** Let

$$I = \int \frac{dx}{(x+1)^3\sqrt{x^2+2x-3}}$$

Put $x+1 = 1/t$ so that

$$dx = -\frac{1}{t^2}dt$$

Therefore

$$I = \int \frac{t^3}{\sqrt{\frac{1}{t^2}-4}}\left(\frac{-1}{t^2}\right)dt$$

$$= -\int \frac{t^2}{\sqrt{1-4t^2}}$$

$$= \frac{1}{4}\int \frac{1-4t^2-1}{\sqrt{1-4t^2}}\,dt$$

$$= \frac{1}{4}\int \sqrt{1-4t^2}\,dt - \frac{1}{4}\int \frac{dt}{\sqrt{1-4t^2}}$$

$$= \frac{1}{4}\times 2\int \sqrt{\frac{1}{4}-t^2}\,dt - \frac{1}{8}\int \frac{dt}{\sqrt{(1/4)-t^2}}$$

$$= \frac{1}{2}\left[\frac{t\sqrt{(1/4)-t^2}}{2}+\frac{1}{8}\mathrm{Sin}^{-1}2t\right]-\frac{1}{8}\mathrm{Sin}^{-1}(2t)+c$$

$$= \frac{1}{8}\left[\left(\frac{1}{x+1}\right)\sqrt{1-\frac{4}{(x+1)^2}}\right]+\frac{1}{16}\mathrm{Sin}^{-1}(2t)-\frac{1}{8}\mathrm{Sin}^{-1}(2t)+c$$

$$= \frac{\sqrt{x^2+2x-3}}{8(x+1)^2}-\frac{1}{16}\mathrm{Sin}^{-1}(2t)+c$$

Therefore

$$f(x) = \mathrm{Sin}^{-1}\left(\frac{2}{x+1}\right)$$

**Answer: (A)**

## *Integer Answer Type Questions*

**1.** If $\int \frac{x^4}{\sqrt{4+x^5}}dx = \frac{p}{q}\sqrt{4+x^5}+c$, then $p+q =$ ______.

**Solution:** Let

$$I = \int \frac{x^4}{\sqrt{4+x^5}}\,dx$$

Put $4+x^5 = t^2$ so that

$$x^4dx = \frac{2}{5}tdt$$

Therefore

$$I = \frac{2}{5}\int dt$$

$$= \frac{2}{5}t+c$$

$$= \frac{2}{5}\sqrt{4+x^5}+c$$

So

$$\frac{p}{q} = \frac{2}{5} \Rightarrow p+q = 7$$

**Answer: 7**

**2.** $\int \frac{(\mathrm{Tan}^{-1}x)^2}{1+x^2}\,dx = \frac{1}{m}(\mathrm{Tan}^{-1}x)^n+c$

where $m+n$ is ______.

**Solution:** We have

$$\int \frac{(\mathrm{Tan}^{-1}x)^2}{1+x^2}dx = \int t^2dt \quad \text{where } t = \mathrm{Tan}^{-1}x$$

$$= \frac{1}{3}t^3+c$$

$$= \frac{1}{3}(\mathrm{Tan}^{-1}x)^3+c$$

So $m+n = 3+3 = 6$.

**Answer: 6**

**3.** If

$$\int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}}dx = \log_e\left(\frac{x^2+1+\sqrt{x^4+3x^2+1}}{x^k}\right)+c,$$

then the value of $k$ is ______.

**Solution:** Let

$$I=\int\frac{x^2-1}{x\sqrt{x^4+3x^2+1}}=\int\frac{1-\dfrac{1}{x^2}}{\sqrt{x^2+\dfrac{1}{x^2}+3}}dx$$

Put $x+(1/x)=t$. Therefore

$$I=\int\frac{dt}{\sqrt{t^2+1}}$$
$$=\text{Sinh}^{-1}t+c$$
$$=\log_e(t+\sqrt{t^2+1})+c$$
$$=\log_e\left(x+\frac{1}{x}\right)+\sqrt{\left(x+\frac{1}{x}\right)^2+1}+c$$
$$=\log_e\frac{(x^2+1+\sqrt{x^4+3x^2+1})}{x}+c$$

Therefore $k=1$.

**Answer: 1**

**4.** If
$$\int\frac{\sin 2x}{\sin 3x\sin 5x}dx=\frac{1}{p}\log_e|\sin 3x|-\frac{1}{q}\log_e|\sin 5x|+c$$
then $|p-q|$ is _____.

**Solution:** We have

$$\int\frac{\sin 2x}{\sin 3x\sin 5x}dx=\int\frac{\sin(5x-3x)}{\sin 3x\sin 5x}dx$$
$$=\int\frac{\sin 5x\cos 3x-\cos 5x\sin 3x}{\sin 3x\sin 5x}dx$$
$$=\int(\cot 3x-\cot 5x)dx$$
$$=\frac{1}{3}\log_e|\sin 3x|-\frac{1}{5}\log_e|\sin 5x|+c$$

Therefore

$$|p-q|=|3-5|=2$$

**Answer: 2**

**5.** If

$$\int\frac{\text{cosec}^2 x-7}{\cos^7 x}dx=-(\cot x)^{\alpha}(\sec x)^{\beta}+c$$

then the value of $\beta-\alpha$ is _____.

**Solution:** We have

$$\int\frac{\text{cosec}^2 x-7}{\cos^7 x}dx=\int\sec^7 x\,\text{cosec}^2 x\,dx-7\int\sec^7 x\,dx$$
$$=(-\cot x)\sec^7 x-\int(-\cot x)7\sec^6 x(\sec x\tan x)dx$$
$$-7\int\sec^7 x\,dx$$
$$=-\cot x\sec^7 x+7\int\sec^7 x\,dx-7\int\sec^7 x\,dx$$
$$=-\cot x\sec^7 x+c$$

Therefore

$$\beta-\alpha=7-1=6$$

**Answer: 6**

**6.** If

$$\int\frac{2x}{(1-x^2)\sqrt{x^4-1}}dx=\left(\frac{x^2+1}{x^2-1}\right)^k+c$$

then $1/k$ is equal to_____.

**Solution:** We have

$$I=\int\frac{-2x}{(x^2-1)^{3/2}\sqrt{x^2+1}}dx$$
$$=\int\frac{-2x}{(x^2-1)^2\sqrt{\dfrac{x^2+1}{x^2-1}}}dx$$

Put

$$\frac{x^2+1}{x^2-1}=t^2$$

Therefore

$$\frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}dx=2t\,dt$$
$$\Rightarrow\frac{-2x}{(x^2-1)^2}dx=t\,dt$$

So

$$I=\int\frac{1}{t}(t)\,dt$$
$$=t+c$$
$$=\sqrt{\frac{x^2+1}{x^2-1}}+c$$

Hence

$$k=\frac{1}{2}\quad\text{or}\quad\frac{1}{k}=2$$

**Answer: 2**

**7.** If

$$\int\frac{\cot^3 x\cos x}{(\sin^5 x+\cos^5 x)^{3/2}}dx=\frac{-1}{2}\frac{(1+\tan 5x)^{2/5}}{\tan^{\alpha} x}+c$$

then $\alpha$ is equal to _____.

**Solution:** Let

$$I=\int\frac{\cot^3 x\cos x}{(\sin^5 x+\cos^5 x)^{3/5}}dx$$

$$=\int\frac{\cos^4 x}{\sin^3 x(\sin^5 x+\cos^5 x)^{3/5}}dx$$

$$=\int\frac{\cos^4 x}{\sin^6 x(1+\cot^5 x)^{3/5}}dx$$

$$=\int\frac{\sec^2 x}{\tan^6 x(1+\cot^5 x)^{3/5}}dx$$

Put $\tan=t$. Then

$$I=\int\frac{1}{t^6[1+(1/t^5)]^{3/5}}dt$$

$$=\int\frac{1}{t^3(1+t^5)^{3/5}}dt$$

$$=\int t^{-3}(1+t^5)^{-3/5}dt$$

Here

$$\frac{m+1}{n}+p=\frac{-3+1}{5}-\frac{3}{5}=-1 \quad \text{(integer)}$$

Put $1+t^5=z^5t^5$ (Binomial differential). Then

$$t^5=\frac{1}{z^5-1}$$

$$1+t^5=z^5t^5=\frac{z^5}{z^5-1}$$

and

$$dt=\frac{-1}{5}(z^5-1)^{(-1/5)-1}(5z^4)dz$$

$$=-z^4(z^5-1)^{-6/5}dz$$

So

$$I=\int(z^5-1)^{3/5}\left(\frac{z^5}{z^5-1}\right)^{-3/5}(-z^4)(z^5-1)^{-6/5}dz$$

$$=-\int z\,dz$$

$$=-\frac{z^2}{2}+c$$

$$=\frac{-1}{2}\left(\frac{1+t^5}{t^5}\right)^{2/5}+c$$

$$=\frac{-1}{2}\frac{(1+\tan^5 x)^{2/5}}{\tan^2 x}+c$$

Therefore $\alpha=2$.

**Answer: 2**

**8.** If

$$\int\frac{x^2}{1+x^6}\mathrm{Tan}^{-1}(x^3)dx=\frac{1}{k}[\mathrm{Tan}^{-1}(x^m)]^n+c$$

then $mn-k$ is equal to _____.

**Solution:** Let

$$I=\int\frac{x^2}{1+x^6}\mathrm{Tan}^{-1}(x^3)dx$$

Put $\mathrm{Tan}^{-1}(x^3)=t$ so that

$$\frac{3x^2}{1+x^6}dx=dt$$

Therefore

$$I=\int\frac{t}{3}dt$$

$$=\frac{1}{6}t^2+c$$

$$=\frac{1}{6}(\mathrm{Tan}^{-1}x^3)^2+c$$

So

$$mn-k=6-6=0$$

**Answer: 0**

**9.** If

$$\int\frac{e^x}{e^{2x}+6e^x+5}dx=\frac{1}{4}\log_e\left(\frac{e^x+a}{e^x+b}\right)+c$$

then $|a-b|$ is _____.

**Solution:** We have

$$I=\int\frac{e^x}{e^{2x}+6e^x+5}dx$$

$$=\int\frac{dt}{(t+1)(t+5)} \quad \text{where } t=e^x$$

$$=\frac{1}{4}\int\left(\frac{1}{t+1}-\frac{1}{t+5}\right)dt$$

$$=\frac{1}{4}\log_e\left(\frac{e^x+1}{e^x+5}\right)+c$$

Therefore

$$|a-b|=4$$

**Answer: 4**

**10.** If

$$\int\frac{2x}{(x^2+1)(x^2+2)}dx=\log_e\left(\frac{x^2+a}{x^2+b}\right)+c$$

then $b-a$ is _____.

**Solution:** Let

$$\int \frac{2x}{(x^2+1)(x^2+2)}dx = \int \frac{dt}{(t+1)(t+2)} \quad \text{where } t = x^2$$

$$= \int \left( \frac{1}{t+1} - \frac{1}{t+2} \right) dt$$

$$= \log_e \left( \frac{x^2+1}{x^2+2} \right)$$

Therefore

$$b - a = 2 - 1 = 1$$

**Answer: 1**

## EXERCISES

To have a grip over integration, the student has to practice problems on various methods. That is why we are supplying the student with a number of problems for evaluation. Hence, the exercise contains only subjective problems.

**1.** Evaluate $\int \frac{xe^x}{(1+x)^2} dx$.

**2.** Evaluate $\int \frac{xe^x}{\sqrt{1+e^x}} dx$

**Hint:** Take $u = x, dv = \frac{e^x}{\sqrt{1+e^x}}$ so that $v = 2\sqrt{1+e^x}$ and use integration by parts.

**3.** Show that

$$\int \frac{dx}{\sqrt{1+e^x+e^{2x}}} = \log_e \left( \frac{-1+e^x+\sqrt{1+e^x+e^{2x}}}{1+e^x+\sqrt{1+e^x+e^{2x}}} \right) + c$$

**4.** Show that

$$\int \sqrt{2+\tan^2 x}\, dx = \mathrm{Tan}^{-1}\left( \frac{\tan x}{\sqrt{2+\tan^2 x}} \right) + \log_e(\tan x + \sqrt{2+\tan^2 x}) + c$$

**Hint:** Put $t = \tan x$.

**5.** Evaluate $\int (x^3 - 2x^2 + 5)e^{3x} dx$.

**6.** Evaluate $\int \frac{\log_e x - 1}{(\log_e x)^2} dx$.

**7.** Show that

$$\int \frac{x \log x}{(x^2-1)^{3/2}} dx = \mathrm{Tan}^{-1}\sqrt{x^2-1} - \frac{\log_e x}{\sqrt{x^2-1}} + c$$

**8.** Prove that

$$\int \frac{\sqrt{1+x^2}}{2+x^2} dx = \frac{1}{2\sqrt{2}} \log_e \left( \frac{\sqrt{2+2x^2}-x}{\sqrt{2+2x^2}+x} \right) + \log_e(x+\sqrt{x^2+1}) + c$$

**9.** Show that

$$\int x^3 \mathrm{Tan}^{-1} x\, dx = \frac{x^4-1}{4}\mathrm{Tan}^{-1} x - \frac{x^3}{12} + \frac{x}{4} + c$$

**Hint:** Use integration by parts.

**10.** Evaluate $\int \frac{\mathrm{Sin}^{-1} x}{\sqrt{1+x}} dx$.

**11.** Show that

$$\int \frac{\sqrt{x^2+1}}{x^4}[\log_e(x^2+1) - 2\log_e x]dx = \frac{(x^2+1)^{3/2}}{9x^3}\left[ 2 - 3\log_e\left(1+\frac{1}{x^2}\right) \right] + c$$

**Hint:** Put $1 + \frac{1}{x^2} = t$.

**12.** Compute $\int \sin x \log_e \tan x\, dx$

**13.** Evaluate $\int \frac{x^3+3x^2+5x}{1+x^2} dx$.

**14.** Evaluate $\int \frac{x^2+1}{\sqrt[3]{x^3+3x+1}} dx$.

**15.** Evaluate $\int \frac{\log_e x}{x\sqrt{1+\log_e x}} dx$.

**16.** Show that

$$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{1}{ab}\mathrm{Tan}^{-1}\left( \frac{a}{b}\tan x \right) + c$$

**Hint:** Divide numerator and denominator with $\cos^2 x$ and put $t = \tan x$.

**17.** Compute $\int \frac{x^2-1}{(x^4+3x^2+1)\mathrm{Tan}^{-1}\left(x+\frac{1}{x}\right)} dx$.

**Hint:** Divide numerator and denominator with $x^2$ and put $t = \mathrm{Tan}^{-1}\left(x + \frac{1}{x}\right)$.

**18.** Evaluate $\int \frac{dx}{x^2-6x+13}$.

**19.** Show that

$$\int (x-1)x^{-2/3}dx = \frac{3}{4}(x-4)x^{1/3}+c$$

**20.** Show that

$$\int \frac{dx}{(x+1)\sqrt{x^2-1}} = \sqrt{\frac{x-1}{x+1}}+c$$

**21.** Prove that

$$\int \frac{dx}{(x^2-4)\sqrt{x+1}} = \frac{1}{4\sqrt{3}}\log_e\left(\frac{u-\sqrt{3}}{u+\sqrt{3}}\right)-\frac{1}{2}\mathrm{Tan}^{-1}u$$

where $u=\sqrt{x+1}$.

**22.** Show that

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}dx = \frac{-2}{x}(\sqrt{1+x+x^2}-1)+$$
$$\log_e\left|2x+1+2\sqrt{1+x+x^2}\right|+c$$

**Hint:** Use Euler's substitution $\sqrt{1+x+x^2}=tx+1$ and carefully simplify.

**23.** Evaluate $\int \frac{dx}{\sqrt{x^2+3x-4}}$.

***Note:*** To evaluate this integral, you can use Euler's substitution

$$\sqrt{(x+4)(x-1)}=(x+4)t$$

But it is easy to see that the given integral is

$$\int \frac{dx}{\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{25}{16}}}$$

which is a standard integral.

**24.** Compute $\int \frac{\sin^3 x}{2+\cos x}dx$.

**25.** Evaluate $\int \frac{\cos^3 x}{\sin^4 x}dx$.

**26.** Evaluate $\int \frac{dx}{\sqrt{5-7x-3x^2}}$.

**27.** Show that

$$\int \frac{dx}{\sqrt{2-3x-x^2}} = \mathrm{Sin}^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$$

**28.** Show that

$$\int \frac{7x+1}{6x^2+x-1}dx = \frac{2}{3}\log_e(3x-1)+\frac{1}{2}\log_e(2x+1)+c$$

**29.** Compute $I=\int \frac{dx}{1+\sin^4 x}$.

**Hint:** Divide numerator and denominator with $\cos^4 x$ and put $t=\tan x$. Then

$$I=\int \frac{t^2+1}{2t^4+2t^2+1}dt$$
$$=\int \frac{1+\frac{1}{t^2}}{2t^2+\frac{1}{t^2}+2}dt$$

Now write

$$1+\frac{1}{t^2}=a\left(\sqrt{2}+\frac{1}{t^2}\right)+b\left(\sqrt{2}-\frac{1}{t^2}\right)$$

Find $a, b$ and then use the substitutions

$$\sqrt{2}t-\frac{1}{t}=u \quad \text{and} \quad \sqrt{2}t+\frac{1}{t}=z$$

***Note:*** This is a very interesting case and the one who attempts the problem is generally considered to be a good student. Do not mind the answer. The procedure is required.

**30.** Show that

$$\int \frac{dx}{\sqrt{3x^2+5x}} = \frac{1}{\sqrt{3}}\log_e\left|6x+5+\sqrt{12x(3x+5)}\right|+c$$

**31.** Show that

$$\int \frac{dx}{x\sqrt{x^2+4x-4}} = \frac{1}{2}\mathrm{Sin}^{-1}\left(\frac{x-2}{x\sqrt{2}}\right)+c$$

**Hint:** Put $x=1/t$.

**32.** Compute $\int (2x-x^2)^{-3/2}dx$.

**Hint:** Put $2-x=t^2x$.

**33.** Evaluate $\int \frac{\sqrt{x^2+2x}}{x}dx$.

**34.** Show that

$$\int \sqrt{2x-x^2}\,dx = \frac{(x-1)}{2}\sqrt{2x-x^2}+\frac{1}{2}\mathrm{Sin}^{-1}(x-1)+c$$

**35.** Show that

$$\int \frac{dx}{4-5\sin x} = \frac{1}{3}\log_e\left|\frac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}\right|+c$$

**Hint:** Put $t = \tan(x/2)$.

**36.** Evaluate $\int \dfrac{x^2 \mathrm{Tan}^{-1} x}{1+x^2} dx$.

**37.** Compute $\int x^3 (\log_e x)^2 dx$.

**38.** Show that

$$\int \frac{\sqrt{x^6+1}}{x^{10}} \left[\log_e\left(\frac{x^6+1}{x^6}\right)\right] dx = \frac{-1}{6}\left[\frac{2}{3}t^{3/2}\log_e t - \frac{4}{9}t^{3/2}\right] + c$$

where $t = \dfrac{x^6+1}{x^6}$.

**39.** Evaluate $\int \dfrac{\log_e x}{(1+\log_e x)^2} dx$.

**Hint:** Put $\log_e x = t$

**40.** Evaluate $\int \dfrac{\sqrt{\cos 2x}}{\sin x} dx$.

**Hint:** Write $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ and put $t^2 = 1-\tan^2 x$

**41.** Evaluate $\int \dfrac{dx}{(x^2+1)^2}$.

**Hint:** Use integration by parts or put the substitution $x = \tan\theta$.

**42.** Evaluate $\int \dfrac{x^4}{(x^2+1)^2} dx$.

**Hint:** We have

$$\frac{x^4}{(x^2+1)^2} = \frac{x^4-1+1}{(x^2+1)^2} = \frac{x^2-1}{x^2+1} + \frac{1}{(x^2+1)^2} = 1 - \frac{2}{x^2+1} + \frac{1}{(x^2+1)^2}$$

**43.** Compute $\int \dfrac{x^2+x+1}{(x^2+1)^2} dx$.

**44.** Evaluate $\int \dfrac{1+x\sin x+\cos x}{x(1+\cos x)} dx$.

**45.** Compute $\int \dfrac{\cos x + x\sin x}{x(x+\cos x)} dx$.

**46.** Evaluate $\int \dfrac{x^2+1}{x^3+1} dx$.

**Hint:** $\dfrac{x^2+1}{x^3+1} = \dfrac{A}{x+1} + \dfrac{Bx+c}{x^2-x+1}$

**47.** Show that

(a) $\displaystyle\int \frac{x^2+1}{x^4-x^2+1} dx = \mathrm{Tan}^{-1}\left(\frac{x^2-1}{x}\right) + c$

(b) $\displaystyle\int \frac{x^2-1}{x^4-x^2+1} dx = \frac{1}{2\sqrt{3}} \log_e \left|\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right| + c$

**Hint:** For (a), divide numerator and denominator by $x^2$ and put $x-(1/x)=t$.

For (b), put $x+(1/x)=t$.

**48.** Show that

$$\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}} dx = \frac{\sqrt{x^4+x^2+1}}{x} + c$$

**Hint:** Divide numerator and denominator by $x^3$ and put substitution $x^2 + \dfrac{1}{x^2} + 1 = t^2$.

**49.** Evaluate $\int \dfrac{1-x^2}{1+x^2} \dfrac{dx}{\sqrt{x^4+1}}$.

**50.** Compute $\int \dfrac{1-x^2}{1+x^2} \dfrac{dx}{\sqrt{1+x^2+x^4}}$.

## ANSWERS

**1.** $\dfrac{e^x}{x+1} + c$

**2.** $2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\log_e\left(\dfrac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + c$

**5.** $e^{3x}\left(\dfrac{1}{3}x^3 - x^2 + \dfrac{2}{3}x + \dfrac{13}{9}\right) + c$

**6.** $\dfrac{x}{\log_e x} + c$

**10.** $2\sqrt{1+x}\,\mathrm{Sin}^{-1}x+4\sqrt{1-x}+c$

**12.** $-\cos x\log_e(\tan x)+\log_e\left|\tan\dfrac{x}{2}\right|+c$

**13.** $\dfrac{x^2}{2}+3x+2\log_e(x^2+1)-3\mathrm{Tan}^{-1}x+c$

**14.** $\dfrac{1}{2}(x^3+3x+1)^{2/3}+c$

**15.** $\dfrac{2}{3}(\log_e x-2)\sqrt{1+\log_e x}+c$

**17.** $\log_e\left|\mathrm{Tan}^{-1}\left(x+\dfrac{1}{x}\right)\right|+c$

**18.** $\dfrac{1}{2}\mathrm{Tan}^{-1}\left(\dfrac{x-3}{2}\right)+c$

**23.** $\log_e\left|\dfrac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right|+c$

**24.** $\dfrac{\cos^2 x}{2}-2\cos x+3\log_e(2+\cos x)+c$

**25.** $\dfrac{-1}{3\sin^3 x}+\dfrac{1}{\sin x}+c$

**26.** $\dfrac{1}{\sqrt{3}}\mathrm{Sin}^{-1}\left(\dfrac{6x+7}{\sqrt{109}}\right)+c$

**32.** $\dfrac{x-1}{\sqrt{2x-x^2}}+c$

**33.** $\sqrt{x^2+2x}+\log_e\left|x+1+\sqrt{x^2+2x}\right|+c$

**36.** $x\,\mathrm{Tan}^{-1}x-\dfrac{1}{2}\log_e(1+x^2)-\dfrac{1}{2}(\mathrm{Tan}^{-1}x)^2+c$

**37.** $\dfrac{x^4}{4}(\log_e x)^2-\dfrac{x^4}{8}\log_e x+\dfrac{x^4}{32}+c$

**39.** $\dfrac{x}{1+\log_e x}+c$

**40.** $-\dfrac{1}{2}\log_e\left|\dfrac{1+t}{1-t}\right|+\dfrac{1}{\sqrt{2}}\log_e\left|\dfrac{\sqrt{2}+t}{\sqrt{2}-t}\right|$ where $t=\sqrt{1-\tan^2 x}$

**41.** $\dfrac{x}{2(x^2+1)}+\dfrac{1}{2}\mathrm{Tan}^{-1}x+c$

**42.** $x-\dfrac{3}{2}\mathrm{Tan}^{-1}x+\dfrac{x}{2(1+x^2)}+c$

**43.** $\mathrm{Tan}^{-1}x-\dfrac{1}{2(x^2+1)}$

**44.** $\log_e|x|+2\log_e\left|\sec\dfrac{x}{2}\right|+c$

**45.** $\log_e\left|\dfrac{x}{x+\cos x}\right|+c$

**46.**
$\dfrac{2}{3}\log_e|x+1|+\dfrac{1}{6}\log_e(x^2-x+1)+\dfrac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right)+c$

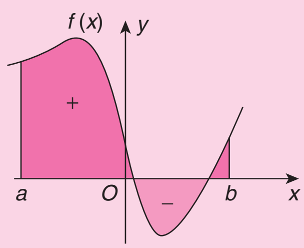
**49.** $-\dfrac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\dfrac{\sqrt{x^4+1}}{x\sqrt{2}}+c$

**50.** $\mathrm{Sin}^{-1}\dfrac{x}{1+x^2}+c$

# Definite Integral, Areas and Differential Equations

# 5

Definite Integral, Areas and Differential Equations




Contents



A **differential equation** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.
**Integration** is an important concept in mathematics and, together with its inverse, **differentiation**, is one of the two main operations in calculus.

## 5.1 | Definite Integral

To find the area of a region bounded by an irregular curve, we subdivide the region into rectangles of small width; the sum of the areas of these rectangles gives an approximate value to the area of the given region. As we reduce the width of the rectangle, the sum of the areas gives a good approximation to the area of the region. This idea led to the definition of definite integral by Bernard Riemann (1826–1866), a nineteenth century mathematician. In this chapter, we state the theorems (without proofs) and provide many problems to improve the working skills of the student.

**DEFINITION 5.1** Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function. Suppose

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

is a partition of $[a, b]$. Suppose $t_i \in [x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Then

$$S(f, P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

(it is customary to write $\Delta x_i$ for $x_i - x_{i-1}$) is called a Riemann intermediate sum of $f$ associated with the partition $P$. If $S(f, P)$ converges to a limit $\alpha$ as $\max\{\Delta x_i, i = 1, 2, \ldots, n\}$ tends to zero then we say that $f$ is Riemann integrable, or simply integrable, over $[a, b]$ and $\alpha$ is the (Riemann) integral of $f$ over $[a, b]$. We denote $\alpha$ by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx$$

Here $x$ is only a dummy variable and can be replaced by any other variable. For example

$$\int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(y)\,dy$$

In this case, $f$ is called the *integrand* and $\int_a^b f(x)\,dx$ is called the (definite) integral of $f$ over $[a, b]$.

**THEOREM 5.1** If $f : [a, b] \to \mathbb{R}$ is an integrable function, then $f$ is bounded.

**DEFINITION 5.2** Suppose $f : [a, b] \to \mathbb{R}$ is integrable so that $f$ is bounded (by Theorem 5.1). Let

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

be a partition of $[a, b]$. Write

$$M_i = \text{l.u.b}\,\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

and

$$m_i = \text{g.l.b.}\,\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

Then

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

is called the (Riemann) *lower sum of f over* $[a, b]$ and

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

is called the (Riemann) *upper sum of f over* $[a, b]$.

It may be noted that $L(f, P) \le S(f, P) \le U(f, P)$.

From the definition of integrability, the following can be easily proved.

**THEOREM 5.2** Suppose $f$ and $g$ are defined and Riemann integrable over $[a, b]$. Then

**(i)** $\lambda \in \mathbb{R} \Rightarrow \lambda f$ is integrable over $[a, b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$.

**(ii)** $f + g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

**(iii)** $f(x) \le g(x)\ \forall\ x \in [a, b] \Rightarrow \int_a^b f \le \int_a^b g$.

***Note:*** If $f_1, f_2, \ldots, f_n$ are integrable over $[a, b]$, then from (ii) of Theorem 5.2, it follows that $f_1 + f_2 + \cdots + f_n$ is integrable over $[a, b]$ and

$$\int_a^b (f_1 + f_2 + \cdots + f_n) = \sum_{i=1}^{n} \int_a^b f_i$$

## Examples

1. The constant function 1 is integrable over $[a, b]$ and

$$\int_a^b 1 = (b-a)(S(f, P)) = b - a$$

for every partition $P$ of $[a, b]$ and hence $S(f, p) \to b - a$ as $\text{Max}\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} \to 0$.

2. Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then $f$ is not integrable over $[a, b]$. This is because

$$S(f, P) = \begin{cases} 0 & \text{if we take } t_i \in [x_{i-1}, x_i] \text{ as irrational} \\ b-a & \text{if we take } t_i \in [x_{i-1}, x_i] \text{ as rational} \end{cases}$$

so that $S(f, P)$ does not converge as $\text{Max}\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ tends to zero.

3. The identify function $f : [a, b] \to \mathbb{R}$ given by $f(x) = x$ for all $x \in [a, b]$ is integrable over $[a, b]$ and

$$\int_a^b f = \frac{b^2 - a^2}{2}$$

***Proof:*** Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n\}$ be a partition of $[a, b]$ and $t_i \in [x_{i-1}, x_i]$ so that

$$S(f, P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} t_i(x_i - x_{i-1})$$

Now

$$\begin{aligned} \frac{b^2 - a^2}{2} &= \left(\frac{b+a}{2}\right)(b-a) \\ &= \left(\frac{x_n + x_0}{2}\right)(x_n - x_0) \\ &= \frac{x_n^2 - x_0^2}{2} \end{aligned}$$

$$= \sum_{i=1}^{n} \frac{(x_i^2 - x_{i-1}^2)}{2}$$

$$= \sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2}(x_i - x_{i-1})$$

Hence

$$\left| S(f, P) - \frac{b^2 - a^2}{2} \right| = \left| \sum_{i=1}^{n} \left( t_i - \frac{x_i + x_{i-1}}{2}(x_i - x_{i-1}) \right) \right|$$

$$\leq \left| \sum_{i=1}^{n} \frac{\in}{2}(x_i - x_{i-1}) \right| \quad \text{if} \quad \Delta x_i < \frac{\in}{2}$$

$$= \frac{\in}{2}(b - a)$$

Hence

$$S(f, P) \to \frac{b^2 - a^2}{2}$$

## 5.1.1 Geometrical Interpretation of the Definite Integral

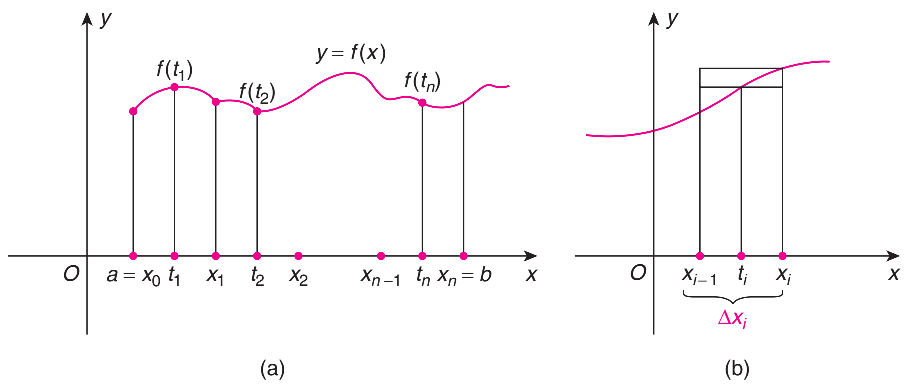



**FIGURE 5.1**

Let $f: [a, b] \to [0, \infty)$ be a function, $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then (see Fig. 5.1)

$$f(t_i)\,\Delta x_i = f(t_i)(x_i - x_{i-1})$$

$$= \text{Area of the rectangle with width } \Delta x_i \text{ and height } f(t_i)$$

Hence

$$S(f, P) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

$$= \text{Sum of the areas of the rectangles with width } \Delta x_i = x_i - x_{i-1} \text{ and height } f(t_i)$$

Thus, the area $A$ enclosed by the $x$-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$ is approximately equal to $S(f, P)$. When the width of the rectangles becomes smaller, that is when Max $\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ is small, the sum of the areas or $S(f, P)$ is very nearly equal to $A$. If $f$ is integrable, then $S(f, P)$ converges to $\int_a^b f(x)\,dx$ and hence

$$A = \int_a^b f(x)\,dx$$

Thus, definite integral of a non-negative function $f$, when integrable, maybe interpreted over $[a, b]$ as the area enclosed by the curve $y = f(x)$, the lines $x = a$, $x = b$ and the $x$-axis.

As already mentioned in the beginning of this section, we are going to state some theorems (without proofs) whose validity, we assume.

**THEOREM 5.3** Suppose $a \le c < d \le b$ and $f:[a,b] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [c,d] \\ 0 & \text{otherwise} \end{cases}$$

Then $f$ is integrable over $[a,b]$ and

$$\int_a^b f = \int_c^d 1 = d - c$$

From Theorem 5.3 it follows that every step function $f$ on $[a,b]$ is integrable over $[a,b]$. In fact we have the following theorem.

**THEOREM 5.4** Suppose $f:[a,b] \to \mathbb{R}$ is a step function. Hence, $f$ can be written as

$$f = \sum_{i=1}^{m} \lambda_i x_{J_i}$$

where $J_i$ is a sub-interval of $[a,b]$ and $J_1, J_2, \ldots, J_m$ are disjoint. Then $f$ is integrable and

$$\begin{aligned}\int_a^b f &= \sum_{i=1}^{m} \lambda_i \ell(J_i) \\ &= \sum_{i=1}^{m} \lambda_i (d_i - c_i) \quad \text{if } J_i = [c_i, d_i]\end{aligned}$$

where $\ell(J_i)$ is the length of the interval $J_i$.

***Note:*** In part (2) of the example given after Theorem 5.2, the function is not a step function, even though it has two values. That is,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is not a step function.

**THEOREM 5.5** Suppose $a < c < b$ and $f$ is integrable over $[a,c]$ and $[c,b]$. Then $f$ is integrable over $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

In fact, if $a_0 < a_1 < a_2 < \cdots < a_n$ and $f$ is integrable over each of $[a_{i-1}, a_i]$ $(i = 1, 2, 3, \ldots, n)$, then $f$ is integrable over $[a_0, a_n]$ and

$$\int_{a_0}^{a_n} f = \int_{a_0}^{a_1} f + \int_{a_1}^{a_2} f + \cdots + \int_{a_{n-1}}^{a_n} f$$

**THEOREM 5.6** Suppose $f:[a,b]\to\mathbb{R}$ is integrable and $\lambda_1, \lambda_2, \ldots, \lambda_n \in (a, b)$. Suppose $g(x) = f(x)$ for $x \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. Then $g$ is integrable over $[a, b]$ and

$$\int_a^b g = \int_a^b f$$

In other words, if two functions $f$ and $g$ differ only at a finite number of points in $[a, b]$ and if one of them is integrable, then the other function is also integrable and will have the same integral.

The following theorem is *useful to decide the integrability* of a function.

**THEOREM 5.7 (RIEMANN INTEGRABILITY CRITERION)** $f:[a,b]\to\mathbb{R}$ is integrable over $[a,b]$ if and only if to each $\in > 0$, there exists a partition $P$ of $[a,b]$ such that

$$|U(f,P)-L(f,P)| < \in$$

A good application of Riemann integrable criterion is the following.

**THEOREM 5.8** Every continuous function on $[a, b]$ is integrable over $[a, b]$.

Theorem 5.8 is very important, application wise, since many of the functions we come across are continuous.

**QUICK LOOK 1**

1. Every polynomial function being continuous for all real $x$, is integrable over any interval $[a, b]$.
2. The functions $\sin x$ and $\cos x$ being continuous on any interval $[a, b]$, are integrable over $[a, b]$.
3. The function $\log x$ is continuous on any closed interval $[a, b]$, $0 < a < b$. Hence, $\log x$ is integrable over $[a, b]$.
4. Since $e^x$ is continuous for all real $x$, $e^x$ is integrable over any closed interval $[a, b]$.

***Note:***

1. Even though Theorem 5.8 is useful application wise, it *suffers from one drawback*, namely it does not give the value of the integral $\int_a^b f$.
2. Theorem 5.8 remains true even if $f$ is discontinuous at a finite number of points in $(a, b)$.

**THEOREM 5.9**
**(i)** Every increasing function on $[a, b]$ is integrable.
**(ii)** Every decreasing function on $[a, b]$ is integrable over $[a, b]$.

***Note:*** Here again, the integrability of a monotonic function $f$ is guaranteed, but does not specify the value of $\int_a^b f$.

**THEOREM 5.10 (ADDITION THEOREM)** Suppose $f:[a,b]\to\mathbb{R}$ is a function and $a < c < b$. Then $f$ is integrable over $[a, b]$ if and only if $f$ is integrable over $[a, c]$ and $[c, b]$. In this case,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

**COROLLARY 5.1** If $a \le c < d \le b$ and $f$ is integrable over $[a, b]$, then $f$ is integrable over $[c, d]$.

**DEFINITION 5.3** Suppose $a < b$ and $f$ is integrable over $[a, b]$. Then, we define

$$\int_b^a f = -\int_a^b f \quad \text{and} \quad \int_a^a f = 0$$

As a consequence of this definition and earlier results, we have the following theorem.

**THEOREM 5.11** Suppose $f$ is integrable over $[a, b]$ and $\alpha, \beta, \gamma \in [a, b]$. Then

$$\int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f = \int_\alpha^\alpha f = 0$$

***Note:***

**1.** Suppose $f:[0, 1] \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} 1/x & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $f$ is continuous on $[c, 1]$ for every $c \in (0, 1]$ and hence it is integrable over $(c, 1]$ if $0 < c \le 1$. But $f$ is *not* integrable over $[0, 1]$, since it is not bounded on $[0, 1]$.

**2.** Define $f:[0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

This function $f$ is bounded on $[0, 1]$ and is continuous on $[0, 1]$ except at $x = 0$ and hence it is integrable over $[0, 1]$.

The following theorem is called *mean value theorem* for integrals.

**THEOREM 5.12 (MEAN VALUE THEOREM FOR INTEGRALS)** Suppose $a < b$ and $f$ be continuous on $[a, b]$. Then there exists $c \in [a, b]$ such that

$$\int_a^b f = f(c)\,(b-a)$$

The following theorem is very important in view of its applicability. This theorem is useful in evaluating definite integrals. It is also known as *Newton–Leibnitz theorem*.

**THEOREM 5.13 (FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS)** Suppose

**(i)** $f:[a, b] \to \mathbb{R}$ is integrable.

**(ii)** $F:[a,b] \to \mathbb{R}$ is continuous.

**(iii)** $F'(x) = f(x)$ in $(a, b)$.

Then

$$\int_a^b f = F(b) - F(a)$$

***Note:***

1. It is customary to denote $F(b) - F(a)$ by $[F(x)]_a^b$.
2. Theorem 5.13 can also be stated as follows: If $F(x)$ is an *antiderivative* of $f(x)$ on $[a, b]$, then

$$\int_a^b f = F(b) - F(a)$$

3. $\int_a^b F' = F(b) - F(a)$ if $F'$ is integrable.
4. Theorem 5.13 remains true if $F'(x) = f(x)$ on $[a, b]$ except at a finite number of points in $[a, b]$.

Using Theorems 5.8 and 5.13 we can prove the following examples.

## Examples

1. $\dfrac{x^2}{2}$ is an antiderivative of $x$ on $[a, b]$ so that

$$\int_a^b x\,dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2}$$

2. $\dfrac{x^{n+1}}{n+1}(n \neq -1)$ is an antiderivation of $x^n$ so that

$$\int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{1}{n+1}(b^{n+1} - a^{n+1})$$

where $a > 0, b > 0$.

3. Antiderivative of $e^x$ is $e^x$. This implies

$$\int_a^b e^x\,dx = e^b - e^a$$

4. $\sin x$ is an antiderivative of $\cos x$. This implies

$$\int_a^b \cos x\,dx = \sin b - \sin a$$

5. Let $\operatorname{sign} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

Now $|x|$ is an antiderivative of sign $x$ except at $x = 0$ and sign $x$ is a step function and hence is integrable. Therefore, from Theorem 5.13 and Note (4) under it, we have

$$\int_{-2}^{2} \operatorname{sign} x\,dx = [|x|]_{-2}^{2} = 2 - 2 = 0$$

6. Suppose $F(x) = \sqrt{x}$ on $[0, 1]$ so that

$$F'(x) = \frac{1}{2\sqrt{x}} \text{ on } (0,\ 1]$$

Let

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $F'(x) = f(x)$ on $(0, 1)$ and $f$ is not integrable over $[0, 1]$ since it is not bounded. Hence, Fundamental Theorem of Integral Calculus (Theorem 5.13) is not applicable in this situation.

**DEFINITION 5.4** Suppose $f : [a,b] \to \mathbb{R}$ is integrable and $x \in [a, b]$. Then $f$ is integrable over $[a, x]$. We define

$$F(x) = \int_a^x f \quad \forall\ x \in [a,b]$$

$F(x)$ is called the indefinite integral of $f$ over $[a, b]$.

***Note:*** We may choose any $c \in [a, b]$ and define $G(x) = \int_c^x f$ which is also an indefinite integral. Also $F$ and $G$ differ by a constant.

We have the following theorem about the indefinite integral.

**THEOREM 5.14** Suppose $f:[a,b]\to\mathbb{R}$ is integrable. Then the indefinite integral $F(x)=\int_a^x f$ is continuous over $[a,b]$.

**THEOREM 5.15 (FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS – SECOND FORM)** If $f$ is continuous on $[a,b]$, then

**(i)** $F(x)=\int_a^x f(t)\,dt$ is differentiable.

**(ii)** $F'(x)=f(x)\ \forall\ x\in[a,b]$.

***Note:*** This theorem gives the impression that the indefinite integral $F(x)$ is an antiderivative of $f(x)$. But this is not true in view of the following example.

**Example**

$F(x)=\int_{-1}^{x}\operatorname{sign} t\,dt=|x|-1$ is the indefinite integral of the signum function "sign $x$" on $[-1, 1]$. But $F(x)$ is not differentiable at "0" and hence is not an antiderivative of sign $x$ on $[-1, 1]$.

As an application of the Fundamental Theorem of Integral Calculus – Second Form, we have the following theorem known as substitution theorem which is useful in the evaluation process of definite integrals.

**THEOREM 5.16 (SUBSTITUTION THEOREM)** Suppose

**(i)** $g:[a,b]\to\mathbb{R}$ is continuously differentiable.

**(ii)** $g([a,b])\subset[c,d]$.

**(iii)** $f:[c,d]\to\mathbb{R}$ is continuous.

Then $(f\circ g)\cdot g'$ is integrable over $[a,b]$ and

$$\int_a^b (f\circ g)g'=\int_{g(a)}^{g(b)} f$$

That is,

$$\int_a^b f(g(t))g'(t)\,dt=\int_{g(a)}^{g(b)} f(x)\,dx$$

As a consequence of Theorem 5.16, we have the following properties of definite integral listed below as $P_1, P_2, \ldots, P_6$.

## 5.1.2 Some Properties of the Definite Integral

**$P_1$:** If $f$ is integrable over $[0, a]$, then

$$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

***Proof:*** Write $g(x) = a - x$. Then

$$\int_0^a f(a-x)\,dx = -\int_a^0 f(a-x)\,dx$$
$$= \int_a^0 f(g(x))g'(x) \quad (\because g'(x) = -1)$$
$$= \int_{g(a)}^{g(0)} f(x)\,dx$$
$$= \int_0^a f(x)\,dx$$

## Example 5.1

Evaluate $\int_0^1 x(1-x)^9\,dx$.

***Solution:*** Let

$$I = \int_0^1 x(1-x)^9\,dx$$
$$= \int_0^1 (1-x)[1-(1-x)]^9$$
$$= \int_0^1 (1-x)x^9$$
$$= \int_0^1 (x^9 - x^{10})\,dx$$
$$= \frac{1}{10}[x^{10}]_0^1 - \frac{1}{11}[x^{11}]_0^1$$
$$= \frac{1}{10} - \frac{1}{11}$$
$$= \frac{1}{110}$$

## Example 5.2

Evaluate $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$ where $n$ is rational.

***Solution:*** Let

$$I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$$
$$= \int_0^{\pi/2} \frac{\sin^n(\frac{\pi}{2} - x)}{\sin^n(\frac{\pi}{2} - x) + \cos^n(\frac{\pi}{2} - x)}\,dx$$
$$= \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx$$

Therefore

$$2I = \int_0^{\pi/2} \left( \frac{\sin^n x}{\sin^n x + \cos^n x} + \frac{\cos^n x}{\cos^n x + \sin^n x} \right) dx$$
$$= \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

This implies

$$I = \frac{\pi}{4}$$

## Example 5.3

Show that

$$\int_0^{\pi} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi/2} f(\sin x)\,dx$$

***Solution:*** Let

$$\begin{aligned} I &= \int_0^{\pi} x\,f(\sin x)\,dx \\ &= \int_0^{\pi} (\pi - x)\,f[\sin(\pi - x)]\,dx \\ &= \pi\int_0^{\pi} f(\sin x)\,dx - I \end{aligned}$$

Therefore

$$2I = \pi\int_0^{\pi} f(\sin x)\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/2} f(\sin x)\,dx$$

**$P_2$:** If $f$ is integrable over $[0, 2a]$ then

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$$

and further

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^{a} f(x)\,dx & \text{if } f(2a - x) = f(x) \\ 0 & \text{if } f(2a - x) = -f(x) \end{cases}$$

***Proof:*** Let

$$\begin{aligned} I &= \int_0^{2a} f(x)\,dx \\ &= \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx \quad \text{(By Theorem 5.10)} \\ &= \int_0^{a} f(x)\,dx + \int_a^{0} f(2a - t)(-1)\,dt \quad \text{where } t = 2a - x \\ &= \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - t)\,dt \\ &= \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx \\ &= \begin{cases} 2\displaystyle\int_0^{a} f(x)\,dx & \text{if } f(2a - x) = f(x) \\ 0 & \text{if } f(2a - x) = -f(x) \end{cases} \end{aligned}$$

## Example 5.4

Evaluate $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx.$

***Solution:*** Let

$$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx - I$$

Therefore

$$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}$$

$$= \pi\int_0^{2\pi/2} \frac{\sin x}{1+\cos^2 x}dx$$

$$= 2\pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx \quad \left(\because \frac{\sin(2\frac{\pi}{2}-x)}{1+\cos^2(2\frac{\pi}{2}-x)} = \frac{\sin x}{1+\cos^2 x}\right)$$

$$= 2\pi\int_1^{0} \frac{1}{1+t^2}(-1)dt \quad \text{where } t = \cos x$$

$$= 2\pi\int_0^{1} \frac{dt}{1+t^2}$$

$$= 2\pi\left[\mathrm{Tan}^{-1}t\right]_0^1$$

$$= 2\pi(\mathrm{Tan}^{-1}1 - \mathrm{Tan}^{-1}0)$$

$$= \frac{\pi^2}{2}$$

Hence

$$2I = \frac{\pi^2}{2}$$

$$\Rightarrow I = \frac{\pi^2}{4}$$

**P$_3$:** Suppose $f$ is integrable over $[-a, a]$. Then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)dx & \text{if } f \text{ is an even function} \\ 0 & \text{if } f \text{ is an odd function} \end{cases}$$

***Proof:*** We have

$$I = \int_{-a}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad \text{(By Theorem 5.10)}$$

Therefore

$$I = \int_a^0 f(-t)(-dt) + \int_0^a f(x)\,dx \quad \text{where } x = -t$$

$$= \int_0^a f(-t)\,dt + \int_0^a f(x)\,dx$$

If $f$ is even

$$I = \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

If $f$ is odd, then

$$I = \int_0^a -f(t)dt + \int_0^a f(x)dx = 0$$

## Example 5.5

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$.

***Solution:*** Let

$$I = \int_{-1}^{1} \frac{dx}{1+x^2}$$

We know that $1/(1+x^2)$ is an even function. Therefore

$$\begin{aligned} I &= 2\int_0^1 \frac{dx}{1+x^2} \\ &= 2\left[\mathrm{Tan}^{-1} x\right]_0^1 \\ &= 2(\mathrm{Tan}^{-1} 1 - \mathrm{Tan}^{-1} 0) \\ &= 2\left(\frac{\pi}{4} - 0\right) \\ &= \frac{\pi}{2} \end{aligned}$$

## Example 5.6

Show that

$$\int_{-1/2}^{1/2} \cos x \log_e\left(\frac{1+x}{1-x}\right) dx = 0$$

***Solution:*** Let

$$f(x) = \cos x \log_e\left(\frac{1+x}{1-x}\right) \quad \text{for } \frac{-1}{2} \le x \le \frac{1}{2}$$

Now

$$\begin{aligned} f(-x) &= \cos(-x)\log_e\left(\frac{1-x}{1+x}\right) \\ &= \cos x\left[\log_e(1-x) - \log_e(1+x)\right] \\ &= -\cos x \log_e\left(\frac{1+x}{1-x}\right) \\ &= -f(x) \end{aligned}$$

Therefore, $f$ is an odd function and hence

$$\int_{-1/2}^{1/2} f(x)dx = 0$$

## Example 5.7

If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are continuous functions then show that

$$\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)][g(x) - g(-x)]dx = 0$$

***Solution:*** Let

$$Q(x) = [f(x) + f(-x)][g(x) - g(-x)]$$

so that

$$\begin{aligned} Q(-x) &= [f(-x) + f(x)][g(-x) - g(x)] \\ &= -Q(x) \end{aligned}$$

Therefore $Q$ is an odd function. Hence

$$\int_{-\pi/2}^{\pi/2} Q(x)dx = 0$$

**P$_4$:** Suppose $f$ is a periodic function on $\mathbb{R}$ with period $T$. That is

$$f(x+T) = f(x)\ \forall\ x \in \mathbb{R}$$

Then for any positive integer $n$,

$$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

***Proof:*** We have

$$\int_0^{nT} f(x)\,dx = \int_0^T f(x)\,dx + \int_T^{2T} f(x)\,dx + \cdots + \int_{(m-1)T}^{mT} f(x)\,dx + \cdots + \int_{(n-1)T}^{nT} f(x)\,dx$$

Let

$$I_m = \int_{(m-1)T}^{mT} f(x)\,dx, \quad m = 1, 2, 3, \ldots, n$$

Put $t = x-(m-1)T$. Then

$$x = (m-1)T \Rightarrow t = 0$$
$$x = mT \Rightarrow t = T$$

Hence

$$I_m = \int_0^T f[t+(m-1)T]\,dt$$
$$= \int_0^T f(t)\,dt \quad [\because (m-1)T \text{ is also period of } f \text{ when } m = 2, 3, \ldots, n]$$

Therefore

$$\int_0^{nT} f(x)\,dx = \sum_{m=1}^{n} I_m$$
$$= \sum_{m=1}^{n} \int_{(m-1)T}^{mT} f(x)\,dx$$
$$= \sum_{m=1}^{n} \int_0^T f(x)\,dx$$
$$= n\int_0^T f(x)\,dx$$

## Example 5.8

Evaluate $\int_0^{100} (x-[x])dx$.

**Solution:** Since $x - [x]$ is of period 1, we have

$$\int_0^{100} (x-[x])dx = 100\int_0^1 (x-[x])dx$$

$$= 100\int_0^1 x\,dx = 50$$

**P$_5$:** If $f$ is integrable over $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

***Proof:*** Let $t = a + b - x$ so that $dx = -dt$. Now

$$x = a \Rightarrow t = b$$

and $$x = b \Rightarrow t = a$$

Therefore

$$\int_a^b f(a+b-x)dx = \int_b^a f(t)(-1)dt = \int_a^b f(t)dt$$

**P$_6$:** This property implies

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$$

***Proof:*** Using P$_5$ we have

$$I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \int_a^b \frac{f(x+b-x)}{f(a+b-x)+f(x)}dx$$

Therefore

$$2I = \int_a^b 1\,dx = b-a$$

$$\Rightarrow I = \frac{b-a}{2}$$

## Example 5.9

Evaluate $\int_{10}^{100} \frac{\log_e x}{\log_e x + \log_e(110-x)}dx$.

**Solution:** We have

$$\int_{10}^{100} \frac{\log_e x}{\log_e x + \log_e(110-x)}dx = \frac{100-10}{2} = 45$$

**Theorem 5.17 (Composite Theorem)** Suppose $f:[a,b]\to\mathbb{R}$ is integrable, $f([a,b])\subset[c,d]$ and $g:[c,d]\to\mathbb{R}$ is continuous. Then $g\circ f$ is integrable over $[a,b]$.

As an application of Theorem 5.17, we have the following theorem.

**Theorem 5.18** If $f:[a,b]\to\mathbb{R}$ is integrable, then $|f|$ is integrable over $[a,b]$. Further, if $|f(x)|\le M\ \forall\ x\in[a,b]$, then

$$\left|\int_a^b f\right|\le\int_a^b |f|\le M(b-a)$$

**Proof** Let $g(x)=|x|$ so that

$$|f|(x)=|f(x)|=g(f(x))$$

Hence by Theorem 5.17, the result follows. ■

***Note:*** The converse of Theorem 5.18 is not true. That is, if $|f|$ is integrable, then $f$ need not be integrable. For this, consider

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ -1 & \text{if } x \text{ is irrational}\end{cases}$$

on $[0, 1]$. Then $f$ is not integrable over $[0, 1]$ whereas $|f(x)|=1\ \forall\ x\in[0,1]$ is integrable.

The following is another important application of the Composite Theorem.

**Theorem 5.19 (Product Theorem)** Suppose $f$ and $g$ are defined and integrable over $[a, b]$. Then $f^2$, $g^2$ and $fg$ are integrable over $[a, b]$.

**Proof** Define $h(t)=t^2$. Then

$$f^2(x)=(h\circ f)(x)$$

Hence, by Composite Theorem, $f^2$ is integrable. Similarly, $g^2$ is integrable and

$$f\cdot g=\frac{1}{2}[(f+g)^2-f^2-g^2]$$

is integrable. ■

As a nice application of the product theorem, we have the following important theorem known as *Integration by parts* which is very useful in evaluating definite integrals.

**Theorem 5.20 (Integration By Parts)** Suppose $F$ and $G$ are differentiable on $[a, b]$, $f=F'$, $g=G'$ and $f, g$ are integrable over $[a, b]$. Then

$$\int_a^b (fG)(x)dx=[(FG)(x)]_a^b-\int_a^b (Fg)(x)dx$$

$$=[F(b)\,G(b)-F(a)\,G(a)]-\int_a^b (Fg)(x)dx$$

***Note:*** In the integration by parts formula, if we take $u=G$ and $dv=f$, we have

$$\int_a^b u\,dv = [u(x)v(x)]_a^b - \int_a^b v\,du$$

Hence, the student can follow the usual formula for integration by parts in the indefinite integration with $a$ and $b$ as limits of integration.

## Examples

**1.** $$\int_0^1 xe^x\,dx = \left[xe^x\right]_0^1 - \int_0^1 e^x(1)\,dx$$
$$= e - 0 - \left[e^x\right]_0^1$$
$$= e - (e-1) = 1$$

**2.** $$\int_1^e x\log x\,dx = \left[\frac{x^2}{2}\log x\right]_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$
$$= \frac{e^2}{2} - \frac{1}{4}\left[x^2\right]_1^e$$
$$= \frac{e^2}{2} - \left[\frac{e^2-1}{4}\right]$$
$$= \frac{e^2}{4} + \frac{1}{4}$$

The following theorem, called *Leibnitz Rule*, is useful in many problems.

**Theorem 5.21 (Leibnitz Rule)** Suppose $f:[a,b]\to\mathbb{R}$ is continuous and $g,h:[a,b]\to[a,b]$ are differentiable. Then

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))h'(x) - f(g(x))g'(x)$$

That is, if

$$Q(x) = \int_{g(x)}^{h(x)} f(t)\,dt$$

then

$$Q'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

**Proof** Write $F(t) = \int_a^t f(x)\,dx$ so that $F$ is the indefinite integral of $f$ (see Definition 5.4). Now $F'(t) = f(t)$. Hence

$$\int_{g(x)}^{h(x)} f = \int_a^{h(x)} f - \int_a^{g(x)} f = F(h(x)) - F(g(x))$$

Therefore

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f\right) = F'(h(x))h'(x) - F'(g(x))g'(x)$$
$$= f(h(x))h'(x) - f(g(x))g'(x) \quad (\because F' = f)$$

■

## Example

Let $x \in [0, 1]$. Take $g(x) = 0$, $h(x) = x^2$ and $f(x) = 1/(x^3 + 1)$. Then

$$\frac{d}{dx}\left(\int_0^{x^2} \frac{dt}{1+t^3}\right) = \frac{1}{1+x^6}(2x) - 0 = \frac{2x}{1+x^6}$$

Some of the important consequences of Leibnitz Rule are listed in the following theorem.

**THEOREM 5.22**

**(i)** If $f:[0,1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f \ \forall\, x \in [0,1]$, then $f(x) = 0 \ \forall\, x \in (0,1)$.

**(ii)** If $f:[a,b] \to \mathbb{R}$ is non-negative and continuous such that $\int_a^b f(x)\,dx = 0$, then $f(x) = 0 \ \forall\, x \in [a,b]$.

**(iii)** If $f$ and $g$ are integrable over $[a, b]$ and $\int_a^b f^2(x)\,dx = 0$, then $\int_a^b (fg)(x)\,dx = 0$.

**PROOF**

**(i)** Write $h(x) = \int_0^x f(t)\,dt$ so that $h'(x) = f(x)$. Now

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^x f(t)\,dt + \int_x^1 f(t)\,dt \\ &= h(x) + h(x) \quad \text{(By hypothesis)} \\ &= 2h(x)\end{aligned}$$

Therefore, $2h(x)$ is constant. Hence

$$0 = 2h'(x) = 2f(x)$$
$$\Rightarrow f(x) = 0 \ \forall\, x \in (0, 1)$$

**(ii)** Let $F(x) = \int_a^x f(t)\,dt$ be the indefinite integral of $f$. Then $F$ is continuous (Theorems 5.14 and 5.15) and $F'(x) = f(x) \geq 0$. Hence $F(x)$ is increasing on $[a, b]$. Further

$$F(a) = 0 = F(b) \quad \text{(By hypothesis)}$$

Hence $F(x) = 0 \ \forall\, x \in [a,b]$. Consequently, $f(x) = F'(x) = 0$ for all $x \in [a, b]$ [since $F(x)$ is the constant function 0].

**(iii)** Suppose $f$ and $g$ are integrable over $[a, b]$ and

$$\int_a^b f^2(x)\,dx = 0$$

Now, $\lambda > 0$ implies

$$\int_a^b (f \pm \lambda g)^2 \geq 0$$

$$\Rightarrow \int_a^b f^2 + \lambda^2 \int_a^b g^2 \pm 2\lambda \int_a^b fg \geq 0$$

Therefore

$$\lambda^2 \int_a^b g^2 \pm 2\lambda \int_a^b fg \geq 0 \quad \left(\because \int_a^b f^2 = 0\right)$$

$$\lambda \int_a^b g^2 \geq 2\left|\int_a^b fg\right|$$

Since this is true for every $\lambda > 0$, it follows that

$$\int_a^b fg = 0$$

Even though $\dfrac{1}{\sqrt{1-x^2}}$ is not defined at $x = 1$, we write

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \text{Sin}^{-1} 1 - \text{Sin}^{-1} 0 = \frac{\pi}{2}$$

To overcome this, the following concept is being introduced.

**DEFINITION 5.5** **(i)** Suppose $f : (a, b] \to \mathbb{R}$ is defined and is integrable over $[c, b]$ for every $c \in (a, b]$ and let

$$F(c) = \int_c^b f(x)\, dx$$

for $c \in (a, b]$. If $\lim_{c \to a} F(c)$ exists finitely (that is the limit is finite), then we say that $f$ is integrable over $[a, b]$ and write

$$\int_a^b f(x)dx = \lim_{c \to a} F(c)$$

**(ii)** Suppose $f : [a, b) \to \mathbb{R}$ is defined and is integrable over $[a, c]$ for every $c \in [a, b)$ and let

$$F(c) = \int_a^c f(x)\, dx$$

If $\lim_{c \to b} F(c)$ exists finitely, then we say that $f$ is integrable over $[a, b]$ and write

$$\int_a^b f(x)dx = \lim_{c \to b} F(c)$$

### Example

Suppose

$$f(x)=\frac{1}{\sqrt{1-x^2}},\ x\in[0,1)$$

Then $f$ is integrable over $[0, c)$ for all $c \in [0, 1)$ and

$$F(c)=\int_0^c \frac{dx}{\sqrt{1-x^2}}$$
$$=\left[\operatorname{Sin}^{-1}x\right]_0^c$$
$$=\operatorname{Sin}^{-1}c-\operatorname{Sin}^{-1}0$$
$$=\operatorname{Sin}^{-1}c$$

Also $F(c)=\operatorname{Sin}^{-1}c\to\operatorname{Sin}^{-1}1=\pi/2$ as $c\to 1-0$. Therefore

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2}$$

### Example

Suppose $f(x)=1/x^2$, $x\in(0,1]$. Then $f$ is integrable over $(c,1]\ \forall\ c\in(0,1]$ and

$$F(c)=\int_c^1 \frac{1}{x^2}dx$$
$$=\left[\frac{-1}{x}\right]_c^1$$
$$=-\left[1-(1/c)\right]$$
$$=(1/c)-1\to\infty \quad \text{as } c\to 0+0$$

Thus $\lim_{c\to 0+} F(c)$ is not finite and hence $f(x)=1/x^2$ is not integrable over [0, 1]. Since $\lim_{c\to 0+} F(c)=\infty$, in this case we write

$$\int_0^1 \frac{1}{x^2}dx=\infty$$

The value of $\int_0^{\pi/2} \sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx$ can be evaluated by repeating the integration by parts which is given in the following theorem. This formula is also known as *Wallis Formula* and is useful in some of the area problems.

**THEOREM 5.23 (WALLIS FORMULA)**

**(i)** If $n\ge 2$ is an integer, then

$$\int_0^{\pi/2}\sin^n x\,dx=\frac{[(n-1)(n-3)(n-5)\cdots 2 \text{ or } 1]}{[n(n-2)(n-4)\cdots 2 \text{ or } 1]}\times q$$

where $q=\pi/2$ if $n$ is even, otherwise $q=1$.

**(ii)** If $m, n$ are positive integers, then

$$\int_0^{\pi/2}\sin^m x\cos^n x=\frac{[(m-1)(m-3)\cdots 2 \text{ or } 1][(n-1)(n-2)\cdots 2 \text{ or } 1]}{[(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1]}\times p$$

where $p=\pi/2$ if both $m$ and $n$ are even, otherwise $p=1$.

***Note:*** Actually (i) is a consequence of (ii) if we take $n=0$ in (ii).

***Caution:*** In writing the factors in the Numerator and Denominator of the Wallis Formula, one should stop before entering into zero factor or negative factor.

## Examples

1. $\int_0^{\pi/2} \sin^7 x\,dx = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)} = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3} = \frac{16}{35}$

2. $$\int_0^{\pi/2} \sin^8 x\,dx = \frac{(8-1)(8-3)(8-5)(8-7)}{8(8-2)(8-4)(8-6)} \times \frac{\pi}{2}$$
$$= \frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2} \times \frac{\pi}{2}$$
$$= \frac{35\pi}{256}$$

3. $$\int_0^{\pi/2} \sin^6 x \cos^4 x\,dx = \frac{[(6-1)(6-3)(6-5)][(4-1)(4-3)]}{10(10-2)(10-4)(10-6)(10-8)} \times \frac{\pi}{2}$$
$$= \frac{(5\cdot 3\cdot 1)(3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2} \times \frac{\pi}{2}$$
$$= \frac{3\pi}{512}$$

### 5.1.2 Definite Integral as a Limit of a Sum

Sometimes definite integral can be used to find the limit of a sum. This is because, for a given function $f$ on $[a, b]$ and a partition $P$ of $[a, b]$, $S(f, P)$ (see Definition 5.1) is a finite sum and as the maximum of $\Delta x_i$ of $P$ tends to zero, the sum tends to $\int_b^b f$. For example, suppose $f : [0, 1] \to \mathbb{R}$ is integrable. Let $n$ be $a$ positive integer. Consider the partition

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$$

of $[0, 1]$. That is the interval $[0, 1]$ is divided into $n$ equal parts. Here $x_i = i/n$ for $i = 1, 2, \ldots, n$ and $x_{r-1} - x_r = 1/n$ for $r = 1, 2, \ldots, n$ so that

$$\|P\| = \max\{\Delta x_i, \Delta x_2, \ldots, \Delta x_n\} \quad (\text{where } \Delta x_i = x_i - x_{i-1})$$
$$= \frac{1}{n}$$

The division points $x_r$ are equally spaced in $[0, 1]$. We select $t_r = 1/r$ in $[x_{r-1}, x_r] = [(1/(r-1)), 1/r]$. Hence,

$$S(f, P) = f(t_1)(x_1 - x_0) + f(t_2)(x_2 - x_1) + \cdots + f(t_n)(x_n - x_{n-1})$$
$$= f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\left(\frac{1}{n}\right)$$
$$= \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + \left(\frac{n}{n}\right)\right]$$

Since $f$ is integrable over $[0, 1,]$,

$$S(f, P) \to \int_a^b f$$

Thus

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right] = \int_a^b f$$

### 5.1.3 Method of Finding the Limit of a Sum as a Definite Integral

Here, we give a procedure which can be done mechanically to obtain limit of a sum.

***Step 1:*** Write the general term $r$th or $(r+1)$th term of the sum whichever contains $r$.

***Step 2:*** Replace $n$ with $1/h$ in the general term and simplify it.

***Step 3:*** Alot one $h$ in the numerator to $dx$ and in the remaining replace $rh$ with $x$.

***Step 4:*** **(i)** If the number of terms in the sum is $n-1$ or $n$ or $n+1$, then the limits of integration are from 0 to 1.
**(ii)** If the number of terms in the sum is $2n-1$ or $2n$ or $2n+1$, the limits of integration are from 0 to $n$.

In this way we find the function and its limits of integration so that the definite integral obtained is the limit of the sum.

**Example 5.10**

Evaluate $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{4n}}\right)$.

***Solution:***

***Step 1:*** $T_r = r\text{th term} = \frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{r}}$

***Step 2:*** Put $n=\frac{1}{h}$ so that

$$T_r=\sqrt{\frac{h}{r}}=\frac{h}{\sqrt{rh}}$$

***Step 3:*** Replace $rh$ with $x$.

***Step 4:*** Since there are $4n$ terms in the sum, the required limit is

$$\int_0^4\frac{dx}{\sqrt{x}}=\lim_{t\to 0}\int_t^4\frac{dx}{\sqrt{x}}\quad\left(\because \frac{1}{\sqrt{x}}\text{ is not defined at } x=0\right)$$

$$=\lim_{t\to 0+}\left[2\sqrt{x}\right]_t^4$$

$$=2\sqrt{4}-0$$

$$=4$$

**Example 5.11**

Evaluate $\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+2n}\right)$.

***Solution:***

***Step 1:*** $(r+1)$th term $T_{r+1}=\frac{1}{n+r}$.

***Step 2:*** Put $n=\frac{1}{h}$ so that $T_{r+1}=\frac{h}{1+rh}$.

***Step 3:*** Put $rh=x$.

***Step 4:*** Since there are $2n+1$ terms in the given sum, the required limit is

$$\int_0^2\frac{dx}{1+x}=\left[\log_e(1+x)\right]_0^2$$

$$=\log 3-\log 1$$

$$=\log 3$$

***Note:*** If the sum contains $kn$ terms, then the limit is $\log_e(k+1)$.

**Example 5.12**

Show that $\lim\limits_{n\to\infty}\frac{1}{n}[(n+1)(n+2)\cdots(n+n)]^{1/n}=\frac{4}{e}$.

***Solution:***

Let

$$A=\lim_{n\to\infty}\frac{1}{n}[(n+1)(n+2)\cdots(n+n)]^{1/n}$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

Therefore

$$\log A=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\cdots+\left(1+\frac{n}{n}\right)\right]$$

Now

$$r\text{th term}=\frac{1}{n}\log\left(1+\frac{r}{n}\right)$$

Put $n=1/h$ so that

$$r\text{th term} = h\log(1+x)$$

where $x = rh$. Therefore

$$\log_e A = \int_0^1 \log_e(1+x)\,dx$$

$$= [x\log_e(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \log_e 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right)dx$$

$$= \log 2 - (1 - \log_e 2)$$

$$= 2\log_e 2 - 1$$

So

$$A = e^{2\log_e 2 - 1} = \frac{4}{e}$$

## Example 5.13

If $S_n = \dfrac{1}{1+\sqrt{n}} + \dfrac{1}{2+\sqrt{2n}} + \dfrac{1}{3+\sqrt{3n}} + \cdots + \dfrac{1}{n+\sqrt{n^2}}$, then find $\lim\limits_{n\to\infty} S_n$.

***Solution:***

$$T_r = r\text{th term} = \frac{1}{r+\sqrt{rn}}$$

Put $n = 1/h$, so that

$$\frac{\sqrt{h}}{r\sqrt{h}+\sqrt{r}} = \frac{h}{rh+\sqrt{rh}} = \frac{h}{x+\sqrt{x}}$$

Now

$$\text{Required limit} = \int_0^1 \frac{dx}{x+\sqrt{x}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

$$= \lim_{t\to 0+} \int_t^1 \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

$$= 2\log_e(\sqrt{1}+1) - \lim_{t\to 0+} 2\log_e(\sqrt{t}+1)$$

$$\left(\because \frac{1}{x+\sqrt{x}} \text{ is not defined at } x = 0\right)$$

$$= 2\log_e 2 - 2\log_e 1 = 2\log_e 2$$

Therefore

$$\lim_{n\to\infty} S_n = \log_e 4$$

## Example 5.14

Show that $\lim\limits_{n\to\infty}\left(\dfrac{1}{n} + \dfrac{n^2}{(n+1)^3} + \dfrac{n^2}{(n+2)^3} + \cdots + \dfrac{1}{8n}\right) = \dfrac{3}{8}$

***Solution:***

$$T_{r+1} = (r+1)\text{th term} = \frac{n^2}{(n+r)^3}$$

Put $n = 1/h$ so that

$$T_{r+1} = \frac{h}{(1+rh)^3} = \frac{h}{(1+x)^3}$$

Therefore, the required limit is

$$\int_0^1 \frac{dx}{(1+x)^3} = \left[\frac{(1+x)^{-3+1}}{-3+1}\right]_0^1$$

$$= -\frac{1}{2}\times\left[\frac{1}{(1+1)^2} - 1\right]$$

$$= \frac{-1}{2}\,\frac{(1-4)}{4}$$

$$= \frac{3}{8}$$

# 5.2 | Areas

We have already observed that the area enclosed by the $x$-axis, the lines $x = a, x = b$ and a curve $y = f(x)$ represented by a non-negative function $f(x)$ can be regarded as the definite integral $\int_a^b f(x)\,dx$. This is definitely the case when $f$ is continuous.

1. In other words, if $f:[a, b] \to \mathbb{R}$ is continuous and non-negative, then the area $A$ of the region enclosed by the lines $x = a, x = b$, the $x$-axis and the curve $y = f(x)$ is given by (see Fig. 5.2)

$$A = \int_a^b f(x)\,dx$$

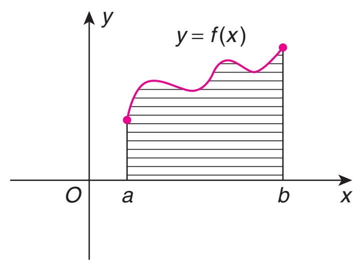



FIGURE 5.2

2. If $f:[a, b] \to \mathbb{R}$ is continuous and non-positive, that is $f(x) \le 0$ for all $x \in [a, b]$, then the area $B$ of the region enclosed by the $x$-axis, the lines $x = a, x = b$ and the curve $y = f(x)$ is same as the area of the region enclosed by the curve $y = -f(x)$ and the lines $x = a, x = b$ and $x$-axis, since both the regions are mirror reflections of each other about the $x$-axis. Hence

$$B = \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx = \left|\int_a^b f(x)\,dx\right|$$

See Fig. 5.3.

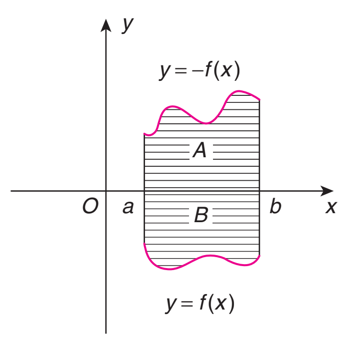



FIGURE 5.3

3. Suppose $a < c < b$, $f:[a, b] \to \mathbb{R}$ is continuous, $f(x) \ge 0 \;\forall\; x \in [a, c]$ and $f(x) \le 0 \;\forall\; x \in [c, b]$. Then, the area of the region enclosed by the $x$-axis, the lines $x = a, x = c$ and $x = b$ and the curve $y = f(x)$ can be split into two parts (see Fig. 5.4).

   **(a)** Area $A$ enclosed by the $x$-axis, the lines $x = a,\ x = c$ and the curve $y = f(x), a \le x \le c$ [so that $f(x) \ge 0$]

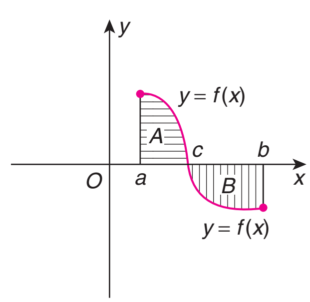



FIGURE 5.4

**(b)** Area $B$ enclosed by the $x$-axis, the lines $x = c$, $x = b$ and the curve $y = f(x), c \le x \le b$ [so that $f(x) \le 0$]. Hence the area of the total region is

$$\begin{aligned} A + B &= \int_a^c f(x)\,dx + \int_c^b -f(x)\,dx \\ &= \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right| \\ &= \int_a^c |f(x)|\,dx + \int_c^b |f(x)|\,dx \\ &= \int_a^b |f(x)|\,dx \end{aligned}$$

**4.** The area $A$ enclosed between the $y$-axis, the lines $y = a$, $y = b$ and the curve $x = f(y)$ is given by (Fig. 5.5)

$$A = \int_a^b f(y)\,dy = \int_a^b x\,dy$$

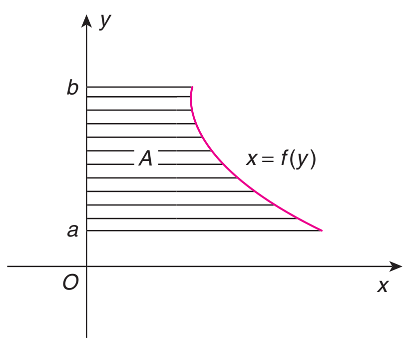



FIGURE 5.5

**5.** Suppose $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}$ are continuous functions and $0 \le f(x) \le g(x)$ for $x \in [a, b]$. Then the area $A$ of the region bounded by the curves $y = g(x)$, $y = f(x)$ and the lines $x = a$, $x = b$ is given by (see Fig. 5.6)

$$\begin{aligned} A &= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \\ &= \int_a^b (g - f)(x)\,dx \end{aligned}$$

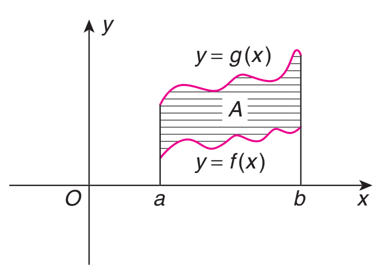



FIGURE 5.6

***Note:*** If the two curves intersect in the points $(x_1, y_1)$ and $(x_2, y_2)$, then

$$A = \int_{x_1}^{x_2} (g - f)(x)\,dx$$

While computing areas, it is always useful if a rough diagram (figure) is drawn. For this the following tips will be helpful.

**1. Symmetry:**

**(a)** If the powers of $x$ in the curve equation are all even, then the curve is symmetric about $y$-axis. That is, the equation of the curve does not change if $x$ is replaced by $-x$.

**(b)** If the equation of the curve does not change by replacing $y$ with $-y$, then the curve is symmetric about $x$-axis (i.e., the powers of $y$ are all even).

**2.** Find the points of intersection of the curve(s) with the coordinate axes.

**3.** Observe whether there are any limitations for the abscissa (ordinate) of points on the curve.

**4.** If for any point $(x, y)$ on the curve, $\lim_{x\to\pm\infty} y = 0$ and $y \neq 0$ for all points on the curve, then the curve approaches nearer and nearer to $x$-axis but will never meet it. Similarly, if $\lim_{y\to\pm\infty} x = 0$ and $x \neq 0$ for all points, the curve approaches $y$-axis without meeting it.

## Example 5.15

Show that the area bounded by the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(called ellipse)}$$

is $\pi ab$ square units.

***Solution:*** See Fig. 5.7.

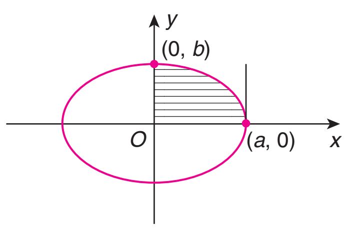



FIGURE 5.7

**(i)** Since powers of both $x$ and $y$ are even, the curve is symmetric about both coordinate axes.

**(ii)** $-a \le x \le a$ and $-b \le y \le b$.

**(iii)** Curve meets the axes in $(\pm a, 0)$ and $(0, \pm b)$.

Therefore

Area bounded by the curve

$$= 4(\text{Area in the first quadrant})$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left\{\left[\frac{x\sqrt{a^2 - x^2}}{2}\right]_0^a + \frac{a^2}{2}\left[\text{Sin}^{-1}\frac{x}{a}\right]_0^a\right\}$$

$$= \frac{4b}{a}\left\{(0 - 0) + \frac{a^2}{2}(\text{Sin}^{-1}1 - \text{Sin}^{-1}0)\right\}$$

$$= \frac{4b}{a} \times \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \pi ab$$

## Example 5.16

Find the area included between the parabola $y = 2x - x^2$ and the $x$-axis.

***Solution:*** We have

$$y = 2x - x^2 = 1 - (x-1)^2 \Rightarrow (x-1)^2 = -(y-1)$$

This is a downward parabola with vertex at (1, 1) and meeting the $x$-axis in (0, 0) and (2, 0). The required area (shaded region in Fig. 5.8) is

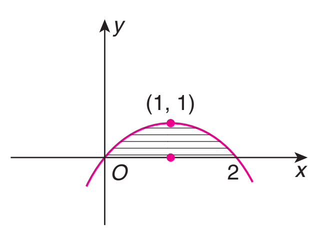



FIGURE 5.8

$$\int_0^2 y\,dx = \int_0^2 (2x - x^2)\,dx$$

$$= \left[x^2\right]_0^2 - \frac{1}{3}\left[x^3\right]_0^2$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

## Example 5.17

Compute the area bounded by $y = e^x$, $x$-axis and the lines $x = 1$ and $x - 2$.

***Solution:*** The required area is (see shaded region in Fig. 5.9)

$$\int_1^2 e^x\,dx = \left[e^x\right]_1^2 = e^2 - e$$

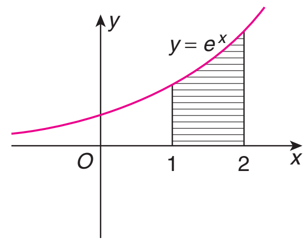



FIGURE 5.9

## Example 5.18

Show that area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $16ab/3$ square units.

***Solution:*** The two curves intersect in the points (0, 0) and $(h, k)$ where $h = 4a^{1/3}b^{2/3}$ and $k = 4a^{2/3}b^{1/3}$ (see Fig. 5.10). Now

$$0 \le x \le h \Rightarrow y = \sqrt{4ax} \ge \frac{x^2}{4b} = y$$

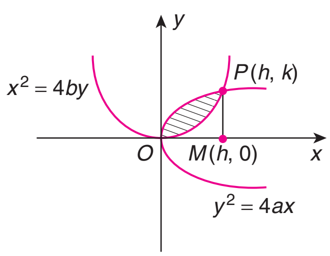



FIGURE 5.10

Therefore the required area (shaded one in Fig. 5.10) is

$$\int_0^h \left(\sqrt{4ax} - \frac{x^2}{4b}\right)dx = 2\sqrt{a} \times \tfrac{2}{3}\left[x^{3/2}\right]_0^h - \frac{1}{12b}\left[x^3\right]_0^h$$

$$= \frac{4\sqrt{a}}{3}\cdot h^{3/2} - \frac{h^3}{12b}$$

$$= \frac{4\sqrt{a}}{3}(4a^{1/3}b^{2/3})^{3/2} - \frac{(4a^{1/3}b^{2/3})^3}{12b}$$

$$= \frac{4\sqrt{a}}{3} \times 8\sqrt{a}\,b - \frac{64ab^2}{12b}$$

$$= \frac{32}{3}(ab) - \frac{16}{3}(ab)$$

$$= \frac{16}{3}(ab)$$

## Example 5.19

Compute the area enclosed between the curves $y = \sin x$ and $y = \cos x$ for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

***Solution:*** See Fig. 5.11. Here

$$\cos x \le \sin x \quad \text{for } x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

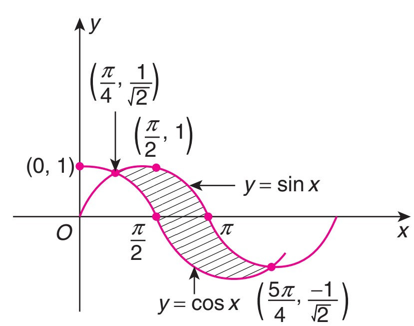



FIGURE 5.11

Therefore, the required area is

$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$$

$$= -\left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\right) + \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2} + \sqrt{2}$$

$$= 2\sqrt{2}$$

## Example 5.20

Compute the area of the whole region enclosed by the curve $a^2y^2 = x^2(a^2 - x^2)$.

***Solution:***

1. Since the powers of $x$ and $y$ are even, the curve is symmetric about both the axes.
2. The curve meets $x$-axis in $(-a, 0)$, $(0, 0)$ and $(a, 0)$.

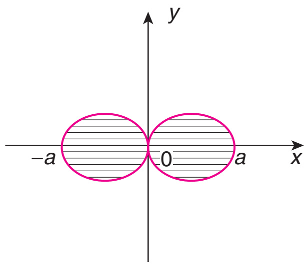



FIGURE 5.12

3. The curve meets $y$-axis at $(0, 0)$ only.
4. $y^2 = \dfrac{x^2}{a^2}(a^2 - x^2) \Rightarrow -a \le x \le a.$

A rough diagram is shown in Fig. 5.12. Therefore, the required area is

$$4\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\, dx = \frac{-2}{a} \times \frac{2}{3}\left[(a^2 - x^2)^{3/2}\right]_0^a$$

$$= \frac{-4}{3a}\left[0 - a^3\right] = \frac{4a^2}{3}$$

## 5.3 | Differential Equations

Differential equations have applications in many branches like Physics, Chemistry, Chemical Engineering, Biology, Economics, Geology, etc. Therefore in-depth study of differential equations assumed great importance in all modern scientific research. James Bernoulli (1654–1705), Joseph Louis Lagrange (1736–1813), and Leibnitz (1646–1716) are some of the early mathematicians responsible for the development of differential equations.

In this section, we will study some basic concepts regarding differential equations, learn how to form differential equations and find the solutions of ordinary differential equations. We strictly confine to the types of differential equations which are included in IIT-JEE syllabus.

**DEFINITION 5.6** Equation involving derivative (derivatives) of a dependent variable with respect to one or more independent variable(s) is called a *differential equation*.

If there is only one independent variable, then the equation is called an *ordinary differential equation*.

## Examples

The following are ordinary differential equations:

1. $\dfrac{dy}{dx} + \cos x = 0$
2. $x\dfrac{dy}{dx} + y = 0$
3. $\dfrac{d^2y}{dx^2} + y = 0$

**DEFINITION 5.7** **Order and Degree** The highest order of the derivatives that are present (or occur) in the differential equation is called the *order of the differential equation* and the degree of the highest order derivative present in the equation (when the equation is a polynomial equation in derivatives) is called the *degree of the differential equation*.

## Examples

1. $\left(\dfrac{dy}{dx}\right)^2+\left(\dfrac{dy}{dx}\right)+1=0$ is of order 1 and degree 2.

2. $\dfrac{d^2y}{dx^2}+y=0$ is of order 2 and degree 1.

3. $\left(\dfrac{d^2y}{dx^2}\right)^3+2\left(\dfrac{dy}{dx}\right)^2+\dfrac{dy}{dx}+y=0$ is of order 2 and degree 3.

4. $\dfrac{d^2y}{dx^2}+\sin\left(\dfrac{dy}{dx}\right)=0$ is of order 2 but the *degree is not defined* because the given equation is not a polynomial equation in derivatives.

5. $1+\left(\dfrac{d^2y}{dx^2}\right)^3=\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^4$ is of order 2 and degree 3.

**DEFINITION 5.8** **Solution, General Solution, Particular Solution**

1. Any relation between the independent and dependent variables satisfying a differential equation is called a *solution of the differential equation.*
2. A solution which contains as many arbitrary constants as the order of the equation is called *general solution* of the given equation.
3. Any solution obtained from the general solution by giving values to arbitrary constants (i.e., solution free from arbitrary constants) is called a *particular solution*.

### 5.3.1 Procedure to Form Differential Equation When General Solution is Given

***Step 1:*** Write the general solution or the equation representing the family of curves.
***Step 2:*** Differentiate the given equation as many times as the number of arbitrary constants that are present in the given equation.
***Step 3:*** Eliminate the arbitrary constants from the equations written in Step 1 and Step 2. Then the differential equation obtained is the equation representing the curves mentioned in Step 1.

## Example 5.21

Form the differential equation of all lines of the form $y = mx$.

***Solution:*** Given that $y = mx$ ($m$ is arbitrary constant). Therefore

$$\frac{dy}{dx}=m$$

Substituting the value of $m = dy/dx$ in $y = mx$, we have

$$x\frac{dy}{dx}=y$$

which is the differential equation representing the set of all lines $y = mx$.

## Example 5.22

Find the differential equation of all circles in $xy$-plane touching $y$-axis at the origin.

***Solution:*** Since $y$-axis is a tangent to a circle at (0, 0), the centre of the circle lies on the $x$-axis. Hence any circle touching $y$-axis at (0, 0) is of the form

$$(x-a)^2 + y^2 = a^2$$

or
$$x^2 + y^2 - 2ax = 0 \tag{5.1}$$

where "$a$" is the arbitrary constant. Differentiating Eq. (5.1) w.r.t. $x$, we get

$$x + y\frac{dy}{dx} = a \tag{5.2}$$

Eliminating $a$ from Eqs. (5.1) and (5.2) we have

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$$

Therefore

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

is the differential equation of all circles touching $y$-axis at the origin.

## QUICK LOOK 2

The equation

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

is the differential equation of all circles touching the $x$-axis at the origin because equation of such a circle is of the form

$$x^2 + (y-a)^2 = a^2$$

## Example 5.23

Form the differential equation whose general solution is $y = Ae^x + Be^{-x}$ ($A$, $B$ are arbitrary constants).

***Solution:*** We have

$$y = Ae^x + Be^{-x} \tag{5.3}$$

Differentiating w.r.t. $x$ we get

$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

and
$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x} = y \tag{5.4}$$

From Eqs. (5.3) and (5.4) we have

$$\frac{d^2y}{dx^2} - y = 0$$

which is the differential equation whose general solution is $y = Ae^x + Be^{-x}$.

## Example 5.24

Form the differential equation whose general solution is $y = A\cos x + B\sin x$, where $A$ and $B$ are arbitrary constants.

***Solution:*** We have

$$y = A\cos x + B\sin x \tag{5.5}$$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

and
$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x = -y \tag{5.6}$$

From Eqs. (5.5) and (5.6), we get that

$$\frac{d^2y}{dx^2} + y = 0$$

is the differential equation whose general solution is $y = A\cos x + B\sin x$.

## Example 5.25

Form the differential equation from the equation $y^2 = 4ax$ ($a$ is an arbitrary non-zero constant).

***Solution:*** We have

$$y^2 = 4ax \tag{5.7}$$

Differentiating both sides w.r.t. $x$, we have

$$y\frac{dy}{dx} = 2a \qquad (5.8)$$

Eliminating $a$ from Eqs. (5.7) and (5.8), we have

$$y^2 = 2x\left(y\frac{dy}{dx}\right) = 2xy\frac{dy}{dx}$$

### 5.3.2 Variables Separable or Separation of Variables

In this section we solve equations of the form $f(x)dx + g(y)dy = 0$. The solution is $\int f(x)dx + \int g(y)dy = c$ where $c$ is an arbitrary constant.

#### Example 5.26

Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x dy = 0$.

***Solution:*** The given equation can be written as

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0 \quad \text{(Variables Separable)}$$

Therefore

$$\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = c$$

$$\Rightarrow \log|\tan x| + \log|\tan y| = c$$

$$\Rightarrow |\tan x \tan y| = e^c \quad \text{or} \quad \tan x \tan y = c$$

#### Example 5.27

Solve $\dfrac{dy}{dx} = e^{x+y}$.

***Solution:*** We have

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

Therefore

$$e^{-y}\, dy = e^x dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int e^{-y}\, dy = \int e^x\, dx + c$$

$$\Rightarrow -e^{-y} = e^x + c$$

$$\Rightarrow e^x + e^{-y} = c'$$

#### Example 5.28

Solve $(y \log y)\, dx - xdy = 0$.

***Solution:*** We have

$$\frac{dy}{y\log y} - \frac{dx}{x} = 0$$

Integrating both sides, we get

$$\int \frac{dy}{y\log y} - \int \frac{dx}{x} = c$$

$$\Rightarrow \log\log y - \log x = c$$

$$\Rightarrow \log\left(\frac{\log y}{x}\right) = c$$

$$\Rightarrow \log y = cx$$

$$\Rightarrow y = e^{cx}$$

#### Example 5.29

Solve $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

***Solution:*** We have

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + c$$

$$\Rightarrow \text{Tan}^{-1} y = \text{Tan}^{-1} x + c$$

## Example 5.30

Solve $\dfrac{dy}{dx} = \dfrac{xy+y}{xy+x}$.

***Solution:*** We have

$$\frac{dy}{dx} = \frac{y(x+1)}{x(y+1)}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)dy = \left(\frac{x+1}{x}\right)dx \quad \text{(Variables Separable)}$$

Integrating both sides we get

$$\int \left(\frac{y+1}{y}\right)dy = \int \frac{x+1}{x}\,dx + c$$

$$\Rightarrow y + \log y = x + \log x + c$$

$$\Rightarrow y - x = \log\left(\frac{x}{y}\right) + c$$

## Example 5.31

Solve $\dfrac{dy}{dx} - x\tan(y-x) = 1$.

***Solution:*** Put $y - x = z$. Differentiating w.r.t. $x$ we have

$$\frac{dy}{dx} - 1 = \frac{dz}{dx}$$

$$\Rightarrow \frac{dz}{dx} = x \tan z$$

$$\Rightarrow (\cot z)\,dz = x\,dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \cot z\,dz = \int x\,dx + c$$

$$\Rightarrow \log(\sin z) = \frac{1}{2}x^2 + c$$

$$\Rightarrow \sin z = ke^{x^2/2}$$

$$\Rightarrow \sin(y-x) = ke^{x^2/2}$$

### 5.3.3 Homogeneous Differential Equation

**DEFINITION 5.9** **(Homogeneous Function)** A function $f(x, y)$ is called a homogeneous function of degree $n$ ($n$ need not be an integer) if

$$f(tx, ty) = t^n f(x, y)$$

for any non-zero $t$.

## Examples

1. $f(x, y) = ax^2 + 2hxy + by^2$, where at least one of $a, h, b$ is non-zero, is a homogeneous function of degree 2.
2. $f(x, y) = \dfrac{x^{3/2} + y^{3/2}}{x+y}$ is a homogeneous function of degree 1/2.
3. $f(x, y) = \dfrac{x^2 + xy + y^2}{x^2 - y^2}$ is a homogeneous function of degree zero.
4. $f(x, y) = \dfrac{x+y}{x^3 + y^3 + x^2 y}$ is a homogeneous function of degree –2.

**DEFINITION 5.10** **Homogeneous Differential Equation** Equation of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where both $f$ and $g$ are homogeneous functions of the same degree is called a *homogeneous differential equation*.

To solve a homogeneous differential equation, use the substitution $y = vx$ so that the equation will reduce to Variables Separable form which can be solved. Finally in the solution replace $v$ with $y/x$.

## Example 5.32

Solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$.

***Solution:*** Put $y = vx$. Therefore

$$v + x\frac{dv}{dx} = \frac{dy}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$= \frac{-(1+v^2)}{2v}$$

Now

$$\left(\frac{2v}{1+v^2}\right)dv + \frac{dx}{x} = 0 \quad \text{(Variables Separable)}$$

Integrating both sides we get

$$\int \frac{2v}{1+v^2}dv + \int \frac{dx}{x} = c$$

$$\Rightarrow \log(1+v^2) + \log x = c$$

$$\Rightarrow (1+v^2)x = k$$

$$\Rightarrow x^2 + y^2 = kx$$

## Example 5.33

Solve $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$.

***Solution:*** Put $y = vx$ so that

$$v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{-v^4}{1+v^3}$$

$$\Rightarrow \left(\frac{1+v^3}{v^4}\right)dv + \frac{dx}{x} = 0$$

Integrating we get

$$\int \frac{1+v^3}{v^4}dv + \int \frac{dx}{x} = c$$

$$\Rightarrow -\frac{1}{3v^3} + \log v + \log x = c$$

$$\Rightarrow vx = k\,e^{1/3v^3}$$

$$\Rightarrow y = k\,e^{x^3/3y^3}$$

## Example 5.34

Given that $y = \pi/2$ when $x = 1$, solve

$$y\cos\frac{y}{x}(x\,dy - y\,dx) + x\sin\frac{y}{x}(x\,dy + y\,dx) = 0$$

***Solution:*** We have

$$y\cos\frac{y}{x}\left(x\frac{dy}{dx} - y\right) + x\sin\frac{y}{x}\left(x\frac{dy}{dx} + y\right) = 0$$

Re-arranging the terms we get

$$\frac{dy}{dx}\left(xy\cos\frac{y}{x} + x^2\sin\frac{y}{x}\right) = y^2\cos\frac{y}{x} - xy\sin\frac{y}{x}$$

Put $y = vx$. Then

$$\left(v + x\frac{dv}{dx}\right)(v\cos v + \sin v) = v^2\cos v - v\sin v$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2\cos v - v\sin v}{v\cos v + \sin v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-2v\sin v}{v\cos v + \sin v}$$

$$\Rightarrow \left(\frac{v\cos v + \sin v}{v\sin v}\right)dv + \frac{2}{x}dx = 0$$

Integrating we get

$$\int \frac{v\cos v + \sin v}{v\sin v}dv + 2\int\frac{dx}{x} = c$$

$$\Rightarrow \int\left(\frac{\cos v}{\sin v} + \frac{1}{v}\right)dv + 2\int\frac{dx}{x} = c$$

$$\Rightarrow \log(\sin v) + \log v + 2\log x = c$$

$$\Rightarrow (vx^2)\sin v = k$$

$$\Rightarrow (xy)\sin\left(\frac{y}{x}\right) = k$$

Since $y = \pi/2$ when $x = 1$, we have $k = \pi/2$. Therefore, the solution is $(xy)\sin(y/x) = \pi/2$.

## Example 5.35

Find the equation of the curve passing through the point $(4, -2)$ and satisfying the equation

$$\frac{dy}{dx} = \frac{y(x + y^3)}{x(y^3 - x)}$$

***Solution:*** The given equation can be written as

$$(xy^3 - x^2)dy - (xy + y^4)dx = 0$$

$$\Rightarrow y^3(x\,dy - y\,dx) - x(x\,dy + y\,dx) = 0$$

$$\Rightarrow y^3\frac{(x\,dy - y\,dx)}{x^2} - \frac{1}{x}(x\,dy + y\,dx) = 0$$

$$\Rightarrow y^3\,d\left(\frac{y}{x}\right) - \frac{1}{x}d(xy) = 0$$

$$\Rightarrow \frac{y^3}{xy^2}d\left(\frac{y}{x}\right) - \frac{1}{x^2y^2}d(xy) = 0$$

$$\Rightarrow \frac{y}{x}d\left(\frac{y}{x}\right) - \frac{d(xy)}{x^2y^2} = 0$$

On integration we have

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{xy} = c$$

Since the curve passes through the point $(4, -2)$ we have

$$\frac{1}{2}\left(\frac{4}{16}\right) + \frac{1}{(4)(-2)} = c$$

Therefore $c = 0$. Hence the equation of the curve is

$$\frac{y^2}{2x^2} + \frac{1}{xy} = 0$$

$$\Rightarrow y^3 = -2x$$

## Example 5.36

Solve $\dfrac{dy}{dx} = \dfrac{y^3 + 3x^2y}{x^3 + 3xy^2}$.

***Solution:*** Put $y = vx$. Therefore

$$v + x\frac{dv}{dx} = \frac{v^3 + 3v}{1 + 3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^3 + 3v}{1 + 3v^2} - v$$

$$= \frac{2v - 2v^3}{1 + 3v^2}$$

Hence

$$\left(\frac{1 + 3v^2}{v(1 - v^2)}\right)dv - \frac{2}{x}dx = 0$$

Integrating we get

$$\int\frac{1 + 3v^2}{v(1 - v)(1 + v)}dv - 2\int\frac{dx}{x} = c$$

$$\Rightarrow \int\left(\frac{1}{v} + \frac{2}{1 - v} - \frac{2}{1 + v}\right)dv - 2\int\frac{dx}{x} = c$$

$$\Rightarrow \log v - 2\log(1 - v) - 2\log(1 + v) - 2\log x = c$$

$$\Rightarrow \log\left(\frac{(1 - v)^2(1 + v)^2x^2}{v}\right) = -c$$

$$\Rightarrow (1 - v^2)^2x^2 = kv$$

$$\Rightarrow \frac{(x^2 - y^2)^2x^2}{x^4} = \frac{ky}{x}$$

$$\Rightarrow (x^2 - y^2)^2 = kxy$$

## Example 5.37

Solve $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y.$

***Solution:*** The given equation can be written as

$$\frac{dy}{dx}+\frac{y^2}{x^2}=\frac{y}{x}$$

Put $y = vx$. We have

$$v+x\frac{dv}{dx}+v^2=v$$

$$\Rightarrow \frac{dv}{v^2}+\frac{dx}{x}=0$$

Integrating both sides we get

$$\int\frac{dv}{v^2}+\int\frac{dx}{x}=c$$

$$\Rightarrow -\frac{1}{v}+\log x=c$$

$$\Rightarrow x=k\,e^{x/y}$$

## Example 5.38

Solve $\dfrac{dy}{dx}+\dfrac{y(x+y)}{x^2}=0.$

***Solution:*** Put $y = vx$. We have

$$v+x\frac{dv}{dx}+v(1+v)=0$$

$$\Rightarrow x\frac{dv}{dx}+2v+v^2=0$$

$$\Rightarrow \frac{1}{v(2+v)}dv+\frac{dx}{x}=0$$

Integrating we get

$$\int\frac{1}{v(2+v)}dv+\int\frac{dx}{x}=c$$

$$\Rightarrow \frac{1}{2}\int\left(\frac{1}{v}-\frac{1}{2+v}\right)dv+\int\frac{dx}{x}=c$$

$$\Rightarrow \log v-\log(2+v)+2\log x=c$$

$$\Rightarrow vx^2=k\,(2+v)$$

$$\Rightarrow yx=k\left(2+\frac{y}{x}\right)$$

$$\Rightarrow yx^2=k\,(2x+y)$$

### 5.3.4 Linear Differential Equation

Suppose $P$ and $Q$ are integrable functions of $x$. Then the differential equation

$$\frac{dy}{dx}+Py=Q$$

is called *linear equation of first degree*. To solve linear equations, multiply both sides with $e^{\int P\,dx}$ so that we have

$$\frac{d}{dx}(y\,e^{\int P\,dx})=Q\,e^{\int P\,dx}$$

Therefore

$$y\,e^{\int Pdx}=\int Q\,e^{\int Pdx}\,dx+c$$

is the solution. Here, $e^{\int P\,dx}$ is called the *integrating factor* (I. F.)

***Note:*** If $P$ and $Q$ are integrable functions of $y$, then the equation

$$\frac{dx}{dy}+Px=Q$$

is also called a linear equation whose solution is

$$x\,e^{\int P\,dy}=\int Q\,e^{\int P\,dy}\,dy+c$$

## Example 5.39

Solve $\frac{dy}{dx} + xy = x$.

***Solution:*** Here $P = x, Q = x$ so that integrating factor is

$$\text{I.F.} = e^{\int P\,dx} = e^{\int x\,dx} = e^{x^2/2}$$

Therefore the solution is

$$ye^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c$$

$$\Rightarrow ye^{x^2/2} = \int x e^{x^2/2} dx + c = e^{x^2/2} + c$$

Therefore $y = 1 + c e^{-x^2/2}$ is the solution.

## Example 5.40

Solve $\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{1}{(1+x^2)^2}$.

***Solution:*** Here

$$P = \frac{2x}{1+x^2}, \quad Q = \frac{1}{(1+x^2)^2}$$

The integrating factor is

$$\begin{aligned}\text{I.F.} &= e^{\int P\,dx}\\ &= e^{\int \frac{2x}{1+x^2}dx}\\ &= e^{\log(1+x^2)}\\ &= 1+x^2\end{aligned}$$

The solution is

$$y e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c$$

This implies

$$\begin{aligned}y(1+x^2) &= \int \frac{1}{(1+x^2)^2}(1+x^2)\,dx + c\\ &= \int \frac{dx}{1+x^2} + c\\ &= \text{Tan}^{-1} x + c\end{aligned}$$

So, the solution is

$$y(1+x^2) = \text{Tan}^{-1} x + c$$

## Example 5.41

Solve $(\sin y)\frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$.

***Solution:*** Put $\cos y = z$. Therefore

$$(\sin y)\frac{dy}{dx} = \frac{-dz}{dx}$$

$$\Rightarrow \frac{-dz}{dx} = \cos x(2z - \sin^2 x)$$

$$\Rightarrow \frac{dz}{dx} + (2\cos x)z = \cos x \sin^2 x \qquad \text{(Linear equation)}$$

Here the integrating factor is

$$\text{I.F.} = e^{\int 2\cos x\,dx} = e^{2\sin x}$$

Therefore the solution is

$$\begin{aligned}ze^{2\sin x} &= \int (\cos x \sin^2 x) e^{2\sin x}\,dx + c\\ &= \int t^2 e^{2t}\,dt + c \quad \text{where } t = \sin x\\ &= \frac{1}{2}t^2 e^{2t} - \int \frac{1}{2} e^{2t}(2t)\,dt + c\\ &= \frac{t^2 e^{2t}}{2} - \int t e^{2t}\,dt + c\\ &= \frac{t^2 e^{2t}}{2} - \frac{1}{2} t e^{2t} + \int \frac{1}{2} e^{2t}\,dt + c\\ &= \frac{t^2 e^{2t}}{2} - \frac{te^{2t}}{2} + \frac{e^{2t}}{4} + c\\ &= \frac{e^{2t}}{4}(2t^2 - 2t + 1) + c\end{aligned}$$

So the final solution is

$$(\cos y)e^{2\sin x} = \frac{e^{2\sin x}}{4}(2\sin^2 x - 2\sin x + 1) + c$$

## Example 5.42

Solve $\cos^3 x \dfrac{dy}{dx} + y\cos x = \sin x$.

***Solution:*** Given equation can be written as

$$\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x \qquad \text{(Linear equation)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \sec^2 x} = e^{\tan x}$$

The solution is

$$\begin{aligned} y e^{\tan x} &= \int \tan x \sec^2 x\, e^{\tan x}\, dx + c \\ &= \int t e^t\, dt + c \quad \text{where } t = \tan x \\ &= t e^t - \int e^t (1)\, dt + c \\ &= e^t (t-1) + c \end{aligned}$$

Hence

$$y e^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

## Example 5.43

Solve $(\cos^2 x)\dfrac{dy}{dx} - (\tan 2x) y = \cos^4 x,\ \dfrac{-\pi}{4} < x < \dfrac{\pi}{4}$.

***Solution:*** We have

$$\frac{dy}{dx} - \left(\frac{2\tan x}{1-\tan^2 x}\right)\sec^2 x\, y = \cos^2 x$$

Here

$$P = -\frac{2\tan x}{1-\tan^2 x}\sec^2 x$$

Therefore

$$\begin{aligned} \int P\, dx &= \int \frac{-2t}{1-t^2} dt \quad \text{where } t = \tan x \\ &= \log(1-t^2) \quad (\text{Note that } |t| < 1) \end{aligned}$$

The integrating factor is

$$\begin{aligned} \text{I.F.} &= e^{\int P\, dx} \\ &= e^{\log(1-t^2)} \\ &= 1 - t^2 \\ &= 1 - \tan^2 x \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} y(1-\tan^2 x) &= \int \cos^2 x (1-\tan^2 x)\, dx + c \\ &= \int (\cos^2 x - \sin^2 x)\, dx + c \\ &= \int \cos 2x\, dx + c \\ &= \frac{1}{2}\sin 2x + c \end{aligned}$$

## Example 5.44

Solve $y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$.

***Solution:*** The given equation is

$$y + y + x\frac{dy}{dx} = x(\sin x + \log x)$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \sin x + \log x \qquad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{2}{x} dx} = x^2$$

Therefore the solution is

$$\begin{aligned} yx^2 &= \int x^2 (\sin x + \log x)\, dx + c \\ &= \int x^2 \sin x\, dx + \int x^2 \log x\, dx + c \\ &= x^2(-\cos x) - \int (-\cos x) 2x\, dx + \frac{x^3}{3}\log x - \int \frac{x^3}{3}\cdot\frac{1}{x} dx + c \\ &= -x^2 \cos x + 2\int x\cos x + \frac{x^3}{3}\log x - \frac{1}{9}x^3 + c \\ &= -x^2 \cos x + 2(\cos x + x\sin x) + \frac{x^3}{3}\log_e x - \frac{x^3}{9} + c \end{aligned}$$

**Example 5.45**

Solve $dx + x\,dy = e^{-y}\sec^2 y\,dy$.

***Solution:*** The given equation is

$$\frac{dx}{dy} + x = e^{-y}\sec^2 y \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int 1 dy} = e^y$$

Therefore, the solution is

$$xe^y = \int (e^{-y}\sec^2 y)e^y\,dy + c = \tan y + c$$

**Example 5.46**

Solve $(1+xy)\dfrac{dy}{dx} + y^3 = 0$.

***Solution:*** The given equation is

$$\frac{dx}{dy} + \frac{1+xy}{y^3} = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y^2} = -\frac{1}{y^3} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int dy/y^2} = e^{-(1/y)}$$

Therefore, the solution is

$$x e^{-(1/y)} = \int -\frac{1}{y^3} e^{-(1/y)}\,dy + c$$

$$= \int t e^t\,dt + c \quad \text{where } t = -\frac{1}{y}$$

$$= e^t(t-1) + c$$

Hence

$$x e^{-(1/y)} = e^{-(1/y)}\left(-\frac{1}{y} - 1\right) + c$$

or

$$x = c e^{1/y} - \left(1 + \frac{1}{y}\right)$$

## 5.3.5 Extended Form of Linear Equation (Bernoulli's Equation)

The equation

$$\frac{dy}{dx} + Py = Qy^n$$

where $n$ is rational, is called *Bernoulli's equation*. The value $n = 0$ gives linear equation. To solve it, divide the equation by $y^n$ and then use the substitution $y^{1-n} = z$ so that the resulting equation is linear in $z$.

**Example 5.47**

Solve $3\dfrac{dy}{dx} + \dfrac{2y}{x+1} = \dfrac{x^3}{y^2}$.

***Solution:*** Multiplying both sides with $y^2$ we get

$$3y^2\frac{dy}{dx} + \frac{2y^3}{x+1} = x^3$$

Put $y^3 = z$. Therefore

$$\frac{dz}{dx} + \frac{2}{x+1}z = x^3 \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{2}{x+1}dx} = e^{2\log|x+1|} = (x+1)^2$$

Therefore we have

$$z(x+1)^2 = \int x^3(x+1)^2\,dx + c$$

$$= \int x^3(x^2+2x+1)\,dx + c$$

So the solution is

$$y^3(x+1)^2 = \frac{1}{6}x^6 + \frac{2}{5}x^5 + \frac{1}{4}x^4 + c$$

## Example 5.48

Solve $x\dfrac{dy}{dx} + y\log y = xye^x$.

***Solution:*** Dividing both sides with $xy$ we get

$$\frac{1}{y}\frac{dy}{dx} + \frac{\log y}{x} = e^x$$

Put $\log y = z$. Therefore

$$\frac{dz}{dx} + \frac{z}{x} = e^x \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = x$$

Therefore

$$zx = \int x e^x \, dx + c = e^x(x-1) + c$$

The solution is

$$x \log y = e^x(x-1) + c$$

## Example 5.49

Solve $\dfrac{dy}{dx} + y\cot x = y^2 \sin^2 x$.

***Solution:*** Dividing both sides with $y^2$, we have

$$y^{-2}\frac{dy}{dx} + \frac{1}{y}\cot x = \sin^2 x$$

Put $1/y = z$. Therefore

$$-\frac{dz}{dx} + z\cot x = \sin^2 x$$

$$\Rightarrow \frac{dz}{dx} + (-\cot x)z = -\sin^2 x \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int -\cot x \, dx} = \frac{1}{\sin x}$$

Hence the solution is

$$z\left(\frac{1}{\sin x}\right) = \int -\sin^2 x\left(\frac{1}{\sin x}\right)dx + c$$
$$= \cos x + c$$

So

$$\frac{1}{y} = \sin x \cos x + c \sin x$$

## Example 5.50

Solve $\dfrac{dy}{dx} + \dfrac{1}{x}\sin 2y = x^3 \cos^2 y$.

***Solution:*** Dividing both sides with $\cos^2 y$, we have

$$\sec^2 y \frac{dy}{dx} + \frac{1}{x}(\sin 2y)\sec^2 y = x^3$$

$$\Rightarrow (\sec^2 y)\frac{dy}{dx} + \frac{2}{x}(\tan y) = x^3$$

Put $\tan y = z$. Then

$$\frac{dz}{dx} + \left(\frac{2}{x}\right)z = x^3 \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{2}{x}dx}$$
$$= e^{\log(x^2)}$$
$$= x^2$$

The solution is

$$z x^2 \int x^3(x^2)\,dx + c$$

$$\Rightarrow x^2 \tan y = \frac{1}{6}x^6 + c$$

## WORKED-OUT PROBLEMS

### Single Correct Choice Type Questions

**1.** $\int_0^{10} e^{x-[x]}\,dx$ ([·] denotes integral part) is equal to

(A) $10(e-1)$ (B) $\frac{e-1}{10}$ (C) $\frac{e^{10}-1}{10}$ (D) $\frac{e^{10}-1}{e-1}$

**Solution:** Let

$$I=\int_0^{10} e^{x-[x]}\,dx$$

Since $x-[x]$ is a function with least period unity, we have (by $P_4$)

$$I=10\int_0^1 e^{x-[x]}\,dx$$
$$=10\int_0^1 e^x\,dx \quad (\because 0\le x<1 \Rightarrow [x]=0)$$
$$=10(e-1)$$

**Answer: (A)**

**2.** $\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$ is equal to

(A) $\pi$ (B) $\frac{\pi}{2}$ (C) 0 (D) $-\pi$

**Solution:** Let

$$I=\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$$
$$=\int_0^{\pi}\frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}dx \quad (\text{By } P_1)$$
$$=\int_0^{\pi}\frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}dx$$

Therefore

$$2I=\int_0^{\pi}\frac{e^{\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}dx=\int_0^{\pi}1dx=\pi$$

This implies

$$I=\frac{\pi}{2}$$

**Answer: (B)**

**3.** $\int_0^{\pi}\frac{x^2\sin 2x\sin[(\pi/2)\cos x]}{2x-\pi}dx=$

(A) $\frac{4}{\pi^2}$ (B) $\frac{\pi^2}{4}$ (C) $\frac{\pi^2}{8}$ (D) $\frac{8}{\pi}$

**Solution:** Let $I$ be the given integral. Note that for $x=\pi/2$, the integrand is not defined. Put

$$t=\frac{\pi}{2}-x$$

Then

$$I=\int_{\pi/2}^{-\pi/2}\frac{\left(\frac{\pi}{2}-t\right)^2\sin(\pi-2t)\sin\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}-t\right)\right)(-1)\,dt}{-2t}$$
$$=-\int_{-\pi/2}^{\pi/2}\frac{\left(t^2-t\pi+\frac{\pi^2}{4}\right)\sin 2t\sin\left(\frac{\pi}{2}\sin t\right)}{2t}dt$$

Observe that

$$\frac{\left(t^2+\frac{\pi^2}{4}\right)\sin 2t\sin\left(\frac{\pi}{2}\sin t\right)}{2t}$$

is an odd function (see $P_3$). Therefore

$$I=-\frac{1}{2}\int_{-\pi/2}^{\pi/2}(-\pi)\sin 2t\sin\left(\frac{\pi}{2}\sin t\right)dt$$
$$=\frac{1}{2}\times 2\pi\int_0^{\pi/2}\sin 2t\sin\left(\frac{\pi}{2}\sin t\right)dt$$
$$=\pi\int_0^{\pi/2}2\sin t\cos t\sin\left(\frac{\pi}{2}\sin t\right)dt$$

Put $(\pi/2)\sin t=z$ so that $\cos t\,dt=(2/\pi)\,dz$. Therefore

$$I=\pi\int_0^{\pi/2}2\left(\frac{2z}{\pi}\right)\sin z\left(\frac{2}{\pi}\right)dz$$
$$=\frac{8}{\pi}\int_0^{\pi/2}z\sin z\,dz$$
$$=\frac{8}{\pi}\left[[\sin z]_0^{\pi/2}-[z\cos z]_0^{\pi/2}\right]$$

$$= \frac{8}{\pi}(1-0)$$
$$= \frac{8}{\pi}$$

**Answer: (D)**

**4.** If $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{12}$, then $x$ equals

(A) 4 (B) log 2 (C) 2 (D) 2log 2

**Solution:** Let

$$\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{12}$$
$$\Rightarrow \left[\mathrm{Sec}^{-1} t\right]_{\sqrt{2}}^{x} = \frac{\pi}{12}$$
$$\Rightarrow \mathrm{Sec}^{-1} x - \mathrm{Sec}^{-1}\sqrt{2} = \frac{\pi}{12}$$
$$\Rightarrow \mathrm{Sec}^{-1} x - \frac{\pi}{4} = \frac{\pi}{12}$$
$$\Rightarrow \mathrm{Sec}^{-1} x = \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}$$
$$\Rightarrow x = \sec\frac{\pi}{3} = 2$$

**Answer: (C)**

**5.** Solution of the equation

$$\int_{\log_e 2}^{x} \frac{dt}{\sqrt{e^t-1}} = \frac{\pi}{6}$$

is

(A) 4 (B) 2 (C) $\log_e 2$ (D) $\log_e 4$

**Solution:** Let

$$I = \int_{\log_e 2}^{x} \frac{dt}{\sqrt{e^t-1}}$$
$$= \int_{\log_e 2}^{x} \frac{e^t}{e^t\sqrt{e^t-1}}dt$$

Put $e^t - 1 = z^2$. Then $e^t dt = 2z\,dz$. Now

$$t = \log_e 2 \Rightarrow z = 1$$
$$t = x \Rightarrow z = \sqrt{e^x - 1}$$

Therefore

$$\int_{1}^{\sqrt{e^x-1}} \frac{1}{(z^2+1)z}(2z)dz = 2\int_{1}^{\sqrt{e^x-1}} \frac{dz}{z^2+1}$$
$$= 2(\mathrm{Tan}^{-1}\sqrt{e^x-1} - \mathrm{Tan}^{-1}1)$$
$$= \frac{\pi}{6} \quad \text{(By hypothesis)}$$

Hence

$$e^x - 1 = \left(\tan\frac{\pi}{3}\right)^2 = 3$$
$$\Rightarrow e^x = 4 \quad \text{or} \quad x = \log_e 4$$

**Answer: (D)**

**6.** The value of the integral

$$\int_{-1}^{1} \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}dx$$

is

(A) $\frac{\pi}{2}$ (B) 0 (C) $\frac{\pi}{4}$ (D) 2

**Solution:** We know that $x^7 - 3x^5 + 7x^3 - x$ is an odd function and $\cos^2 x$ is an even function. Hence the integrand is an odd function. Therefore, the value of the integral is zero.

**Answer: (B)**

**7.** $\int_{1}^{e^3} \frac{dx}{x\sqrt{1+\log_e x}} =$

(A) 2 (B) 4 (C) $2\sqrt{2}$ (D) $4\sqrt{2}$

**Solution:** Let

$$I = \int_{1}^{e^3} \frac{dx}{x\sqrt{1+\log_e x}}$$

Put $\log_e x = t$. Therefore $x = e^t$ so that $dx = e^t dt$. Now

$$x = 1 \Rightarrow t = 0$$
$$x = e^3 \Rightarrow t = 3$$

So

$$I = \int_{0}^{3} \frac{1}{e^t\sqrt{1+t}}e^t\,dt$$
$$= \int_{0}^{3} \frac{dt}{\sqrt{1+t}}$$

$$= \left[\frac{(1+t)^{-(1/2)+1}}{-(1/2)+1}\right]_0^3$$
$$= 2\left[\sqrt{1+t}\right]_0^3$$
$$= 2(\sqrt{1+3}-1)$$
$$= 2(2-1)$$
$$= 2$$

**Answer: (A)**

**8.** $\displaystyle\int_1^e \frac{1+\log_{10} x}{x}\,dx =$

(A) $\frac{1}{2}\log_{10} e$ (B) $\frac{1+\log_{10} e}{2}$

(C) $\frac{1}{2}\log_{10} e + 1$ (D) $2\log_{10} e$

**Solution:** We have

$$I = \int_1^e \frac{1+\log_{10} x}{x}$$
$$= \int_1^e \frac{1}{x}dx + \int_1^e \frac{\log_e x \cdot \log_{10} e}{x}dx$$
$$= [\log_e x]_1^e + (\log_{10} e)\frac{1}{2}\left[(\log_e x)^2\right]_1^e$$
$$= (1-0) + \frac{1}{2}\log_{10} e\,[(\log_e e)^2 - (\log_e 1)^2]$$
$$= 1 + \frac{1}{2}\log_{10} e$$

**Answer: (C)**

**9.** If $\displaystyle\int_0^\pi x f(\sin x)\,dx = k\int_0^{\pi/2} f(\sin x)\,dx$, the value of $k$ is

(A) $\frac{\pi}{2}$ (B) $\pi$ (C) $\frac{\pi}{3}$ (D) $2\pi$

**Solution:** We have

$$I = \int_0^\pi x f(\sin x)\,dx$$
$$= \int_0^\pi (\pi - x) f(\sin(\pi - x))\,dx$$
$$= \pi\int_0^\pi f(\sin x)\,dx - I$$

So

$$2I = \pi\int_0^\pi f(\sin x)\,dx$$
$$= \pi \times 2\int_0^{\pi/2} f(\sin x)\,dx \quad [\because \sin(\pi - x) = \sin x]$$

Hence

$$I = \pi\int_0^{\pi/2} f(\sin x)\,dx$$

So $k = \pi$.

**Answer: (B)**

**10.** $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx =$

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{2}$ (C) $\frac{\pi^2}{2\sqrt{2}}$ (D) $\frac{\pi^2}{4\sqrt{2}}$

**Solution:** We have

$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$$
$$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$
$$= \int_0^\pi \frac{\pi \sin x}{1+\cos^2 x}dx - I$$

Therefore

$$2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$$
$$= \pi \times 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx \quad [\because f(2a - x) = f(x)]$$

Hence

$$I = \pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$$
$$= \pi\int_1^0 \frac{-dt}{1+t^2} \quad \text{where } t = \cos x$$
$$= \pi\int_0^1 \frac{dt}{1+t^2}$$
$$= \pi\left[\mathrm{Tan}^{-1} t\right]_0^1$$

$$= \pi(\mathrm{Tan}^{-1}1 - \mathrm{Tan}^{-1}0)$$
$$= \frac{\pi^2}{4}$$

**Answer: (A)**

**11.** $\displaystyle\int_{1/e}^{\tan x} \frac{t}{1+t^2}dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} =$

(A) $2(\tan e - 1)$ (B) $2\tan e$
(C) 1 (D) $\tan e + \cot e$

**Solution:** Let

$$F(x) = \int_{1/e}^{\tan x} \frac{t}{1+t^2}dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$$

Then

$$F'(x) = \left(\frac{\tan x}{1+\tan^2 x}\right)\sec^2 x + \frac{1}{\cot x(1+\cot^2 x)}(-\mathrm{cosec}^2 x)$$
$$= \tan x - (1/\cot x)$$
$$= 0$$

Therefore $F$ is a constant function. Now

$$F\left(\frac{\pi}{4}\right) = \int_{1/e}^{1} \frac{t}{1+t^2}dt + \int_{1/e}^{1} \frac{1}{t(1+t^2)}dt$$
$$= \int_{1/e}^{1} \frac{t^2+1}{t(1+t^2)}dt$$
$$= [\log_e t]_{1/e}^{1}$$
$$= 0 - (0 - \log_e e)$$
$$= 1$$

Hence $F(x) = 1$.

**Answer: (C)**

**12.** $\displaystyle\int_0^{\sin^2 x} (\mathrm{Sin}^{-1}\sqrt{t})dt + \int_0^{\cos^2 x} (\mathrm{Cos}^{-1}\sqrt{t})dt =$

(A) $\pi$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2\sqrt{2}}$

**Solution:** Let

$$F(x) = \int_0^{\sin^2 x} (\mathrm{Sin}^{-1}\sqrt{t})dt + \int_0^{\cos^2 x} (\mathrm{Cos}^{-1}\sqrt{t})dt$$

Then

$$F'(x) = (\mathrm{Sin}^{-1}\sqrt{\sin^2 x})(2\sin x\cos x)$$
$$+ (\mathrm{Cos}^{-1}\sqrt{\cos^2 x})(-2\sin x\cos x)$$
$$= 2x\sin x\cos x - 2x\sin x\cos x$$
$$= 0$$

So $F(x)$ is a constant function. Now

$$F\left(\frac{\pi}{4}\right) = \int_0^{1/2} \mathrm{Sin}^{-1}\sqrt{t}\,dt + \int_0^{1/2} \mathrm{Cos}^{-1}\sqrt{t}\,dt$$
$$= \int_0^{1/2} (\mathrm{Sin}^{-1}\sqrt{t} + \mathrm{Cos}^{-1}\sqrt{t})dt$$
$$= \int_0^{1/2} \frac{\pi}{2}dt$$
$$= \frac{\pi}{4}$$

Therefore

$$F(x) = \frac{\pi}{4}$$

**Answer: (C)**

**13.** $\displaystyle\int_0^{\log_e 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx =$

(A) $3+\pi$ (B) $3-\pi$ (C) $4+\pi$ (D) $4-\pi$

**Solution:** Let

$$I = \int_0^{\log_e 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx$$

Put $e^x - 1 = t^2$ $(\because e^x > 1 \text{ for } x > 0)$. Therefore

$$e^x\,dx = 2t\,dt$$

Now

$$I = \int_0^2 \frac{t}{t^2+4}(2t)dt$$
$$= 2\int_0^2 \frac{t^2}{t^2+4}dt$$
$$= 2\int_0^2 \frac{t^2+4-4}{t^2+4}dt$$
$$= 2\int_0^2 \left(1 - \frac{4}{t^2+4}\right)dt$$
$$= 4 - \frac{8}{2}\left[\mathrm{Tan}^{-1}\left(\frac{t}{2}\right)\right]_0^2$$

$$= 4 - 4\,(\mathrm{Tan}^{-1}1 - \mathrm{Tan}^{-1}0)$$
$$= 4 - 4\left(\frac{\pi}{4} - 0\right)$$
$$= 4 - \pi$$

**Answer: (D)**

**14.** $\displaystyle\int_{-\pi/3}^{-\pi/6} \frac{dx}{1+\tan^4 x} =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{\pi}{2}$

**Solution:** Let

$$I = \int_{-\pi/3}^{-\pi/6} \frac{dx}{1+\tan^4 x}$$
$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\tan^4(-x)}$$
$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\tan^4 x}$$
$$= \int_{\pi/6}^{\pi/3} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}dx$$
$$= \int_{\pi/6}^{\pi/3} \frac{\cos^4\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}{\sin^4\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)+\cos^4\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}dx$$
$$= \int_{\pi/6}^{\pi/3} \frac{\sin^4 x}{\cos^4 x + \sin^4 x}dx$$

Therefore

$$2I = \int_{\pi/6}^{\pi/3} \frac{\cos^4 x + \sin^4 x}{\cos^4 x + \sin^4 x}dx$$
$$= \int_{\pi/6}^{\pi/3} 1\,dx$$
$$= \frac{\pi}{3} - \frac{\pi}{6}$$
$$= \frac{\pi}{6}$$

Hence

$$I = \frac{\pi}{12}$$

**Answer: (C)**

**15.** If

$$\int_0^x e^{zx}\cdot e^{-z^2}\,dz = f(x)\int_0^x e^{-z^2/4}\,dz$$

then

$$\int e^x\left(\log_e(f(x)) + \frac{x}{2}\right)dx =$$

(A) $\frac{xe^x}{2}+c$ (B) $\frac{x^2e^x}{4}+c$

(C) $\frac{x^2e^x}{2}+c$ (D) $\frac{xe^x}{4}+c$

**Solution:** Let

$$I = \int_0^x e^{zx}\cdot e^{-z^2}\,dz$$
$$= \int_0^x e^{zx-z^2}\,dz$$
$$= \int_0^x e^{\frac{x^2}{4}-\left(z-\frac{x}{2}\right)^2}\,dz$$
$$= e^{x^2/4}\int_0^x e^{-\left(z-\frac{x}{2}\right)^2}\,dz$$

Put $z-(x/2)=t$ so that $dz = dt$. Also

$$z = 0 \Rightarrow t = -x/2$$

and $$z = x \Rightarrow t = x/2$$

Therefore

$$I = e^{x^2/4}\int_{-x/2}^{x/2} e^{-t^2}\,dt$$
$$= e^{x^2/4}(2)\int_0^{x/2} e^{-t^2}\,dt$$
$$= e^{x^2/4}(2)\int_0^x e^{-u^2/4}\left(\frac{1}{2}\right)du \quad \text{where } t = \frac{u}{2}$$
$$= e^{x^2/4}\int_0^x e^{-u^2/4}\,du$$
$$= e^{x^2/4}\int_0^x e^{-z^2/4}\,dz$$

Therefore

$$f(x) = e^{x^2/4}$$

so that

$$\int e^x\left(\log_e(f(x))+\frac{x}{2}\right)dx=\int e^x\left(\frac{x^2}{4}+\frac{x}{2}\right)dx$$

$$=\frac{x^2e^x}{4}+c$$

**Answer: (B)**

**16.** If

$$I_1=\int_{-4}^{-5} e^{(x+5)^2}\,dx$$

and $$I_2=3\int_{1/3}^{2/3} e^{9[x-(2/3)]^2}\,dx$$

then the value of $I_1+I_2$ is

(A) 0 (B) 1 (C) $e^{-1}$ (D) $e$

**Solution:** Given

$$I_1=\int_{-4}^{-5} e^{(x+5)^2}\,dx$$

Put $t=x+5$ so that

$$I_1=\int_1^0 e^{t^2}\,dt=-\int_0^1 e^{t^2}\,dt$$

Again

$$I_2=3\int_{1/3}^{2/3} e^{9[x-(2/3)]^2}=3\int_{1/3}^{2/3} e^{(3x-2)^2}\,dx$$

Put $t=-3x+2$ so that $dx=-(1/3)dt$. Also

$$x=\frac{1}{3}\Rightarrow t=1$$

and $$x=\frac{2}{3}\Rightarrow t=0$$

Therefore

$$I_2=3\int_1^0 e^{t^2}\left(-\frac{1}{3}\right)dt=\int_0^1 e^{t^2}\,dt$$

So

$$I_1+I_2=-\int_0^1 e^{t^2}\,dt+\int_0^1 e^{t^2}\,dt=0$$

**Answer: (A)**

**QUICK LOOK**

Generally, if a relation between two integrals is being asked, try to transform both the integrals to have same limits of integration.

**17.** If $I=\int_0^1\frac{e^t}{t+1}dt$, then $\int_{a-1}^{a}\frac{e^{-t}}{t-a-1}dt=$

(A) $Ie^a$ (B) $(-I)e^a$ (C) $(-I)e^{-a}$ (D) $Ie^{-a}$

**Solution:** Let

$$J=\int_{a-1}^{a}\frac{e^{-t}}{t-a-1}dt$$

Put $t=x+a-1$ so that

$$t=a-1\Rightarrow x=0$$

and $$t=a\Rightarrow x=1$$

Therefore

$$J=\int_0^1\frac{e^{-(x+a-1)}}{x-2}dx$$

$$=e^{-a}\int_0^1\frac{e^{1-x}}{x-2}dx$$

Again, put $z=1-x$ so that

$$J=e^{-a}\int_1^0\frac{e^z}{-(1+z)}(-1)\,dz$$

$$=-e^{-a}\int_0^1\frac{e^z}{z+1}dz$$

$$=-e^{-a}(I)$$

$$=(-I)e^{-a}$$

**Answer: (C)**

**18.** $\int_{-2}^{2}|x(x-1)|\,dx=$

(A) $\frac{17}{3}$ (B) $\frac{11}{3}$ (C) $\frac{13}{3}$ (D) $\frac{16}{3}$

**Solution:** We have

$$I=\int_{-2}^{2}|x(x-1)|\,dx$$

$$=\int_{-2}^{0}x(x-1)+\int_0^1(x-x^2)\,dx+\int_1^2(x^2-x)\,dx$$

$$= \frac{1}{3}\left[x^3\right]_{-2}^{0} - \frac{1}{2}\left[x^2\right]_{-2}^{0} + \frac{1}{2}\left[x^2\right]_{0}^{1} - \frac{1}{3}\left[x^3\right]_{0}^{1} + \frac{1}{3}\left[x^3\right]_{1}^{2} - \frac{1}{2}\left[x^2\right]_{1}^{2}$$

$$= \frac{8}{3} + 2 + \frac{1}{2} - \frac{1}{3} + \frac{7}{3} - \frac{3}{2}$$

$$= \frac{8}{3} + 2 + \frac{3-2+14-9}{6}$$

$$= \frac{8}{3} + 2 + 1$$

$$= \frac{17}{3}$$

**Answer: (A)**

**19.** The maximum value of $\int_{a-1}^{a+1} e^{-(x-1)^2}\,dx$ is attained ($a$ is real) at

(A) $a = 2$  (B) $a = 1$

(C) $a = -1$  (D) $a = 0$

**Solution:** Let

$$F(a) = \int_{a-1}^{a+1} e^{-(x-1)^2}\,dx$$

Now by Leibnitz Rule (Theorem 5.21), we have

$$F'(a) = e^{-(a+1-1)^2} - e^{-(a-1-1)^2} = e^{-a^2} - e^{-(a-2)^2}$$

So

$$F'(a) = 0 \Rightarrow e^{a^2} = e^{(a-2)^2} \Rightarrow a = 1$$

Also

$$F''(a) = -2a e^{-a^2} + 2(a-2)e^{-(a-2)^2}$$

$$\Rightarrow F''(1) = -2e^{-1} - 2e^{-1} < 0$$

Therefore $F$ is maximum at $a = 1$.

**Answer: (B)**

**20.** $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = k\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$ where $k$ equals

(A) 2  (B) 4  (C) $\sqrt{2}$  (D) $2\sqrt{2}$

**Solution:** Let

$$I = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx \quad (5.9)$$

$$= \int_0^{\pi/2} f\left(\sin 2\left(\frac{\pi}{2} - x\right)\right)\sin\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \quad (5.10)$$

From Eqs. (5.9) and (5.10), we have

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\,dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right)dx$$

Now, put

$$t = -x + \frac{\pi}{4}$$

so that

$$x = 0 \Rightarrow t = +\frac{\pi}{4}$$

and $$x = \frac{\pi}{2} \Rightarrow t = -\frac{\pi}{4}$$

and $dx = -dt$. Therefore

$$2I = \sqrt{2}\int_{\pi/4}^{-\pi/4} f\left(\sin 2\left(\frac{\pi}{4} - t\right)\right)\sin\left(\frac{\pi}{4} - t + \frac{\pi}{4}\right)(-dt)$$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2t)\cos t\,dt$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2t)\cos t\,dt$$

($\because \cos x$ is an even function). Therefore

$$I = \sqrt{2}\int_0^{\pi/4} f(\cos 2t)\cos t\,dt$$

Hence

$$k = \sqrt{2}$$

**Answer: (C)**

**21.** $\int_0^1 |\sin 2\pi x|\, dx =$

(A) $\frac{1}{\pi}$ (B) $\frac{2}{\pi}$ (C) $\frac{4}{\pi}$ (D) $\pi$

**Solution:** Let

$$I = \int_0^1 |\sin 2\pi x| dx$$

Put $t = 2\pi x$. Now

$$x = 0 \Rightarrow t = 0$$

and $$x = 1 \Rightarrow t = 2\pi$$

Also

$$dx = \frac{1}{2\pi} dt$$

Therefore

$$\begin{aligned} I &= \frac{1}{2\pi}\int_0^{2\pi} |\sin t| dt \\ &= \frac{1}{2\pi}\int_0^{\pi} \sin t\, dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} (-\sin t) dt \\ &= \frac{1}{2\pi}[-\cos t]_0^{\pi} + \frac{1}{2\pi}[\cos t]_{\pi}^{2\pi} \\ &= -\frac{1}{2\pi}(-1-1) + \frac{1}{2\pi}[1-(-1)] \\ &= \frac{1}{\pi} + \frac{1}{\pi} \\ &= \frac{2}{\pi} \end{aligned}$$

**Answer: (B)**

**22.** $\int_0^{\pi/2} (\sin^4 x + \cos^4 x) dx =$

(A) $\frac{\pi}{8}$ (B) $\frac{3\pi}{8}$ (C) $\frac{5\pi}{8}$ (D) $\frac{7\pi}{8}$

**Solution:** Let

$$\begin{aligned} I &= \int_0^{\pi/2} (\sin^4 x + \cos^4 x) dx \\ &= \int_0^{\pi/2} (1 - 2\sin^2 x \cos^2 x) dx \\ &= \int_0^{\pi/2} \left(1 - \frac{1}{2}\sin^2 2x\right) dx \\ &= \int_0^{\pi/2} \left(1 - \frac{1}{2}\frac{(1-\cos 4x)}{2}\right) dx \\ &= \frac{1}{4}\int_0^{\pi/2} (3 + \cos 4x) dx \\ &= \frac{1}{4}\left[(3x)_0^{\pi/2} + \frac{1}{4}(\sin 4x)_0^{\pi/2}\right] \\ &= \frac{1}{4}\left[\frac{3\pi}{2} + \frac{1}{4}(0-0)\right] \\ &= \frac{3\pi}{8} \end{aligned}$$

**Answer: (B)**

> **Try it out** Since $\sin^4 x + \cos^4 x$ is of period $\pi/2$, the value of the integral
>
> $$\int_a^{a+\pi/2} (\sin^4 x + \cos^4 x) dx$$
>
> is also $3\pi/8$.

**23.** If $[t]$ stands for the integral part of $t$, then

$$\int_0^{5\pi/12} [\tan x] dx =$$

(A) $\frac{\pi}{2}$ (B) $\pi$ (C) $\frac{\pi}{4}$ (D) $2\pi$

**Solution:** We have

$$\tan\frac{5\pi}{12} = \tan 75° = 2 + \sqrt{3}$$

and $$\tan 0 = 0$$

$1, 2$ and $3$ are integers between $0$ and $2+\sqrt{3}$. Therefore

$$\begin{aligned} \int_0^{5\pi/12} [\tan x] dx &= \int_{\mathrm{Tan}^{-1} 1}^{\mathrm{Tan}^{-1} 2} [\tan x] dx + \int_{\mathrm{Tan}^{-1} 2}^{\mathrm{Tan}^{-1} 3} [\tan x] dx \\ &\quad + \int_{\mathrm{Tan}^{-1} 3}^{\mathrm{Tan}^{-1}(2+\sqrt{3})} [\tan x] dx \\ &= \int_{\mathrm{Tan}^{-1} 1}^{\mathrm{Tan}^{-1} 2} 1\, dx + \int_{\mathrm{Tan}^{-1} 2}^{\mathrm{Tan}^{-1} 3} 2\, dx + \int_{\mathrm{Tan}^{-1} 3}^{\mathrm{Tan}^{-1}(2+\sqrt{3})} 3 dx \end{aligned}$$

$$= (\mathrm{Tan}^{-1}2 - \mathrm{Tan}^{-1}1) + 2\,(\mathrm{Tan}^{-1}3 - \mathrm{Tan}^{-1}2)$$
$$+ 3\,[\mathrm{Tan}^{-1}(2+\sqrt{3}) - \mathrm{Tan}^{-1}3]$$
$$= -\frac{\pi}{4} - (\mathrm{Tan}^{-1}3 + \mathrm{Tan}^{-1}2) + 3\left(\frac{5\pi}{12}\right)$$
$$= \frac{5\pi}{4} - \frac{\pi}{4} - \left(\mathrm{Tan}^{-1}\left(\frac{3+2}{1-6}\right) + \pi\right) \quad (\because 6 > 1)$$
$$= \pi - [\mathrm{Tan}^{-1}(-1) + \pi]$$
$$= \pi + \frac{\pi}{4} - \pi$$
$$= \frac{\pi}{4}$$

**Answer: (C)**

**24.** $\displaystyle\int_{-1}^{1} \sqrt{\cos x}\, \log_e\left(\frac{1-x}{1+x}\right) dx =$

(A) 0 (B) 1 (C) 1/2 (D) 2

**Solution:** Let

$$I = \int_{-1}^{1} \sqrt{\cos x}\, \log_e\left(\frac{1-x}{1+x}\right) dx$$

Let

$$f(x) = \sqrt{\cos x}\, \log_e\left(\frac{1-x}{1+x}\right)$$

so that

$$f(-x) = \sqrt{\cos(-x)}\, \log_e\left(\frac{1+x}{1-x}\right)$$
$$= \sqrt{\cos x}\left[-\log_e\left(\frac{1-x}{1+x}\right)\right]$$
$$= -f(x)$$

Therefore $f$ is an odd function so that $I = 0$.

**Answer: (A)**

**25.** If

$$\int_0^{\pi} \frac{x^2}{(1+\sin x)^2} dx = A$$

then

$$\int_0^{\pi} \frac{2x^2 \cos^2(x/2)}{(1+\sin x)^2} dx =$$

(A) $A + \pi - \pi^2$ (B) $A - \pi + \pi^2$

(C) $A - \pi - \pi^2$ (D) $A + 2\pi - \pi^2$

**Solution:** Let

$$I = \int_0^{\pi} \frac{2x^2 \cos^2(x/2)}{(1+\sin x)^2} dx$$
$$= \int_0^{\pi} \frac{x^2(1+\cos x)}{(1+\sin x)^2} dx$$
$$= \int_0^{\pi} \frac{x^2}{(1+\sin x)^2} dx + \int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2} dx$$
$$= A + \int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2} dx$$

Therefore

$$I = A + \int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2} dx \qquad (5.11)$$

Let

$$I_1 = \int_0^{\pi} \frac{x^2 \cos x}{(1+\sin x)^2}$$

Take

$$u = x^2$$
$$dv = \frac{\cos x}{(1+\sin x)^2} dx$$

so that

$$v = \frac{-1}{1+\sin x}$$

Now using integration by parts, we have

$$I_1 = \left[x^2\left(\frac{-1}{1+\sin x}\right)\right]_0^{\pi} - \int_0^{\pi} (2x)\left(\frac{-1}{1+\sin x}\right) dx$$
$$I_1 = \pi^2(-1) - 0 + 2\int_0^{\pi} \frac{x}{1+\sin x} dx \qquad (5.12)$$
$$I_1 = -\pi^2 + 2\int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx$$
$$I_1 = -\pi^2 + 2\int_0^{\pi} \frac{\pi}{1+\sin x} - 2\int_0^{\pi} \frac{x}{1+\sin x} dx \qquad (5.13)$$

Adding Eqs. (5.12) and (5.13) we get

$$2I_1 = -2\pi^2 + 2\pi\int_0^{\pi}\frac{dx}{1+\sin x}$$

So

$$I_1 = -\pi^2 + \pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}\,dx$$

$$= -\pi^2 + \pi\left[(\tan x)_0^{\pi} - (\sec x)_0^{\pi}\right]$$

$$= -\pi^2 + \pi\,[0-(-1-1)]$$

$$= -\pi^2 + 2\pi \qquad (5.14)$$

Substituting the value of $I_1$ [from Eq. (5.14)] $= 2\pi - \pi^2$ in Eq. (5.11), we get

$$I = A + 2\pi - \pi^2$$

**Answer: (D)**

**26.** If $a$ and $b$ are real numbers different from zero, then

$$\int_0^{\pi/2}\frac{|ab|}{a^2\cos^2 x + b^2\sin^2 x}\,dx =$$

(A) $\frac{\pi}{2}$ (B) $\pi$ (C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{4}$

**Solution:** Let

$$I = \int_0^{\pi/2}\frac{|ab|}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$

$$= \int_0^{\pi/2}\frac{|ab|\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

$$= \frac{|ab|}{b^2}\int_0^{\pi/2}\frac{\sec^2 x}{(a/b)^2 + \tan^2 x}\,dx$$

$$= \frac{|a|}{|b|}\int_0^{\infty}\frac{dt}{(a/b)^2 + t^2} \quad \text{where } t = \tan x$$

$$= \frac{|a|}{|b|}\times\frac{1}{|a/b|}\left[\mathrm{Tan}^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$$

$$= \mathrm{Tan}^{-1}(\infty) - \mathrm{Tan}^{-1}(0)$$

$$= \frac{\pi}{2}$$

**Answer: (A)**

**27.** $\displaystyle\int_3^{29}\frac{\sqrt[3]{(x-2)^2}}{3+\sqrt[3]{(x-2)^2}}\,dx =$

(A) $4+\frac{3\sqrt{3}}{2}\pi$ (B) $2+\frac{3\sqrt{3}}{2}\pi$

(C) $4+\frac{\sqrt{3}}{2}\pi$ (D) $8+\frac{3\sqrt{3}}{2}\pi$

**Solution:** Put $(x-2)^{2/3} = t$. Therefore

$$x = t^{3/2} + 2$$

$$\Rightarrow dx = \frac{3}{2}t^{1/2}\,dt$$

Now

$$x = 3 \Rightarrow t = 1$$

and $$x = 29 \Rightarrow t = 9$$

Therefore

$$I = \int_3^{29}\frac{\sqrt[3]{(x-2)^2}}{3+\sqrt[3]{(x-2)^2}}\,dx$$

$$= \int_1^9\frac{t}{3+t}\left(\frac{3}{2}\sqrt{t}\right)dt$$

$$= \frac{3}{2}\int_1^9\frac{t\sqrt{t}}{3+t}\,dt$$

Now put $t = z^2$ so that $dt = (2z)\,dz$. Again

$$t = 1 \Rightarrow z = 1$$

and $$t = 9 \Rightarrow z = 3$$

Therefore

$$I = \frac{3}{2}\int_1^3\frac{z^3}{3+z^2}(2z)\,dz$$

$$= 3\int_1^3\frac{z^4}{z^2+3}\,dz$$

$$= 3\int_1^3\frac{z^4-9+9}{z^2+3}\,dz$$

$$= 3\int_1^3\left((z^2-3)+\frac{9}{z^2+3}\right)dz$$

$$= 3\left[\frac{1}{3}\left[z^3\right]_1^3 - 3[z]_1^3\right] + \frac{27}{\sqrt{3}}\left[\mathrm{Tan}^{-1}\frac{z}{\sqrt{3}}\right]_1^3$$

$$= 3\left[\frac{26}{3}-6\right] + \frac{27}{\sqrt{3}}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]$$

$$= 8 + \frac{3\sqrt{3}}{2}\pi$$

**Answer: (D)**

**28.** $\displaystyle\int_0^{1/\sqrt{3}} \frac{dx}{(2x^2+1)\sqrt{x^2+1}} =$

(A) $\dfrac{\pi}{2}$ (B) $\mathrm{Tan}^{-1}2$

(C) $\mathrm{Tan}^{-1}1/2$ (D) $\pi$

**Solution:** Let

$$I = \int_0^{1/\sqrt{3}} \frac{dx}{(2x^2+1)\sqrt{x^2+1}}$$

Put $t = 1/x$ or $x = 1/t$ so that

$$dx = \frac{-1}{t^2}\,dt$$

Now

$$x \to 0+ \Rightarrow t \to +\infty$$

and $$x \to \frac{1}{\sqrt{3}} \Rightarrow t \to \sqrt{3}$$

Therefore

$$I = \int_{\infty}^{\sqrt{3}} \frac{t^3}{(2+t^2)\sqrt{1+t^2}}\left(-\frac{1}{t^2}\right)dt$$

$$= \int_{\sqrt{3}}^{\infty} \frac{t}{(2+t^2)\sqrt{1+t^2}}\,dt$$

Now, put $1+t^2 = z^2$, so that $t\,dt = z\,dz$. Again

$$t = \sqrt{3} \Rightarrow z = 2$$

and $$t \to +\infty \Rightarrow z \to +\infty$$

Hence

$$I = \int_2^{\infty} \frac{dz}{(z^2+1)\,z}(z)$$

$$= \left[\mathrm{Tan}^{-1} z\right]_2^{\infty}$$

$$= \frac{\pi}{2} - \mathrm{Tan}^{-1}2$$

$$= \mathrm{Cot}^{-1}2 + \mathrm{Tan}^{-1}2 - \mathrm{Tan}^{-1}2$$

$$= \mathrm{Cot}^{-1}2 = \mathrm{Tan}^{-1}\frac{1}{2}$$

**Answer: (C)**

**29.** $\displaystyle\int_0^{\pi/2} \frac{dx}{1+(1/6)\sin^2 x} =$

(A) $\dfrac{\pi}{2}\sqrt{\dfrac{6}{7}}$ (B) $\pi\sqrt{\dfrac{6}{7}}$ (C) $\dfrac{2\pi}{7}$ (D) $\dfrac{\pi\sqrt{6}}{7}$

**Solution:** Let

$$I = \int_0^{\pi/2} \frac{6}{6+\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{6\sec^2 x}{6\sec^2 x + \tan^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{6\sec^2 x}{7\tan^2 x + 6}\,dx$$

$$= 6\int_0^{\infty} \frac{dt}{7t^2+6} \qquad \text{where } t = \tan x$$

$$= \frac{6}{7}\times\sqrt{\frac{7}{6}}\times\frac{\pi}{2} = \frac{\pi}{2}\sqrt{\frac{6}{7}}$$

**Answer: (A)**

**30.** $\displaystyle\int_0^{a} \frac{dx}{x+\sqrt{a^2-x^2}} =$

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$ (C) $\pi$ (D) $\dfrac{3\pi}{4}$

**Solution:** We have

$$I = \int_0^{a} \frac{dx}{x+\sqrt{a^2-x^2}}$$

Put $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$. Therefore

$$I = \int_0^{\pi/2} \frac{1}{a\sin\theta + a\cos\theta}(a\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta = \frac{\pi}{4}$$

**Answer: (B)**

**31.** If $I_n = \displaystyle\int_1^{e} (\log_e x)^n\,dx$ ($n$ is a positive integer), then $I_{2012} + (2012)I_{2011} =$

(A) $I_{2011} + (2010)I_{2010}$ (B) $I_{2013} + (2013)I_{2012}$

(C) $I_{2011} + (2011)I_{2009}$ (D) $I_{2012} - (2012)I_{2011}$

**Solution:** We have

$$I_n = \int_1^e (\log_e x)^n$$

Using integration by parts we get

$$I_n = \left[x(\log_e x)^n\right]_1^e - \int_1^e x(n\log_e^{n-1} x)\cdot\frac{1}{x}dx$$

$$= e - n\int_1^e (\log_e x)^{n-1}\,dx$$

$$= e - n\,I_{n-1}$$

Therefore

$$I_n + n\,I_{n-1} = e$$

Hence

$$I_{2012} + (2012)I_{2011} = e$$
$$= I_{2013} + (2013)I_{2012}$$

**Answer: (B)**

**32.** $\int_0^1 (1-x^2)^{3/2}\,dx =$

(A) $\dfrac{3\pi}{4}$ (B) $\dfrac{3\pi}{8}$ (C) $\dfrac{3\pi}{16}$ (D) $\dfrac{3\pi}{32}$

**Solution:** Put $x = \sin\theta$. Then

$$\int_0^1 (1-x^2)^{3/2}\,d\theta = \int_0^{\pi/2} (\cos^3\theta)(\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_0^{\pi/2}\frac{1+2\cos 2\theta+\cos^2 2\theta}{4}\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\,d\theta + \frac{1}{4}\int_0^{\pi/2}\frac{1+\cos 4\theta}{2}\,d\theta$$

$$= \frac{3}{8}\int_0^{\pi/2} d\theta + \frac{1}{4}[\sin 2\theta]_0^{\pi/2} + \frac{1}{8}\times\frac{1}{4}[\sin 4\theta]_0^{\pi/2}$$

$$= \frac{3}{8}\left(\frac{\pi}{2}\right) + \frac{1}{4}[\sin\pi - \sin 0] + \frac{1}{32}[\sin 2\pi - \sin 0]$$

$$= \frac{3\pi}{16} + 0 + 0 = \frac{3\pi}{16}$$

**Answer: (C)**

**33.** If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$, then $f(x)$ increases in

(A) $(-2, 2)$ (B) no value of $x$
(C) $(0, \infty)$ (D) $(-\infty, 0)$

**Solution:** We have

$$f'(x) = e^{-(x^2+1)^2}(2x) - e^{-x^4}(2x)$$
$$= 2xe^{-(x^2+1)^2}[1 - e^{2x^2+1}]$$

We know that $1 < e^{2x^2+1}$ for all real $x$. Therefore $f'(x) > 0$ if $x < 0$. So $f$ increases in $(-\infty, 0)$.

**Answer: (D)**

**34.** If $f(x)$ is differentiable and

$$\int_0^{t^2} xf(x)\,dx = \frac{2}{5}t^5$$

then $f(4/25) =$

(A) $\dfrac{2}{5}$ (B) $-\dfrac{5}{2}$ (C) 1 (D) $\dfrac{5}{2}$

**Solution:** Differentiating the given equation both sides with respect to $t$ we have

$$t^2 f(t^2)(2t) = 2t^4 \Rightarrow f(t^2) = t$$

Put $t = 2/5$ so that

$$f\left(\frac{4}{25}\right) = \frac{2}{5}$$

***Note:*** One may get the doubt why cannot we have $t = -2/5$. You should not take $t = -2/5$ as $t > 0$.

**Answer: (A)**

**35.** Let $f$ be a positive valued function. Let

$$I_1 = \int_{1-k}^{k} x\,f[x(1-x)]\,dx$$

and $$I_2 = \int_{1-k}^{k} f[x(1-x)]dx$$

then $I_1/I_2$ is

(A) 2 (B) $k$ (C) $\dfrac{1}{2}$ (D) 1

**Solution:** We have

$$I_1 = \int_{1-k}^{k} (k+1-k-x)f[(k+1-k-x)(1-k-1+k+x)]\,dx$$

$$\left(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right)$$

$$= \int_{1-k}^{k} (1-x)f((1-x)x)\,dx$$

$$= I_2 - I_1$$

Therefore

$$2I_1 = I_2$$

Hence

$$\frac{I_1}{I_2} = \frac{1}{2}$$

**Answer: (C)**

**36.** If

$$\int_0^x f(t)dt = x + \int_x^1 t\, f(t)\,dt$$

then $f(1)$ is

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) $\frac{-1}{2}$

**Solution:** Differentiating the given equation w.r.t. $x$, we have

$$f(x) = 1 + 0 - x\,f(x)$$

Therefore

$$f(x)(1+x) = 1$$

or

$$f(1) = \frac{1}{2}$$

**Answer: (A)**

**37.** $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx =$

(A) $\frac{\pi}{2}+1$ (B) $\frac{\pi}{2}-1$

(C) $-1$ (D) 1

**Solution:** We have

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \left[\mathrm{Sin}^{-1}x\right]_0^1 + \left[\sqrt{1-x^2}\right]_0^1$$

$$= (\mathrm{Sin}^{-1}1 - \mathrm{Sin}^{-1}0) + (\sqrt{1-1} - \sqrt{1-0})$$

$$= \frac{\pi}{2} - 1$$

***Note:*** The above integral can also be evaluated by using the substitution $x = \cos\theta$.

**Answer: (B)**

**38.** Let $f(x) = x - [x]$, where $[x]$ is the integral part of $x$. Then

$$\int_{-1}^{1} f(x)\,dx =$$

(A) 1 (B) 2 (C) 0 (D) $\frac{1}{2}$

**Solution:** We have

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx$$

$$= \int_{-1}^{0} (x-[x])\,dx + \int_0^1 (x-[x])\,dx$$

$$= \int_{-1}^{0} (x+1)\,dx + \int_0^1 x\,dx$$

$$= \int_{-1}^{1} x\,dx + \int_{-1}^{0} dx$$

$$= \frac{1}{2}\left[x^2\right]_{-1}^{1} + [x]_{-1}^{0}$$

$$= 0 + (0+1) = 1$$

**Answer: (A)**

**39.** If

$$I(m,n) = \int_0^1 t^m (1+t)^n\,dt$$

then the expression for $I(m,n)$ in terms of $I(m+1, n-1)$ is

(A) $\frac{2^n}{m+1} - \frac{n}{m+1} I\,(m+1, n-1)$

(B) $\dfrac{n}{m+1} I\,(m+1, n-1)$

(C) $\dfrac{2^n}{m+1} + \dfrac{n}{m+1} I\,(m+1, n-1)$

(D) $\dfrac{m}{n+1} I\,(m+1, n-1)$

**Solution:** Take $u = (1+t)^n$ an $dv = t^m\,dt$. Hence, using integration by parts we have

$$I\,(m,n) = \left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1 - \int_0^1 \frac{t^{m+1}}{m+1}\cdot n\,(1+t)^{n-1}\,dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} I\,(m+1, n-1)$$

**Answer: (A)**

**40.** If

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx, \quad n = 0, 1, 2, \ldots$$

then

(A) $I_n = \mathrm{I}_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = (20)\pi$

(C) $\sum_{m=1}^{10} I_{2m} = (10)\pi$ (D) $I_n = I_{n+1}$

**Solution:** We have

$$I_n = \int_{-\pi}^{\pi} \frac{\sin n\,(\pi - \pi - x)}{(1+\pi^{\pi-\pi-x})\sin(\pi-\pi-x)}\,dx$$

$$\left(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right)$$

$$= \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^{-x})\sin x}\,dx = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(\pi^x+1)\sin x}\,dx$$

Therefore

$$2I_n = \int_{-\pi}^{\pi} \frac{(1+\pi^x)\sin nx}{(1+\pi^x)\sin x}\,dx$$

$$= \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$$

This is because $\sin nx/\sin x$ is an even function. Hence

$$I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$$

Now

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin\,(n+2)x - \sin nx}{\sin x}\,dx$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \sin x}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \cos(n+1)x\,dx$$

$$= \frac{2}{n+1}[\sin(n+1)x]_0^{\pi}$$

$$= 0$$

Therefore

$$I_{n+2} = I_n$$

**Answer: (A)**

**41.** The value of $\displaystyle\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx,\ a > 0$ is

(A) $\pi$ (B) $a\pi$ (C) $\dfrac{\pi}{2}$ (D) $2\pi$

**Solution:** We have

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(-\pi+\pi-x)}{1+a^{-\pi+\pi-x}}\,dx \quad \left(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right)$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}}$$

$$= \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{a^x+1}\,dx$$

Therefore

$$2I = \int_{-\pi}^{\pi} \frac{(a^x+1)\cos^2 x}{a^x+1}\,dx$$

$$= \int_{-\pi}^{\pi} \cos^2 x$$

$$= 2\int_0^{\pi} \cos^2 x$$

$$= \int_0^{\pi} (1+\cos 2x)\, dx$$

$$= \pi + \frac{1}{2}[\sin 2x]_0^{\pi}$$

$$= \pi + 0$$

$$= \pi$$

Therefore

$$I = \frac{\pi}{2}$$

**Answer: (C)**

**42.** Let $f(x) = \int_1^x \sqrt{2-t^2}$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(A) $\pm 1$  (B) $\pm \frac{1}{\sqrt{2}}$

(C) $\pm \frac{1}{2}$  (D) 0 and 1

**Solution:** We have

$$x^2 - f'(x) = 0$$

$$\Rightarrow x^2 - \sqrt{2-x^2} = 0$$

$$\Rightarrow x^4 = 2 - x^2$$

$$\Rightarrow x^4 + x^2 - 2 = 0$$

$$\Rightarrow (x^2-1)(x^2+2) = 0$$

$$\Rightarrow x^2 = 1 \quad \text{or} \quad x = \pm 1$$

**Answer: (A)**

**43.** If $[x]$ represents integral part of $x$, then the value of the integral

$$\int_{-1/2}^{1/2} \left( [x] + \log_e \left( \frac{1+x}{1-x} \right) \right) dx$$

is

(A) $\frac{-1}{2}$  (B) 0

(C) 1  (D) $-2\log_e 2$

**Solution:** One can notice that $\log_e[(1+x)/(1-x)]$ is an odd function. Therefore the given integral is

$$\int_{-1/2}^{1/2} [x]\, dx = \int_{-1/2}^{0} (-1)\, dx + \int_0^{1/2} 0\, dx$$

$$= -[x]_{-1/2}^{0}$$

$$= -\left(0 + \frac{1}{2}\right)$$

$$= -\frac{1}{2}$$

**Answer: (A)**

**44.** If $f(x) = \int_0^x \cos^4 t\, dt$, then $f(x+\pi)$ equals

(A) $f(x) + f(\pi)$  (B) $f(x) - f(\pi)$

(C) $f(x)\, f(\pi)$  (D) $\frac{f(x)}{f(\pi)}$

**Solution:** We have

$$f(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt$$

$$= \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

$$= f(\pi) + \int_{\pi}^{x+\pi} \cos^4 t\, dt \qquad (5.15)$$

Let

$$I = \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

Put $y = t - \pi$ so that $dy = dt$. Now

$$t = \pi \Rightarrow y = 0$$

and $$t = x + \pi \Rightarrow y = x$$

Therefore

$$I = \int_0^x \cos^4(y+\pi)\, dy$$

$$= \int_0^x \cos^4 y\, dy$$

$$= \int_0^x \cos^4 t\, dt$$

$$= f(x)$$

Substituting the value of $I = f(x)$ in Eq. (5.15), we have

$$f(x+\pi) = f(\pi) + f(x)$$

**Answer: (A)**

**45.** Let $a, b, c$ be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x) \times (ax^2+bx+c)\,dx$$

Then the quadratic equation $ax^2+bx+c=0$ has

(A) no root in $(0, 2)$

(B) at least one root in $(0, 2)$

(C) a double root in $(0, 2)$

(D) two imaginary roots

**Solution:** By hypothesis

$$\int_1^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$$

Define

$$f(x) = \int_1^x (1+\cos^8 t)(at^2+bt+c)\,dt$$

so that $f$ is continuous on [1, 2] and differentiable in (1, 2) and $f(1) = 0 = f(2)$. Therefore, by Rolle's theorem $f'(x)=0$ for at least one value of $x$ in $(1, 2)$. Thus

$$(1+\cos^8 x)(ax^2+bx+c)=0$$

for at least one $x \in (1, 2)$. So $ax^2+bx+c=0$ has at least one root in $(1, 2)$.

**Answer: (B)**

**46.** If $a \neq b$ and

$$af(x)+bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$$

for all $x \neq 0$, then

$$\int_1^2 f(x)\,dx = \frac{1}{a^2-b^2}\left[a(\log 2 - \alpha) + \beta\left(\frac{b}{2}\right)\right]$$

where $\beta - \alpha$ is equal to

(A) 12 (B) 5 (C) 7 (D) 2

**Solution:** We have

$$a\,f(x) + b\,f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \tag{5.16}$$

Replacing $x$ with $1/x$ we get

$$b\,f(x) + a\,f\left(\frac{1}{x}\right) = x - 5 \tag{5.17}$$

From Eqs. (5.16) and (5.17), we obtain

$$(a^2-b^2)f(x) = a\left(\frac{1}{x}-5\right) - b(x-5)$$

Therefore

$$f(x) = \frac{1}{a^2-b^2}\left[a\left(\frac{1}{x}-5\right) - b(x-5)\right]$$

So

$$\int_1^2 f(x)\,dx = \frac{1}{a^2-b^2}\int_1^2 \left(\frac{a}{x} - bx + 5(b-a)\right)dx$$

$$= \frac{1}{a^2-b^2}\left[a\log_e 2 - \frac{b}{2}(4-1) + 5(b-a)(2-1)\right]$$

$$= \frac{1}{a^2-b^2}\left[a\log_e 2 - \frac{3b}{2} + 5(b-a)\right]$$

$$= \frac{1}{a^2-b^2}\left[a(\log_e 2 - 5) + \frac{7b}{2}\right]$$

Therefore $\alpha = 5, \beta = 7$.

**Answer: (D)**

**47.** For $n > 0$,

$$\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx =$$

(A) $\dfrac{\pi^2}{2}$ (B) $\dfrac{\pi^2}{4}$ (C) $\pi^2$ (D) $\dfrac{\pi^2}{2\sqrt{2}}$

**Solution:** Let

$$I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

Therefore

$$2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$= 4\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx \quad [\because f(2a-x) = f(x)]$$

$$= 8\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx \quad [\because f(2a-x) = f(x)]$$

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2$$

So $I = \pi^2$.

**Answer: (C)**

**48.** $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx =$

(A) 2 (B) $\frac{1}{2}$ (C) 3 (D) $\frac{1}{3}$

**Solution:** It is known that

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

Taking $f(x)=\sqrt{x}$, $a$ = 2 and $b$ = 3 in this equation we get

$$\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx = \frac{3-2}{2} = \frac{1}{2}$$

**Answer: (B)**

**49.** $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\,dx =$

(A) $2a$ (B) $a$ (C) $\frac{a}{2}$ (D) 1

**Solution:** Since

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

Now in the present case $a$ = 0 and $b$ =2$a$. Therefore the given integral is

$$\frac{2a-0}{2} = a$$

**Answer: (B)**

**50.** Let

$$F(x) = f(x) + f\left(\frac{1}{x}\right)$$

where

$$f(x) = \int_1^x \frac{\log_e t}{1+t}\,dt$$

Then $F(e)$ is equal to

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) 0

**Solution:** We have

$$F(x) = \int_1^x \frac{\log t}{1+t}\,dt + \int_1^{1/x} \frac{\log t}{1+t}\,dt$$

By Leibnitz Rule,

$$\begin{aligned} F'(x) &= \frac{\log x}{1+x} + \frac{\log(1/x)}{1+(1/x)}\left(-\frac{1}{x^2}\right) \\ &= \frac{\log x}{1+x} + \frac{\log x}{x(1+x)} \\ &= \frac{\log x}{x} \end{aligned}$$

Therefore

$$\begin{aligned} F(x) &= \int \frac{\log x}{x}\,dx \\ &= \frac{1}{2}(\log x)^2 + c \end{aligned}$$

But $F(1)=0 \Rightarrow c=0$. Therefore

$$F(x) = \frac{1}{2}(\log x)^2$$

$$\Rightarrow F(e) = \frac{1}{2}$$

***Note:*** The solution to Problem 50 is quite different from the routine types of solutions.

**Answer: (C)**

**51.** Let

$$f(x) = \frac{e^x}{1+e^x}$$

$$I_1 = \int_{f(-a)}^{f(a)} xg(x(1-x))\,dx$$

$$I_2 = \int_{f(-a)}^{f(a)} g(x(1-x))\,dx$$

then $I_2/I_1$ is

(A) 1 (B) –3 (C) –1 (D) 2

**Solution:** We have

$$\begin{aligned} f(a)+f(-a) &= \frac{e^a}{1+e^a} + \frac{e^{-a}}{1+e^{-a}} \\ &= \frac{e^a}{1+e^a} + \frac{1}{1+e^a} \\ &= 1 \end{aligned}$$

Therefore

$$I_1 = \int_{f(-a)}^{f(a)} (1-x)\, g((1-x)x)\, dx$$
$$= \int_{f(-a)}^{f(a)} g(x(1-x))\, dx - I_1$$
$$= I_2 - I_1$$

So

$$2I_1 = I_2 \quad \Rightarrow \frac{I_2}{I_1} = 2$$

**Answer: (D)**

**52.** $\displaystyle\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}\, dx =$

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) 2 (D) 1

**Solution:** We have

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx = \frac{b-a}{2}$$

Here $f(x) = \sqrt{x},\ a = 3,\ b = 6$. Therefore, the given integral is

$$\frac{6-3}{2} = \frac{3}{2}$$

**Answer: (B)**

**53.** $\displaystyle\int_{-2}^{3} |1-x^2|\, dx =$

(A) $\frac{1}{3}$ (B) $\frac{14}{3}$ (C) $\frac{7}{3}$ (D) $\frac{28}{3}$

**Solution:** Let

$$I = \int_{-2}^{3} |1-x^2|\, dx$$
$$= \int_{-2}^{-1} (x^2-1)\, dx + \int_{-1}^{1} (1-x^2)\, dx + \int_{1}^{3} (x^2-1)\, dx$$
$$= \frac{1}{3}\left[x^3\right]_{-2}^{-1} - [x]_{-2}^{-1} + [x]_{-1}^{1} - \frac{1}{3}\left[x^3\right]_{-1}^{1} + \frac{1}{3}\left[x^3\right]_{1}^{3} - [x]_1^3$$
$$= \frac{1}{3}(-1+8) - (-1+2) + (1+1) - \frac{1}{3}(1+1) + \frac{1}{3}(27-1) - (3-1)$$
$$= \frac{7}{3} - 1 + 2 - \frac{2}{3} + \frac{26}{3} - 2$$
$$= \frac{1}{3}(7-3+6-2+26-6)$$
$$= \frac{1}{3}(39-11)$$
$$= \frac{28}{3}$$

**Answer: (D)**

**54.** Let

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}}\, dx$$

and

$$J = \int_0^1 \frac{\cos x}{\sqrt{x}}\, dx$$

Then which of the following is true?

(A) $I > \frac{2}{3}$ and $J > 2$ (B) $I < \frac{2}{3}$ and $J < 2$

(C) $I < \frac{2}{3}$ and $J > 2$ (D) $I > \frac{2}{3}$ and $J < 2$

**Solution:** We know that

$$0 < x < 1 \Rightarrow \frac{\sin x}{x} < 1 \quad \text{(See Theorem 1.27)}$$

Therefore

$$\frac{\sin x}{\sqrt{x}} < \frac{x}{\sqrt{x}} = \sqrt{x}$$

Integrating we get

$$\int_0^1 \frac{\sin x}{\sqrt{x}}\, dx < \int_0^1 \sqrt{x}\, dx = \frac{2}{3}$$

So $I < 2/3$. Also

$$0 < x < 1 \Rightarrow \cos x < \frac{\sin x}{x} < 1 \quad \text{(See Theorem 1.27)}$$
$$\Rightarrow \frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}}$$

Therefore

$$J = \int_0^1 \frac{\cos x}{\sqrt{x}}\, dx$$
$$< \int_0^1 \frac{1}{\sqrt{x}}\, dx$$

$$= 2\left[x^{1/2}\right]_0^1$$
$$= 2$$

So

$$I < \frac{2}{3} \quad \text{and} \quad J < 2$$

**Answer: (B)**

**55.** Suppose $[x]$ denotes the integral part of $x$. Then the value of

$$\int_1^a [x] f'(x)\,dx, \quad a > 1$$

is

(A) $a\,f(a) - \{f(1) + f(2) + \cdots + f([a])\}$

(B) $[a]\,f(a) - \{f(1) + f(2) + \cdots + f([a])\}$

(C) $[a]\,f([a]) - \{f(1) + f(2) + \cdots + f(a)\}$

(D) $a\,f([a]) - \{f(1) + f(2) + \cdots + f(a)\}$

**Solution:** Let

$$I = \int_1^a [x] f'(x)\,dx$$

Then

$$I = \int_1^2 1\,f'(x)\,dx + \int_2^3 2\,f'(x)dx + \cdots + \int_{[a]}^a [a] f'(x)dx$$
$$= [f(2) - f(1)] + 2[f(3) - f(2)] + \cdots + [a][f(a) - f([a])]$$
$$= [a]\,f(a) - \{f(1) + f(2) + \cdots + f([a])\}$$

**Answer: (B)**

**56.** $\displaystyle\int_0^1 x(1-x)^n\,dx =$

(A) $\dfrac{1}{n+1} + \dfrac{1}{n+2}$ (B) $\dfrac{1}{n+1}$

(C) $\dfrac{1}{n+2}$ (D) $\dfrac{1}{n+1} - \dfrac{1}{n+2}$

**Solution:** We have

$$I = \int_0^1 x(1-x)^n\,dx$$
$$= \int_0^1 (1-x)\,x^n\,dx \quad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$
$$= \int_0^1 x^n dx - \int_0^1 x^{n+1} dx$$
$$= \frac{1}{n+1}\left[x^{n+1}\right]_0^1 - \frac{1}{n+2}\left[x^{n+2}\right]_0^1$$
$$= \frac{1}{n+1} - \frac{1}{n+2}$$

**Answer: (D)**

**57.** Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function which satisfies $f(x) + g(x) = x^2$. Then

$$\int_0^1 f(x)\,g(x)\,dx =$$

(A) $e + \dfrac{e^2}{2} + \dfrac{5}{2}$ (B) $e - \dfrac{e^2}{2} - \dfrac{5}{2}$

(C) $e + \dfrac{e^2}{2} - \dfrac{3}{2}$ (D) $e - \dfrac{e^2}{2} - \dfrac{3}{2}$

**Solution:** We have

$$f'(x) = f(x)$$
$$\Rightarrow f(x) = e^x + c$$

Now

$$f(0) = 1 \Rightarrow c = 0$$

Therefore

$$f(x) = e^x$$

So

$$g(x) = x^2 - e^x \quad [\because f(x) + g(x) = x^2]$$

Hence

$$\int_0^1 f(x)\,g(x)\,dx = \int_0^1 e^x(x^2 - e^x)\,dx$$
$$= \int_0^1 x^2 e^x\,dx - \int_0^1 e^{2x}\,dx$$
$$= \left[x^2 e^x - 2xe^x + 2e^x\right]_0^1 - \frac{1}{2}\left[e^{2x}\right]_0^1$$
$$= (e - 2e + 2e) - 2 - \frac{1}{2}(e^2 - 1)$$
$$= e - \frac{e^2}{2} - \frac{3}{2}$$

**Answer: (D)**

**58.** $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx =$

(A) $\dfrac{\pi^2}{4}$ (B) $\pi^2$ (C) 0 (D) $\dfrac{\pi}{2}$

**Solution:** We have

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

Now $\dfrac{2x}{1+\cos^2 x}$ is an odd function. This implies

$$\int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx = 0$$

Now

$$I = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

$$= 4\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \quad \left(\because \frac{x\sin x}{1+\cos^2 x} \text{ is even}\right)$$

$$= 4\int_{0}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$= 4\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I$$

So

$$2I = 4\pi\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= 8\pi\int_{0}^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= 8\pi\int_{0}^{1} \frac{dt}{1+t^2}$$

$$= 8\pi\left[\mathrm{Tan}^{-1} t\right]_0^1 \quad \text{where } t = \cos x$$

$$= 8\pi\left(\frac{\pi}{4}\right)$$

$$= 2\pi^2$$

This implies

$$I = \pi^2$$

**Answer: (B)**

**59.** If

$$I_n = \int_0^{\pi/4} \tan^n x\,dx$$

then $\lim_{n\to\infty} n\,(I_n + I_{n+2})$ equals

(A) $\dfrac{1}{2}$ (B) 1 (C) $+\infty$ (D) 0

**Solution:** Let

$$I_n = \int_0^{\pi/4} \tan^n x\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - I_{n-2}$$

$$= \frac{1}{n-1}\left[\tan^{n-1} x\right]_0^{\pi/4} - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

Therefore

$$I_n + I_{n-2} = \frac{1}{n-1}$$

Replacing $n$ with $n+2$, we have

$$I_n + I_{n+2} = \frac{1}{n+1}$$

Therefore

$$n\,(I_n + I_{n+2}) = \frac{n}{n+1} = \frac{1}{1+(1/n)}$$

So

$$\lim_{n\to\infty} n\,(I_n + I_{n+2}) = 1$$

**Answer: (B)**

**60.** If $f(a+b-x) = f(x)$, then $\displaystyle\int_a^b x\,f(x)\,dx =$

(A) $\dfrac{a+b}{2}\displaystyle\int_a^b f(a+b+x)\,dx$ (B) $\dfrac{a+b}{2}\displaystyle\int_a^b f(b-x)$

(C) $\dfrac{a+b}{2}\displaystyle\int_a^b f(x)\,dx$ (D) $\dfrac{b-a}{2}\displaystyle\int_a^b f(x)\,dx$

**Solution:** Let

$$I = \int_a^b x\,f(x)\,dx$$

$$= \int_a^b (a+b-x) f(a+b-x)\, dx$$

$$= \int_a^b (a+b-x) f(x)\, dx \quad [\because f(a+b-x) = f(x)]$$

Therefore

$$2I = \int_a^b (a+b) f(x)\, dx = (a+b)\int_a^b f(x) dx$$

This implies

$$I = \frac{a+b}{2}\int_a^b f(x)\, dx$$

**Answer: (C)**

**61.** Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying

$$f(x) = x + \int_0^x f(t)\, dt$$

for all $x\in\mathbb{R}$. Then, the number of elements in the set $S = \{x\in\mathbb{R} \mid f(x)=0\}$ is

(A) 1 (B) 2 (c) 3 (D) 4

**Solution:** Using Leibnitz Rule, we get

$$f'(x) = 1 + f(x)$$

If $y = f(x)$, then we have the differential equation

$$\frac{dy}{dx} - y = 1$$

which is a linear equation. The solution of the differential equation is

$$ye^{-x} = \int 1\, e^{-x} dx + c = -e^{-x} + c$$

This implies

$$y = -1 + c\, e^x$$

or $$f(x) = -1 + ce^x$$

Substituting the value of $f(x)$ in the given equation, we have

$$-1 + ce^x = x + \int_0^x (-1 + ce^t)\, dt$$

$$= x + [-x + c(e^x - 1)]$$

$$= ce^x - c$$

This implies $c = 1$. So

$$f(x) = e^x - 1$$

and $$f(x) = 0 \Leftrightarrow x = 0$$

Hence $S$ has only one element.

**Answer: (A)**

**62.** For each positive integer $n$, define

$$f_n(x) = \text{Min}\left(\frac{x^n}{n!}, \frac{(1-x)^n}{n!}\right)$$

for $0 \le x \le 1$. Let

$$I_n = \int_0^1 f_n(x)\, dx$$

for $n \ge 1$. Then the value of $\sum_{n=1}^{\infty} I_n$ is

(A) $2\sqrt{e} - 3$ (B) $2\sqrt{e} - 2$

(C) $2\sqrt{e} - 1$ (D) $2\sqrt{e}$

**Solution:** Let

$$f_n(x) = \begin{cases} \dfrac{x^n}{n!} & \text{for } 0 \le x \le \dfrac{1}{2} \\[2ex] \dfrac{(1-x)^n}{n!} & \text{for } \dfrac{1}{2} < x \le 1 \end{cases}$$

Therefore

$$I_n = \int_0^{1/2} \frac{x^n}{n!}\, dx + \int_{1/2}^1 \frac{(1-x)^n}{n!}\, dx$$

$$= \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} - \left[0 - \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}\right]$$

$$= 2\frac{(1/2)^{n+1}}{(n+1)!}$$

Hence

$$\sum_{n=1}^{\infty} I_n = 2\left[\frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \frac{(1/2)^4}{4!} + \cdots + \infty\right]$$

$$= 2\left[\left(1 + \frac{(1/2)}{1!} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \cdots + \infty\right) - 1 - \frac{1}{2}\right]$$

$$= 2\left[e^{1/2} - \frac{3}{2}\right] = 2\sqrt{e} - 3$$

**Answer: (A)**

**63.** Let $f(x)=x^3+ax^2+bx+c$, where $a$, $b$, $c$ are real numbers. If $f(x)$ has a local minimum at $x=1$ and a local maximum at $x=-1/3$ and $f(2)=0$, then

$$\int_{-1}^{1} f(x)\,dx =$$

(A) $\frac{14}{3}$ (B) $-\frac{14}{3}$ (C) $\frac{7}{3}$ (D) $-\frac{7}{3}$

**Solution:** We have

$$f(2)=0 \Rightarrow 4a+2b+c=-8 \tag{5.18}$$

$$f'(1)=0 \Rightarrow 2a+b=-3 \tag{5.19}$$

$$f'\left(\frac{-1}{3}\right)=0 \Rightarrow -2a+3b=-1 \tag{5.20}$$

Solving Eqs. (5.18)–(5.20), we have $a=-1$, $b=-1$, and $c=-2$. Therefore

$$f(x)=x^3-x^2-x-2$$

Now

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^3\,dx - \int_{-1}^{1} x^2\,dx - \int_{-1}^{1} x\,dx - 2\int_{-1}^{1} dx$$

$$= 0-\frac{2}{3}(1-0)-0-2(1+1)$$

$$= -\frac{2}{3}-4$$

$$= -\frac{14}{3}$$

**Answer: (B)**

**64.** $$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\frac{1}{\sqrt{4n^2-3^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}\right)=$$

(A) $\frac{1}{4}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

**Solution:** We have

$$T_r = r\text{th term} = \frac{1}{\sqrt{4n^2-r^2}}$$

Put $n=1/h$ so that

$$T_r = \frac{h}{\sqrt{4-x^2}}$$

where $x=rh$. Since the sum contains $n$ terms, the required limit is

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\operatorname{Sin}^{-1}\frac{x}{2}\right]_0^1$$

$$= \operatorname{Sin}^{-1}\frac{1}{2}$$

$$= \frac{\pi}{6}$$

**Answer: (D)**

**65.** If $[t]$ denotes the integral part of $t$, then

$$\int_0^1 \cos(\pi x)\cos([2x]\pi)\,dx =$$

(A) 1 (B) $-1$ (c) $-\frac{2}{\pi}$ (D) $\frac{2}{\pi}$

**Solution:** We have

$$[2x]=\begin{cases} 0 & \text{for } 0\le x<\frac{1}{2} \\ 1 & \text{for } \frac{1}{2}\le x<1 \end{cases}$$

Therefore

$$\int_0^1 \cos(\pi x)\cos([2x]\pi)\,dx = \int_0^{1/2}\cos(\pi x)\,dx + \int_{1/2}^{1}\cos(\pi x)(-1)\,dx$$

$$= \frac{1}{\pi}[\sin(\pi x)]_0^{1/2} - \frac{1}{\pi}[\sin(\pi x)]_{1/2}^{1}$$

$$= \frac{1}{\pi}-\frac{1}{\pi}(0-1)$$

$$= \frac{2}{\pi}$$

**Answer: (D)**

**66.** $$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\operatorname{Cos}^{-1}\left(\frac{2x}{1+x^2}\right)+\operatorname{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)}{e^x+1}\,dx =$$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{4\sqrt{3}}$ (D) $\frac{\pi}{2\sqrt{3}}$

**Solution:** Let $I$ be the given integral. Then

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{[(\pi/2)-2\operatorname{Tan}^{-1}x]+2\operatorname{Tan}^{-1}x}{e^x+1}\,dx$$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{e^x+1}$$

$$= \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{e^{(1/\sqrt{3})-(1/\sqrt{3})-x}+1}\, dx$$

$$= \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{dx}{e^{-x}+1}$$

$$= \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{e^x}{1+e^x}\, dx$$

So

$$2I = \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} 1\, dx$$

$$= \frac{\pi}{2}\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right) = \frac{\pi}{\sqrt{3}}$$

Hence

$$I = \frac{\pi}{2\sqrt{3}}$$

**Answer: (D)**

**67.** $\displaystyle\int_0^{1/2} e^x \left(\mathrm{Sin}^{-1}x - \frac{x}{(1-x^2)^{3/2}}\right) dx =$

(A) $\sqrt{e}\left(\frac{\pi}{6}-\frac{1}{\sqrt{3}}\right)+1$ (B) $\sqrt{e}\left(\frac{\pi}{6}+\frac{2}{\sqrt{3}}\right)+1$

(C) $\sqrt{e}\left(\frac{\pi}{6}-\frac{2}{\sqrt{3}}\right)+1$ (D) $\sqrt{e}\left(\frac{\pi}{6}+\frac{1}{\sqrt{3}}\right)+1$

**Solution:** Let

$$I = \int_0^{1/2} e^x \left(\mathrm{Sin}^{-1}x - \frac{x}{(1-x^2)^{3/2}}\right) dx$$

$$= \int_0^{1/2} e^x \left(\mathrm{Sin}^{-1}x + \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} - \frac{x}{(1-x^2)^{3/2}}\right) dx$$

$$= \int_0^{1/2} e^x \left(\mathrm{Sin}^{-1}x + \frac{1}{\sqrt{1-x^2}}\right) dx - \int_0^{1/2} e^x \left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}}\right) dx$$

$$= \left[e^x \mathrm{Sin}^{-1}x\right]_0^{1/2} - \left[\frac{e^x}{\sqrt{1-x^2}}\right]_0^{1/2}$$

$$= \left(\sqrt{e}\,\frac{\pi}{6} - 0\right) - \left[\frac{\sqrt{e}}{\sqrt{1-(1/4)}} - \frac{1}{1}\right]$$

$$= \frac{\pi\sqrt{e}}{6} - \frac{2\sqrt{e}}{\sqrt{3}} + 1$$

$$= \sqrt{e}\left(\frac{\pi}{6} - \frac{2}{\sqrt{3}}\right) + 1$$

**Answer: (C)**

**68.** The function

$$F(x) = \int_{\pi/6}^{x} (4\sin t + 3\cos t)\, dt$$

attains least value on $[\pi/4, 3\pi/4]$ at $x$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{4}$ (D) $\frac{\pi}{4}$

**Solution:** We have

$$F'(x) = 4\sin x + 3\cos x = 5\sin\left(x + \mathrm{Tan}^{-1}\frac{3}{4}\right)$$

Now

$$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} + \mathrm{Tan}^{-1}\frac{3}{4} \le x + \mathrm{Tan}^{-1}\frac{3}{4} \le \frac{3\pi}{4} + \mathrm{Tan}^{-1}\frac{3}{4}$$

$$\Rightarrow \frac{\pi}{2} < x + \mathrm{Tan}^{-1}\frac{3}{4} < \pi$$

Hence

$$F'(x) = 5\sin\left(x + \mathrm{Tan}^{-1}\frac{3}{4}\right) > 0$$

So $F$ is increasing in $[\pi/4, 3\pi/4]$ and $F$ is least at $x = \pi/4$.

**Answer: (D)**

**69.** $\displaystyle\int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\, dx =$

(A) $\frac{2}{\pi^2}$ (B) $\frac{8}{\pi^2}$ (C) $\frac{2}{\pi}$ (D) $\frac{8}{\pi}$

**Solution:** Let

$$I = \int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\, dx$$

Put $x = (\pi/2) + t$ so that $dx = dt$. Therefore

$$I = \int_{-\pi/2}^{\pi/2} \frac{\left(\frac{\pi}{2}+t\right)\sin(\pi+2t)\sin\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}+t\right)\right)}{2t}\, dt$$

$$= \frac{\pi}{2}\int_{-\pi/2}^{\pi/2} \frac{(-\sin 2t)\sin\left(-\frac{\pi}{2}\sin t\right)}{2t}\, dt + \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right) dt$$

$$= \frac{\pi}{4}(0) + \frac{1}{2} \times 2 \int_0^{\pi/2} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right) dt$$

$$= 2\int_0^{\pi/2} \sin t \cos t \sin\left(\frac{\pi}{2}\sin t\right) dt$$

Now, put $(\pi/2)\sin t = z$, so that $\cos t\, dt = (2/\pi)\, dz$. Therefore

$$I = 2\int_0^{\pi/2} \frac{2z}{\pi}(\sin z) \times \left(\frac{2}{\pi}\right) dz$$

$$= \frac{8}{\pi^2}\int_0^{\pi/2} z \sin z\, dz$$

$$= \frac{8}{\pi^2}[z(-\cos z) + \sin z]_0^{\pi/2}$$

$$= \frac{8}{\pi^2}(0 + 1 - 0)$$

$$= \frac{8}{\pi^2}$$

**Answer: (B)**

**70.** If $f(x) = \int_{1/x}^{\sqrt{x}} \sin(t^2)\, dt$ then $f'(1)$ is

(A) $\frac{3}{2}$ (B) $\sin 1$ (C) $\frac{1}{2}\sin 1$ (D) $\frac{3}{2}\sin 1$

**Solution:** We have

$$f'(x) = [\sin(\sqrt{x})^2]\left(\frac{1}{2\sqrt{x}}\right) - \sin\left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right)$$

(by Leibnitz Rule, Theorem 5.21). Therefore

$$f'(1) = \frac{1}{2}\sin 1 + \sin 1 = \frac{3}{2}\sin 1$$

**Answer: (D)**

**71.** $\int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})} =$

(A) $\frac{1}{\sqrt{11}}\log_e\left(\frac{\sqrt{11}+2}{\sqrt{11}}\right)$ (B) $\frac{1}{\sqrt{11}}\log_e\left(\frac{\sqrt{11}+1}{\sqrt{10}}\right)$

(C) $\frac{1}{\sqrt{10}}\log_e\left(\frac{\sqrt{10}+2}{\sqrt{11}}\right)$ (D) $\frac{1}{\sqrt{10}}\log_e\left(\frac{\sqrt{10}+1}{\sqrt{11}}\right)$

**Solution:** Let

$$I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$$

$$= \int_0^1 \frac{dx}{[5+2(1-x)-2(1-x^2)](1+e^{2-4(1-x)})} \quad (\text{By } P_5)$$

$$= \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{-(2-4x)})}$$

$$= \int_0^1 \frac{e^{2-4x}}{(5+2x-2x^2)(e^{2-4x}+1)} dx$$

Therefore

$$2I = \int_0^1 \frac{1+e^{2-4x}}{(5+2x-2x^2)(1+e^{2-4x})} dx$$

$$= \int_0^1 \frac{dx}{5+2x-2x^2}$$

This implies

$$I = \frac{1}{4}\int_0^1 \frac{dx}{(5/2)+x-x^2}$$

$$= \frac{1}{4}\int_0^1 \frac{dx}{-[x-(1/2)]^2 + (5/2) + (1/4)}$$

$$= \frac{1}{4}\int_0^1 \frac{dx}{(11/4) - [x-(1/2)]^2}$$

$$= \frac{1}{4} \times \frac{1}{2\sqrt{11/2}}\left[\log_e\left|\frac{\frac{\sqrt{11}}{2}+x-\frac{1}{2}}{\frac{\sqrt{11}}{2}-x+\frac{1}{2}}\right|\right]_0^1$$

$$= \frac{1}{4\sqrt{11}}\left[\log_e\left|\frac{\frac{\sqrt{11}}{2}+1-\frac{1}{2}}{\frac{\sqrt{11}}{2}-1+\frac{1}{2}}\right| - \log_e\left|\frac{\frac{\sqrt{11}}{2}-\frac{1}{2}}{\frac{\sqrt{11}}{2}+\frac{1}{2}}\right|\right]$$

$$= \frac{1}{4\sqrt{11}}\left[\log_e\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right) - \log_e\left(\frac{\sqrt{11}-1}{\sqrt{11}+1}\right)\right]$$

$$= \frac{1}{4\sqrt{11}}\log_e\left(\frac{(\sqrt{11}+1)^2}{(\sqrt{11}-1)^2}\right)$$

$$= \frac{1}{2\sqrt{11}}\log_e\frac{\sqrt{11}+1}{\sqrt{11}-1}$$

$$= \frac{1}{2\sqrt{11}}\log_e\left(\frac{(\sqrt{11}+1)^2}{11-1}\right)$$

$$= \frac{1}{\sqrt{11}}\log_e\left(\frac{\sqrt{11}+1}{\sqrt{10}}\right)$$

**Answer: (B)**

**72.** Let

$$f(x)=\begin{cases}\int_0^x \{5+|1-y|\}dy & \text{if } x>2\\ 5x+1 & \text{if } x\le 2\end{cases}$$

Then

(A) $f(x)$ is continuous but not differentiable at $x=2$.
(B) $f(x)$ is not continuous at $x=2$.
(C) $f(x)$ is differentiable everywhere.
(D) The right derivative of $f(x)$ at $x=2$ does not exist.

**Solution:** For $x>2$,

$$\int_0^x (5+|1-y|)dy = \int_0^1 (5+1-y)dy+\int_1^x (5+y-1)dy$$
$$=\int_0^1 (6-y)dy+\int_1^x (4+y)dy$$
$$=6-\frac{1}{2}+4(x-1)+\frac{1}{2}(x^2-1)$$
$$=\frac{1}{2}x^2+4x+1$$

Therefore

$$f(x)=\begin{cases}5x+1 & \text{for } x\le 2\\ (1/2)x^2+4x+1 & \text{for } x>2\end{cases}$$

Now

$$\lim_{x\to 2-0} f(x)=5(2)+1=11$$

and

$$\lim_{x\to 2+0} f(x)=\frac{4}{2}+8+1=11$$

So $f$ is continuous at $x=2$. Again

$$Lf'(2)=5 \quad\text{and}\quad Rf'(2)=2+4=6$$

Therefore $f$ is not differentiable at $x=2$.

**Answer: (A)**

**73.** $\int_{-4}^{3}\left|x^2-4\right|dx=$

(A) $\frac{55}{6}$ (B) $\frac{55}{3}$ (C) $\frac{71}{3}$ (D) $\frac{71}{6}$

**Solution:** We have $x^2-4>0$ if either $x<-2$ or $x>2$. Also

$$x^2-4<0 \Leftrightarrow -2<x<2$$

Therefore

$$\int_{-4}^{3}|x^2-4|dx=\int_{-4}^{-2}(x^2-4)dx+\int_{-2}^{2}(4-x^2)dx+\int_{2}^{3}(x^2-4)dx$$

$$=\left[\frac{x^3}{3}-4x\right]_{-4}^{-2}+\left[4x-\frac{x^3}{3}\right]_{-2}^{2}+\left[\frac{x^3}{3}-4x\right]_{2}^{3}$$
$$=\frac{1}{3}(-8+64)-4(-2+4)+4(2+2)-\frac{1}{3}(8+8)$$
$$+\frac{1}{3}(27-8)-4(3-2)$$
$$=\frac{1}{3}(56-16+19)-8+16-4$$
$$=\frac{59}{3}+4=\frac{71}{3}$$

**Answer: (C)**

**74.** If

$$f(x)=\begin{cases}e^{\cos x}\sin x & \text{for } |x|\le 2\\ 2 & \text{otherwise}\end{cases}$$

then $\int_{-2}^{3} f(x)dx=$

(A) 0 (B) 1 (C) 2 (D) 3

**Solution:** In $[-2, 2]$, the function $e^{\cos x}\sin x$ is an odd function. Therefore

$$\int_{-2}^{3} f(x)dx=\int_{-2}^{2} e^{\cos x}\sin x\,dx+\int_{2}^{3}2\,dx$$
$$=0+2(3-1)=2$$

**Answer: (C)**

**75.** If $\int_{-\pi/3}^{\pi/3}\frac{\pi+4x^3}{2-\cos[|x|+(\pi/3)]}dx=\frac{4\pi}{\sqrt{3}}\mathrm{Tan}^{-1}(k)$, then $k$ equals

(A) 1 (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}$

**Solution:** We know that

$$\frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}$$

is an odd function. Therefore, the given integral equals

$$2\pi\int_0^{\pi/3}\frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$
$$=2\pi\int_{\pi/3}^{2\pi/3}\frac{dt}{2-\cos t} \quad \text{where } t=x+\frac{\pi}{3}$$
$$=2\pi\int_{\pi/3}^{2\pi/3}\frac{1+\tan^2\left(\frac{t}{2}\right)}{2\left(1+\tan^2\frac{t}{2}\right)-\left(1-\tan^2\frac{t}{2}\right)}dt$$

$$= 4\pi \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dz}{1+3z^2} \quad \text{where } z = \tan\frac{t}{2}$$

$$= \frac{4\pi}{\sqrt{3}}\left[\text{Tan}^{-1} z\sqrt{3}\right]_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}}(\text{Tan}^{-1}3 - \text{Tan}^{-1}1)$$

$$= \frac{4\pi}{\sqrt{3}}\text{Tan}^{-1}\frac{1}{2}$$

So $k = 1/2$.

**Answer: (B)**

**76.** $\displaystyle\int_0^{\pi/2} \frac{\sin 8x \log_e(\cot x)}{\cos 2x} dx =$

(A) 0 (B) 1 (C) $\dfrac{\pi}{2}$ (D) $\dfrac{\pi}{4}$

**Solution:** We have

$$I = \int_0^{\pi/2} \frac{\sin 8x \log_e(\cot x)}{\cos 2x} dx$$

$$= \int_0^{\pi/2} \frac{\sin(4\pi - 8x)\log_e\left(\cot\left(\frac{\pi}{2} - x\right)\right)}{\cos 2\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{-\sin 8x \log_e(\tan x)}{-\cos 2x} dx$$

$$= \int_0^{\pi/2} \frac{\sin 8x \log(1/\cot x)}{\cos 2x} dx$$

$$= -I$$

Thus, $2I = 0$ or $I = 0$.

**Answer: (A)**

**77.** $\displaystyle\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} dx =$

(A) $\dfrac{\pi^2}{ab}$ (B) $\dfrac{\pi}{2ab}$ (C) $\dfrac{\pi^2}{2}$ (D) $\dfrac{\pi^2}{2ab}$

**Solution:**

$$I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x} dx$$

$$= \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x} dx$$

$$= \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} - I$$

So

$$2I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \pi \times 2\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Therefore

$$I = \pi\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \pi\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x} dx$$

$$= \frac{\pi}{b^2}\int_0^{\pi/2} \frac{\sec^2 x}{(a/b)^2 + \tan^2 x} dx$$

$$= \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{(a/b)^2 + t^2} \quad \text{where } t = \tan x$$

$$= \frac{\pi}{b^2} \times \frac{1}{a/b}\left[\text{Tan}^{-1}\frac{tb}{a}\right]_0^{\infty}$$

$$= \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi^2}{2ab}$$

**Answer: (D)**

**78.** If

$$f(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{t}}{1+\sin^2\sqrt{t}} dt$$

then $f'(\pi)$ is equal to

(A) 0 (B) $\pi$ (C) $2\pi$ (D) $\dfrac{\pi}{2}$

**Solution:** We have

$$f(x) = \cos x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{t}}{1+\sin^2\sqrt{t}} dt$$

Then

$$f'(x) = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{t}}{1+\sin^2\sqrt{t}} dt + \cos x\left[\frac{\cos x}{1+\sin^2 x}\right](2x)$$

Hence

$$f'(\pi) = 0 + \cos(\pi)\left[\frac{\cos\pi}{1+\sin^2\pi}\right](2\pi) = 2\pi$$

**Answer: (C)**

**79.** $\displaystyle\int_0^1 \mathrm{Cot}^{-1}(1-x+x^2)\,dx =$

(A) $\dfrac{\pi}{2} - \log_e 2$  (B) $\dfrac{\pi}{2} + \log_e 2$

(C) $\pi - \log_e 2$  (D) $\pi + \log_e 2$

**Solution:** We have

$$\begin{aligned}
I &= \int_0^1 \mathrm{Cot}^{-1}(1-x+x^2)\,dx \\
&= \int_0^1 \mathrm{Tan}^{-1}\left(\frac{1}{1-x+x^2}\right)dx \\
&= \int_0^1 \mathrm{Tan}^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right)dx \\
&= \int_0^1 [\mathrm{Tan}^{-1}x - \mathrm{Tan}^{-1}(x-1)]\,dx \\
&= \int_0^1 \mathrm{Tan}^{-1}x\,dx - \int_0^1 \mathrm{Tan}^{-1}(x-1)\,dx \\
&= \int_0^1 \mathrm{Tan}^{-1}x - \int_0^1 \mathrm{Tan}^{-1}(1-x-1)\,dx \\
&= 2\int_0^1 \mathrm{Tan}^{-1}x\,dx \\
&= 2\left\{\left[x\mathrm{Tan}^{-1}x\right]_0^1 - \int_0^1 \frac{x\,dx}{1+x^2}\right\} \\
&= 2\left(\frac{\pi}{4} - 0\right) - 2\times\frac{1}{2}\left[\log_e(1+x^2)\right]_0^1 \\
&= \frac{\pi}{2} - \log_e 2
\end{aligned}$$

**Answer: (A)**

**80.** Consider the function

$$f(x) = \int_0^x [t]\,dt$$

where $x > 0$ and $[t]$ is the integral part of $t$. Then

(A) $f(x)$ is not defined for $x = 1, 2, 3, \ldots$

(B) $f(x)$ is defined for all $x > 0$ but is not continuous at $x = 1, 2, 3, \ldots$

(C) $f(x)$ is continuous for all $x > 0$ but is not differentiable for $x = 1, 2, 3, \ldots$

(D) $f(x)$ is differentiable for all $x > 0$

**Solution:** We have

$$\begin{aligned}
f(x) &= \int_0^1 0\,dt + \int_1^2 1\,dt + \int_2^3 2\,dt + \cdots + \int_{[x]}^x ([x])\,dt \\
&= 0 + (2-1) + 2(3-2) + 3(4-3) + \cdots + [x](x-[x]) \\
&= (1+2+3+\cdots+[x]-1) + x[x] - [x]^2 \\
&= \frac{([x]-1)([x])}{2} + x[x] - [x]^2
\end{aligned}$$

Let $n > 0$ be an integer. Then

$$\begin{aligned}
f(n-0) &= \lim_{h\to 0} f(n-h) \\
&= \frac{(n-2)(n-1)}{2} + \lim_{h\to 0}(n-h)(n-1) - (n-1)^2 \\
&= \frac{(n-2)(n-1)}{2} + n(n-1) - (n-1)^2 \\
&= \frac{n-1}{2}[n-2+2n-2(n-1)] \\
&= \frac{(n-1)n}{2}
\end{aligned}$$

Now

$$\begin{aligned}
f(n+0) &= \lim_{h\to 0} f(n+h) \\
&= \frac{(n-1)n}{2} + \lim_{h\to 0}(n+h)n - n^2 \\
&= \frac{(n-1)n}{2} + n^2 - n^2 \\
&= \frac{(n-1)n}{2}
\end{aligned}$$

Therefore $f$ is continuous at all positive integers. We can see that

$$f'(n-0) = n-1$$

and $$f'(n+0) = n$$

So $f$ is not differentiable at positive integers.

**Answer: (C)**

**81.** $\displaystyle\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2\,dt}{x\sin x} =$

(A) $\dfrac{1}{2}$  (B) 1  (C) 0  (D) 2

**Solution:** We have

$$\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(\cos x^2)(2x)}{\sin x + x\cos x}$$
$$=\lim_{x\to 0}\frac{\cos(x^2)(2)}{(\sin x/x)+\cos x}$$
$$=\frac{2}{1+1}$$
$$=1$$

**Answer: (B)**

**82.** If

$$S_n=\frac{1}{1+\sqrt{n}}+\frac{1}{2+\sqrt{2n}}+\cdots+\frac{1}{n+\sqrt{n^2}}$$

then $\lim\limits_{n\to\infty} S_n$ is equal to

(A) $\log_e 2$ (B) $2+2\log_e 2$
(C) $2+\log_e 2$ (D) $2\log_e 2$

**Solution:** We have

$$T_r = r\text{th term} = \frac{1}{r+\sqrt{rn}}$$

Put $n=1/h$. Then

$$T_r=\frac{\sqrt{h}}{r\sqrt{h}+\sqrt{r}}$$
$$=\frac{h}{rh+\sqrt{rh}}$$

Therefore

$$\lim_{n\to\infty} S_n=\int_0^1\frac{dx}{x+\sqrt{x}}$$
$$=\int_0^1\frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$
$$=\int_0^1\frac{1}{t(t+1)}(2t)dt \quad \text{where } t=\sqrt{x}$$
$$=2\left[\log_e(t+1)\right]_0^1$$
$$=2\log_e 2$$

**Answer: (D)**

**83.** $\lim\limits_{n\to\infty}\sum_{r=0}^{n}\frac{{}^nC_r}{n^r(r+3)}=$

(A) $e-1$ (B) $e$ (C) $e-2$ (D) $e+1$

**Solution:** We know that

$$\int_0^1 x^{r+2}\,dx=\frac{1}{r+3}$$

Therefore

$$\lim_{n\to\infty}\sum_{r=0}^{n}\frac{{}^nC_r}{n^r(r+3)}=\lim_{n\to\infty}\sum_{r=0}^{n}{}^nC_r\frac{1}{n^r}\int_0^1 x^{r+2}\,dx$$
$$=\int_0^1 x^2\left(\lim_{n\to\infty}\sum_{r=0}^{n}{}^nC_r\left(\frac{x}{n}\right)^r\right)dx$$
$$=\int_0^1 x^2\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right)dx$$
$$=\int_0^1 x^2e^x$$
$$=\left[x^2e^x-(2x)e^x+2e^x\right]_0^1$$
$$=(e-2e+2e)-(0-0+2)$$
$$=e-2$$

**Answer: (C)**

**84.** $\int_{-1}^{3/2}|x\sin\pi x|dx=$

(A) $\frac{\pi+1}{\pi^2}$ (B) $\frac{3\pi+1}{\pi^2}$ (C) $\frac{3\pi-1}{\pi^2}$ (D) $\frac{3}{\pi^2}$

**Solution:** Let

$$I=\int_{-1}^{3/2}|x\sin\pi x|dx$$

Put $t=\pi x$ so that

$$x=-1\Rightarrow t=-\pi$$

and $$x=\frac{3}{2}\Rightarrow t=\frac{3\pi}{2}$$

Therefore

$$I=\int_{-\pi}^{3\pi/2}\left|\frac{t}{\pi}\sin t\right|\frac{1}{\pi}dt$$
$$=\frac{1}{\pi^2}\int_{-\pi}^{3\pi/2}|t\sin t|dt$$
$$=\frac{1}{\pi^2}\left[\int_{-\pi}^{\pi}(t\sin t)dt+\int_{\pi}^{3\pi/2}(-t\sin t)dt\right]$$
$$=\frac{1}{\pi^2}\left[2\int_0^{\pi}t\sin t\,dt-\int_{\pi}^{3\pi/2}t\sin t\,dt\right]$$

$$= \frac{1}{\pi^2}\left[2[\sin t - t\cos t]_0^{\pi} - [\sin t - t\cos t]_{\pi}^{3\pi/2}\right]$$

$$= \frac{1}{\pi^2}[2(-\pi\cos\pi - 0) - [-1 - 0 - (0 - \pi\cos\pi)]$$

$$= \frac{1}{\pi^2}(2\pi + 1 + \pi)$$

$$= \frac{3\pi + 1}{\pi^2}$$

**Answer: (B)**

**85.** $\int_0^{1.5} [x^2]dx$ ($[x]$ is the integral part of $x$) is equal to

(A) $\sqrt{2}$ (B) $\sqrt{2}-1$ (C) $2-\sqrt{2}$ (D) $\sqrt{2}+1$

**Solution:** We have

$$0 \le x \le \frac{3}{2} \Rightarrow 0 \le x^2 \le \frac{9}{4}$$

so that $[x^2] = 0, 1$ or $2$. Therefore

$$\int_0^{1.5}[x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{3/2} [x^2]dx$$

$$= 0 + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{3/2} 2\,dx$$

$$= 0 + (\sqrt{2}-1) + 2\left(\frac{3}{2} - \sqrt{2}\right)$$

$$= \sqrt{2} - 1 + 3 - 2\sqrt{2}$$

$$= 2 - \sqrt{2}$$

**Answer: (C)**

**86.** Area of the region bounded by the curves $y = x^2, y = x + 2$ and the $x$-axis is

(A) $\frac{4}{3}$ (B) $\frac{5}{3}$ (C) $\frac{5}{4}$ (D) $\frac{5}{6}$

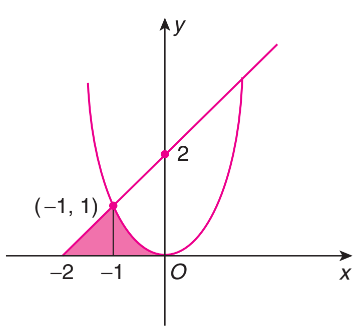



**FIGURE 5.13** Single correct choice type question 86.

**Solution:** The line $y = x + 2$ and the parabola $y = x^2$ intersect in the two points $(-1, 1)$ and $(2, 4)$. The required area is shown as the shaded portion in Fig. 5.13. Mathematically, it is given by

Area required = Area of the triangle with vertices $(-1, 1), (-2, 0), (-1, 0) + \int_{-1}^{0} x^2\,dx$

$$= \frac{1}{2}(1)(1) + \frac{1}{3}\left[x^3\right]_{-1}^{0}$$

$$= \frac{1}{2} + \frac{1}{3}[0 - (-1)]$$

$$= \frac{1}{2} + \frac{1}{3}$$

$$= \frac{5}{6}$$

**Answer: (D)**

**87.** The area of the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$, the $x$-axis and the line $y = x/\sqrt{3}$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) 1

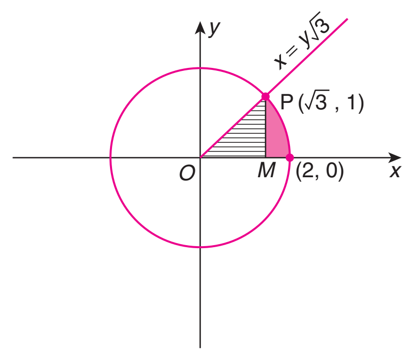



**FIGURE 5.14** Single correct choice type question 87.

**Solution:** The line $y = x/\sqrt{3}$ meets the circle in the first quadrant at $P(\sqrt{3}, 1)$. Therefore

$$\text{Area required} = \Delta OPM + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\,dx$$

$$= \frac{1}{2}\times\sqrt{3}\times 1 + \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\mathrm{Sin}^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$= \frac{\sqrt{3}}{2} + \left[2\mathrm{Sin}^{-1}1 - \frac{\sqrt{3}}{2} - 2\mathrm{Sin}^{-1}\frac{\sqrt{3}}{2}\right]$$

$$= \frac{\sqrt{3}}{2} + \pi - \frac{\sqrt{3}}{2} - 2\times\frac{\pi}{3}$$

$$= \frac{\pi}{3}$$

**Answer: (B)**

**88.** For which values of $m$, does the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals 9/2?

(A) $-4, 1$ (B) $-2, 1$
(C) $2, -1$ (D) $4, -2$

**Solution:** We have

$$y = x - x^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$$

Therefore

$$\left(x - \frac{1}{2}\right)^2 = -\left(y - \frac{1}{4}\right)$$

This equation represents downward parabola with vertex at (1/2, 1/4) and $y \le 1/4$ (see Fig. 5.15). The parabola $y = x - x^2$ meets the $x$-axis in (0, 0) and (1, 0). The line $y = mx$ meets the curve in $(0, 0)$ and $(1-m, m-m^2)$. Note that $1 - m \gtrless 0$. Therefore

$$\begin{aligned}\text{Required area} &= \int_0^{1-m} (x - x^2 - mx)\,dx \\ &= \left(\frac{1-m}{2}\right)\left[x^2\right]_0^{1-m} - \frac{1}{3}\left[x^3\right]_0^{1-m} \\ &= \frac{1-m}{2}[(1-m)^2 - 0] - \frac{1}{3}(1-m)^3 \\ &= \frac{(1-m)^3}{6}\end{aligned}$$

Now

$$\begin{aligned}&\frac{(1-m)^3}{6} = \frac{9}{2} \quad \text{(Given)} \\ &\Rightarrow (1-m)^3 = 27 \\ &\Rightarrow 1 - m = 3 \\ &\Rightarrow m = -2\end{aligned}$$

Also, if $1 - m < 0$, then we have

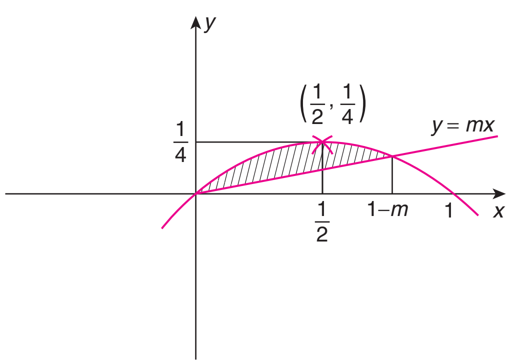



**FIGURE 5.15** Single correct choice type question 88.

$$\begin{aligned}&-\frac{(1-m)^3}{6} = \frac{9}{2} \\ &\Rightarrow 1 - m = -3 \\ &\Rightarrow m = 4\end{aligned}$$

Hence $m = -2, 4$.

**Answer: (D)**

**89.** The area of the figure bounded by the straight lines $x = 0, x = 2$ and the curves $y = 2^x, y = 2x - x^2$ is

(A) $\dfrac{4}{3} + \dfrac{3}{\log_e 2}$ (B) $\log_e 2 + \dfrac{4}{3}$

(C) $\dfrac{3}{\log_e 2} - \dfrac{4}{3}$ (D) $\dfrac{3}{\log_e 2} + \dfrac{2}{3}$

**Solution:** We have

$$\begin{aligned}&y = 2x - x^2 = 1 - (x-1)^2] \\ &\Rightarrow (x-1)^2 = -(y-1)\end{aligned}$$

This is the equation of a downward parabola with vertex at (1, 1) meeting the $x$-axis in (0, 0) and (2, 0). See Fig. 5.16. Hence

$$\begin{aligned}\text{Required area (shaded portion)} &= \int_0^2 [2^x - (2x - x^2)]\,dx \\ &= \left[\frac{2^x}{\log_e 2}\right]_0^2 - \left[x^2\right]_0^2 + \frac{1}{3}\left[x^3\right]_0^2 \\ &= \frac{4-1}{\log_e 2} - 4 + \frac{8}{3} \\ &= \frac{3}{\log_e 2} - \frac{4}{3}\end{aligned}$$

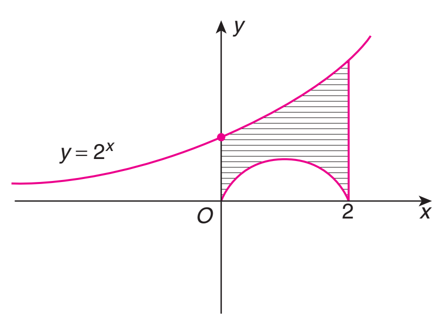



**FIGURE 5.16** Single correct choice type question 89.

**Answer: (C)**

**90.** The area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$ is

(A) 3 (B) $\dfrac{7}{2}$ (C) $\dfrac{9}{2}$ (D) 4

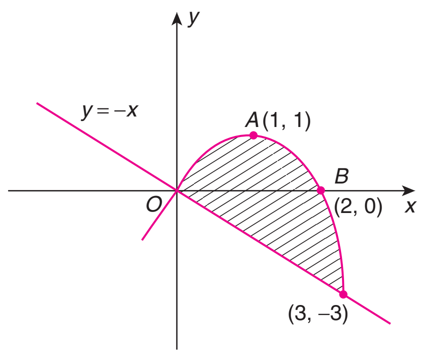



FIGURE 5.17 Single correct choice type question 90.

**Solution:** We have

$$y = 2x - x^2 = 1-(x-1)^2$$
$$\Rightarrow (x-1)^2 = -(y-1)$$

This implies that the curve meets the $x$-axis in (0, 0) and (2, 0). The line $y = -x$ meets the parabola $y = 2x - x^2$ in (0, 0) and (3, −3). See Fig. 5.17. Therefore

$$\begin{aligned}\text{Required area (shaded portion)} &= \int_0^3 [(2x-x^2)-(-x)]\,dx \\ &= \int_0^3 3x\,dx - \int_0^3 x^2\,dx \\ &= \frac{3}{2}\left[x^2\right]_0^3 - \frac{1}{3}\left[x^3\right]_0^3 \\ &= \frac{3}{2}\times 9 - \frac{27}{3} \\ &= \frac{81-54}{6} \\ &= \frac{27}{6} \\ &= \frac{9}{2}\end{aligned}$$

**Answer: (C)**

**91.** The area between the curve $y = 2x^4 - x^2$, the $x$-axis and the ordinates of the two minima of the function $y = 2x^4 - x^2$ is

(A) $\frac{7}{120}$ (B) $\frac{7}{60}$ (C) $\frac{3}{40}$ (D) $\frac{1}{20}$

**Solution:** We have

$$y = 2x^4 - x^2$$

which is symmetric about $y$-axis. The curve meets the $x$-axis in $(-1/\sqrt{2}, 0)$, $(0, 0)$ and $(1/\sqrt{2}, 0)$. Differentiating the equation of the curve w.r.t. $x$ we get

$$\frac{dy}{dx} = 8x^3 - 2x$$

Now

$$\frac{dy}{dx} = 0$$
$$\Rightarrow x(4x^2-1) = 0$$
$$\Rightarrow x = 0, \pm\frac{1}{2}$$

Differentiating again we get

$$\frac{d^2y}{dx^2} = 24x^2 - 2 = 2(12x^2-1)$$
$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=\pm\frac{1}{2}} = 2\left(12\times\frac{1}{4}-1\right) = 2\times 2 > 0$$

So $y$ is minimum at $x = -1/2$ and $1/2$, and $y$ is maximum at $x = 0$. The required area is shown as the shaded portion in Fig. 5.18.

$$\begin{aligned}\text{Required area} &= 2\int_0^{1/2} (-y)\,dx \\ &= 2\int_0^{1/2} x^2 - 2x^4 \\ &= 2\left[\frac{1}{3}\left[x^3\right]_0^{1/2} - \frac{2}{5}\left[x^5\right]_0^{1/2}\right] \\ &= 2\left[\frac{1}{24} - \frac{2}{5}\times\frac{1}{32}\right] \\ &= \frac{1}{12} - \frac{1}{40} \\ &= \frac{10-3}{120} \\ &= \frac{7}{120}\end{aligned}$$

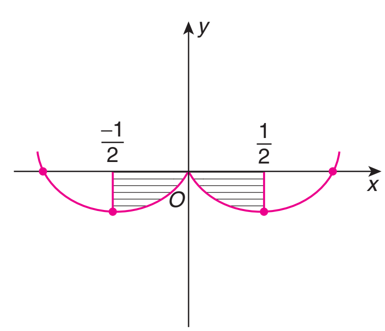



FIGURE 5.18 Single correct choice type question 91.

**Answer: (A)**

**92.** The area of the bounded region enclosed between the curves $y^3 = x^2$ and $y = 2 - x^2$ is

(A) $1\frac{1}{15}$ (B) $2\frac{2}{15}$ (C) $2\frac{4}{15}$ (D) $2\frac{14}{15}$

**Solution:** The curve $y^3 = x^2$ is symmetric about $y$-axis. Also $y \ge 0$. Now

$$y = 2 - x^2$$
$$\Rightarrow x^2 = -(y-2)$$

This represents parabola with vertex $(0, 2)$ and $y \le 2$ (see Fig. 5.19). Again

$$y^3 = x^2 = 2 - y$$
$$\Rightarrow y^3 + y - 2 = 0$$
$$\Rightarrow (y-1)(y^2 + y + 2) = 0$$

This implies $y = 1, x = \pm 1$. Therefore

$$\text{Required area (shaded portion)} = 2\int_0^1 [(2 - x^2) - x^{2/3}]\,dx$$
$$= 4[x]_0^1 - \frac{2}{3}[x^3]_0^1 - \frac{2\times 3}{5}[x^{5/3}]_0^1$$
$$= 4 - \frac{2}{3} - \frac{6}{5}$$
$$= \frac{60 - 10 - 18}{15}$$
$$= \frac{32}{15}$$
$$= 2\frac{2}{15}$$

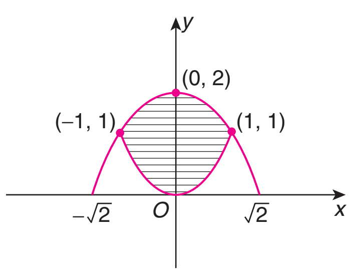



**FIGURE 5.19** Single correct choice type question 92.

**Answer: (B)**

**93.** The area of the region enclosed between the curve $x^2 = 2y$ and the straight line $y = 2$ equals

(A) $\frac{4}{3}$ (B) $\frac{8}{3}$ (C) $\frac{16}{3}$ (D) $\frac{32}{3}$

**Solution:** The line $y = 2$ meets the curve $x^2 = 2y$ in the points $(-2, 2)$ and $(2, 2)$ (see Fig. 5.20). Therefore

$$\text{Area} = 2\int_0^2 \left(2 - \frac{x^2}{2}\right)dx$$
$$= 4[x]_0^2 - \frac{1}{3}[x^3]_0^2$$
$$= 8 - \frac{8}{3}$$
$$= \frac{16}{3}$$

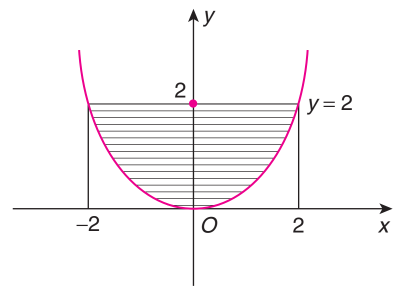



**FIGURE 5.20** Single correct choice type question 93.

**Answer: (C)**

**94.** The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ inside the square formed by the lines $x = 1, y = 1, x = 4, y = 4$ is

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{13}{3}$ (D) $\frac{11}{3}$

**Solution:** The two curves intersect at $(4, 4)$ which is a vertex of the given square (see Fig. 5.21). Therefore

$$\text{Required area (shaded portion)} = \int_1^4 2\sqrt{x}$$
$$-\int_2^4 \frac{x^2}{4}dx - (2-1)\times 1$$
$$= 2\times\frac{2}{3}[x^{3/2}]_1^4 - \frac{1}{12}[x^3]_2^4 - 1$$

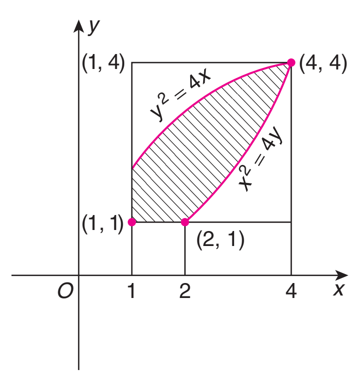



**FIGURE 5.21** Single correct choice type question 94.

$$= \frac{4}{3}\times[8-1]-\frac{1}{12}(64-8)-1$$
$$= \frac{28}{3}-\frac{56}{12}-1$$
$$= \frac{112-56-12}{12}$$
$$= \frac{44}{12}$$
$$= \frac{11}{3}$$

**Answer: (D)**

**95.** Let $f(x) = \text{Max}\{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \le x \le 1$. Then the area of the region bounded by the curve $y = f(x)$, the $x$-axis and the lines $x = 0, x = 1$ is

(A) $\frac{17}{7}$ (B) $\frac{17}{27}$ (C) $\frac{14}{17}$ (D) $\frac{14}{27}$

**Solution:** See. Fig. 5.22. The colored arcs represent the graph of $y = f(x)$. Let $C_1 : y = x^2$; $C_2 : y = (1-x)^2$; $C_3$: $y = 2x(1-x)$. Then $C_2$ and $C_3$ intersect in (1/3, 4/9) and $C_1$ intersect $C_3$ in (2/3, 4/9). Now

$$f(x) = \begin{cases} (1-x)^2 & \text{for } 0 \le x \le 1/3 \\ 2x(1-x) & \text{for } 1/3 \le x \le 2/3 \\ x^2 & \text{for } 2/3 \le x \le 1 \end{cases}$$

Required area (shaded portion)

$$= \int_0^{1/3} (1-x)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} x^2\,dx$$
$$= -\frac{1}{3}\Big[(1-x)^3\Big]_0^{1/3} + \Big[x^2\Big]_{1/3}^{2/3} - \frac{2}{3}\Big[x^3\Big]_{1/3}^{2/3} + \frac{1}{3}\Big[x^3\Big]_{2/3}^{1}$$
$$= -\frac{1}{3}\left[\frac{8}{27}-1\right]+\left(\frac{4}{9}-\frac{1}{9}\right)-\frac{2}{3}\left(\frac{8}{27}-\frac{1}{27}\right)+\frac{1}{3}\left(1-\frac{8}{27}\right)$$

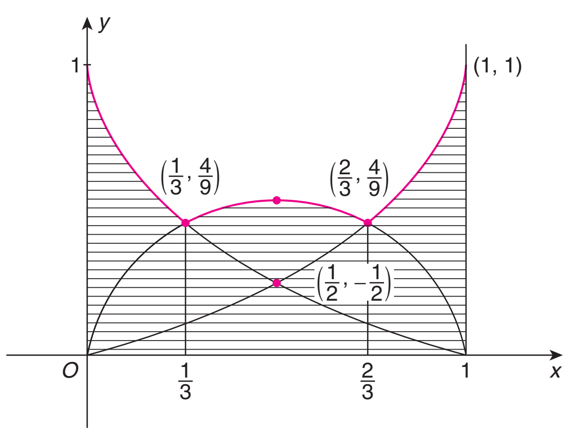



FIGURE 5.22 Single correct choice type question 95.

$$= \frac{19}{81}+\frac{3}{9}-\frac{14}{81}+\frac{19}{81}$$
$$= \frac{19+27-14+19}{81} = \frac{51}{81} = \frac{17}{27}$$

**Answer: (B)**

**96.** The area of the bounded region enclosed between the lines $x = 1/2, x = 2$ and the curves $y = \log_e x$ and $y = 2^x$ is

(A) $\frac{1}{\log_e 2}(4+\sqrt{2})-\frac{5}{2}\log_e 2+\frac{3}{2}$

(B) $\frac{1}{\log_e 2}(4-\sqrt{2})-\frac{5}{2}\log_e 2$

(C) $\frac{1}{\log_e 2}(4-\sqrt{2})-\frac{5}{2}\log_e 2+\frac{3}{2}$

(D) $\frac{1}{\log_e 2}(4+\sqrt{2})-\frac{5}{2}\log_e 2$

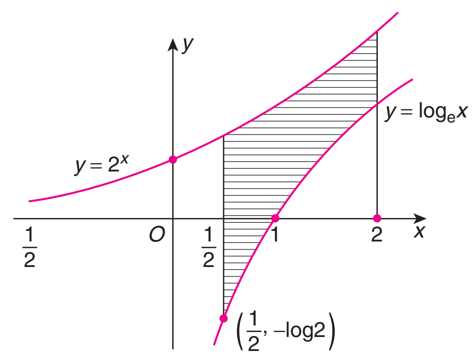



FIGURE 5.23 Single correct choice type question 96.

**Solution:** See Fig. 5.23. The shaded portion is the required area. Now

$$\text{Required area} = \int_{1/2}^{2} (2^x - \log_e x)\,dx$$
$$= \frac{1}{\log_e 2}[2^x]_{1/2}^{2} - [x\log_e x - x]_{1/2}^{2}$$
$$= \frac{1}{\log_e 2}(2^2 - 2^{1/2})$$
$$-\left[(2\log_e 2 - 2) - \left(\frac{1}{2}\log_e\left(\frac{1}{2}\right)-\frac{1}{2}\right)\right]$$
$$= \frac{4-\sqrt{2}}{\log_e 2} - \left[2\log_e 2 + \frac{1}{2}\log_e 2 - 2 + \frac{1}{2}\right]$$
$$= \frac{4-\sqrt{2}}{\log_e 2} - \frac{5}{2}\log_e 2 + \frac{3}{2}$$

**Answer: (C)**

**97.** The $x$-axis divides the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$ in the ratio

(A) $\dfrac{121}{4}$ (B) $\dfrac{111}{4}$ (C) $\dfrac{81}{4}$ (D) $\dfrac{91}{4}$

**Solution:** Let

$$P_1 : y = 4x - x^2 = 4 - (x-2)^2$$

Therefore $P_1$ is a parabola with vertex at $(2, 4)$ and $y \le 4$. Also

$$P_2 : y = x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$$

It is a parabola with vertex at $(1/2, -1/4)$ and $y \ge -1/4$. See Fig. 5.24. $P_1$ and $P_2$ intersect in $(0, 0)$ and $(4/2, 15/4)$. So

$A$ = Area shaded by vertical and horizontal lines

$$\begin{aligned}
&= \int_0^{5/2} [(4x - x^2) - (x^2 - x)]\,dx \\
&= \int_0^{5/2} (5x - 2x^2)\,dx \\
&= \frac{5}{2}\left[x^2\right]_0^{5/2} - \frac{2}{3}\left[x^3\right]_0^{5/2} \\
&= \frac{5}{2}\left(\frac{5}{2}\right)^2 - \frac{2}{3}\left(\frac{5}{2}\right)^3 \\
&= \frac{125}{8} - \frac{125}{12} = \frac{125}{24}
\end{aligned}$$

$B$ = Area (shaded by vertical lines) below the $x$-axis

$$\begin{aligned}
&= \int_0^1 -(x^2 - x)\,dx \\
&= \frac{1}{2}(1) - \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

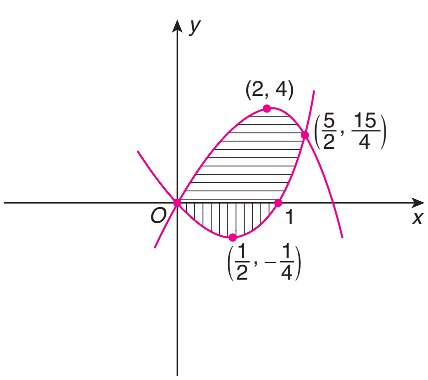



**FIGURE 5.24** Single correct choice type question 97.

Now

$$\begin{aligned}
A - B &= \frac{125}{24} - \frac{1}{6} \\
&= \frac{125 - 4}{24} = \frac{121}{24}
\end{aligned}$$

Therefore

$$\frac{A - B}{B} = \frac{121}{24} \times \frac{6}{1} = \frac{121}{4}$$

**Answer: (A)**

**98.** The area of the figure enclosed by the curve $(y - \mathrm{Sin}^{-1} x)^2 = x - x^2$ is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{2\pi}{3}$

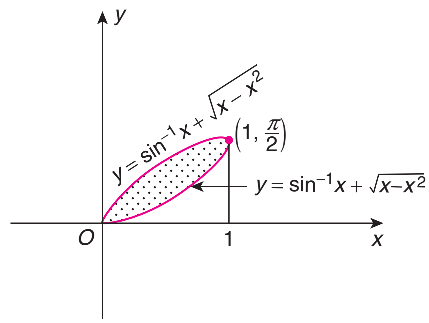



**FIGURE 5.25** Single correct choice type question 98.

**Solution:** Clearly

$$x - x^2 \ge 0 \Leftrightarrow 0 \le x \le 1$$

and $$y = \mathrm{Sin}^{-1} x \pm \sqrt{x - x^2}$$

The equation represents curve consisting of two branches intersecting in $(0, 0)$ and $(1, \pi/2)$ (see Fig. 5.25). Hence the area is

$$\int_0^1 (\mathrm{Sin}^{-1} x + \sqrt{x - x^2} - \mathrm{Sin}^{-1} x + \sqrt{x - x^2})\,dx$$

$$\begin{aligned}
&= 2\int_0^1 \sqrt{x - x^2}\,dx \\
&= 2\int_0^1 \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}\,dx \\
&= 2\left\{\left[\frac{[x - (1/2)]\sqrt{x - x^2}}{2}\right]_0^1 + \frac{1}{8}\left[\mathrm{Sin}^{-1}\left(\frac{x - (1/2)}{1/2}\right)\right]_0^1\right\}
\end{aligned}$$

$$= 2\left[0 + \frac{1}{8}[\mathrm{Sin}^{-1}1 - \mathrm{Sin}^{-1}(-1)]\right]$$
$$= \frac{1}{4}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= \frac{\pi}{4}$$

**Answer: (B)**

**99.** The area bounded by the curve

$$y^2 = 4a(a - |x - a|) \quad (a > 0)$$

is

(A) $\frac{8a^2}{3}$ (B) $\frac{16a^2}{3}$ (C) $\frac{19a^2}{3}$ (D) $\frac{13a^2}{3}$

**Solution:** See Fig. 5.26. We have

$$y^2 = \begin{cases} 4a[a - (a - x)] = 4ax & \text{if } x < a \\ 4a[a - (x - a)] = 4a(2a - x) & \text{if } x \ge a \end{cases}$$

Therefore

$$\text{Area} = 4\int_0^a \sqrt{4ax}\, dx$$
$$= 8\sqrt{a} \times \frac{2}{3}\left[x^{2/3}\right]_0^a$$
$$= \frac{16a^2}{3}$$

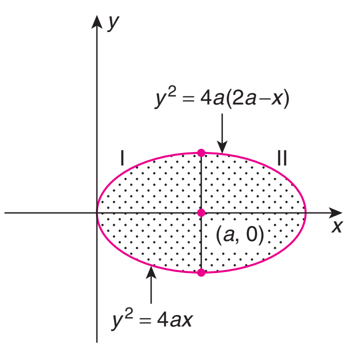



**FIGURE 5.26** Single correct choice type question 99.

***Note:*** If $a = \sqrt{3}$, then the area is 16.

**Answer: (B)**

**100.** The area of the region bounded by the curve

$$32x^3 = (16 - y^2)[41x + 9|x|], \quad x \ne 0$$

is

(A) $18\pi$ (B) $27\pi$ (C) $36\pi$ (D) $45\pi$

**Solution:** See Fig. 5.27.

**Case I:** $x > 0$. Now

$$32x^3 = (16 - y^2)(41x + 9x)$$
$$\frac{32}{50}x^2 = 16 - y^2$$
$$\frac{32}{50}x^2 + y^2 = 16$$

So

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \text{ (upper portion of the ellipse)}$$

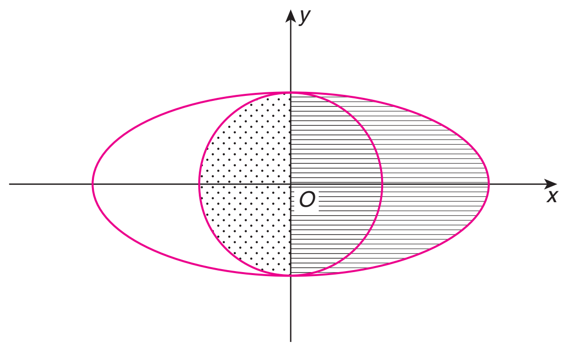



**FIGURE 5.27** Single correct choice type question 100.

**Case II:** $x < 0$. We have

$$32x^3 = (16 - y^2)(41x - 9x)$$
$$x^2 = 16 - y^2$$

So

$$x^2 + y^2 = 16 \text{ (Portion of the circle below } x\text{-axis)}$$

Therefore

Required area = (Area in the first and fourth quadrants of the ellipse) + (Area in the second and third quadrants of the circle)

$$= \frac{\pi(5 \times 4)}{2} + \frac{\pi(4^2)}{2}$$
$$= 10\pi + 8\pi$$
$$= 18\pi$$

***Note:*** The area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $\pi ab$ (see Example 5.15).

**Answer: (A)**

**101.** The area of the region bounded by the curves $y = x^2$, $y = 1/x$ and $x = 1/2$ is $\log_e 2 - p$, where $p$ equals

(A) $\frac{5}{24}$ (B) $-\frac{5}{24}$

(C) $\frac{7}{24}$ (D) $-\frac{7}{24}$

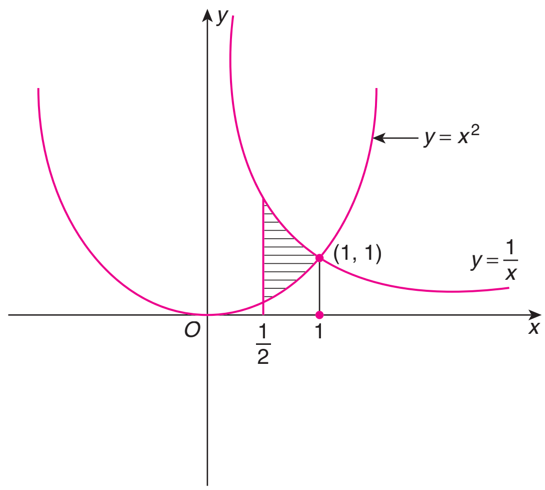



**FIGURE 5.28** Single correct choice type question 101.

**Solution:** The two curves intersect in (–1, 1). See Fig. 5.28. Now

$$\text{Required area (shaded portion)} = \int_{1/2}^{1}\left(\frac{1}{x} - x^2\right)dx$$

$$= [\log_e x]_{1/2}^{1} - \frac{1}{3}[x^3]_{1/2}^{1}$$

$$= 0 - (-\log_e 2) - \frac{1}{3}\left(1 - \frac{1}{8}\right)$$

$$= \log_e 2 - \frac{7}{24}$$

Therefore $p = 7/24$.

**Answer: (C)**

**102.** The area of the region bounded by the curve $y = x^2 - 4x$ and the line $y = -x$ is

(A) $\frac{7}{2}$ (B) 4 (C) $\frac{9}{2}$ (D) 5

**Solution:** The line $y = -x$ meets the parabola

$$(x-2)^2 = y + 4$$

in (0, 0) and (3, –3). See Fig. 5.29. Now

$$\text{Required area} = \int_0^3 [(-x) - (x^2 - 4x)]\,dx$$

$$= \int_0^3 (3x - x^2)\,dx$$

$$= \frac{3}{2}[x^2]_0^3 - \frac{1}{3}[x^3]_0^3$$

$$= \frac{3}{2}(9) - 9$$

$$= \frac{9}{2}$$

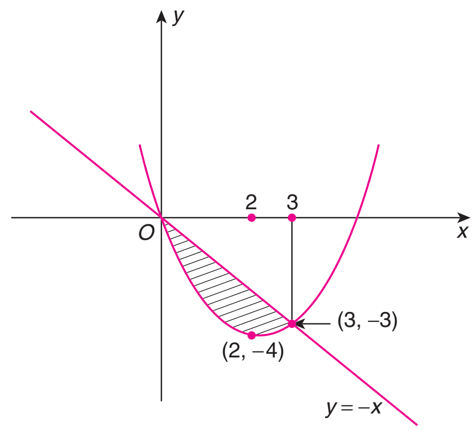



**FIGURE 5.29** Single correct choice type question 102.

**Answer: (C)**

**103.** The area enclosed by the curve $y^2 = x^2 - x^4$ is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$

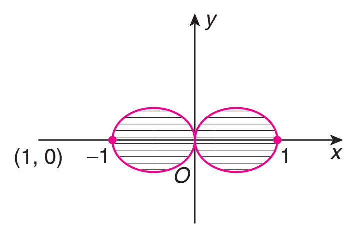



**FIGURE 5.30** Single correct choice type question 103.

**Solution:** Since $x$ and $y$ have even powers, the curve is symmetric about both axes (see Fig. 5.30). The curve cuts $x$-axis in (–1, 0), (0, 0) and (1, 0). Also $-1 \le x \le 1$. Hence the area enclosed by the curve equals four times the area in the first quadrant, that is

$$\text{Area} = 4\int_0^1 x\sqrt{1-x^2}\,dx$$

$$= -2\int_0^1 (-2x)\sqrt{1-x^2}\,dx$$

$$= -2\left[\frac{(1-x^2)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{-4}{3}(0-1)$$

$$= \frac{4}{3}$$

**Answer: (C)**

**104.** The area of the region bounded by the parabola $y^2 = x - 2$ and the line $y = x - 8$ is

(A) $\dfrac{115}{6}$ (B) $\dfrac{125}{3}$ (C) $\dfrac{115}{3}$ (D) $\dfrac{125}{6}$

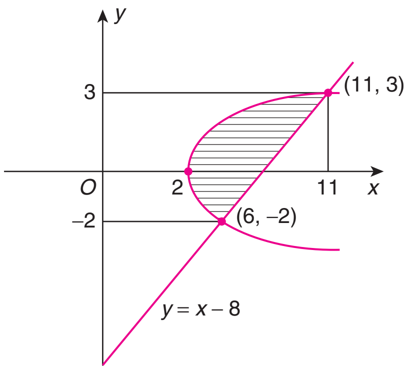



**FIGURE 5.31** Single correct choice type question 104.

**Solution:** See Fig. 5.31. The area is the shaded portion. Therefore

$$
\begin{aligned}
\text{Area} &= \int_{-2}^{3} [(y+8)-(y^2+2)]\,dy \\
&= \frac{1}{2}\left[y^2\right]_{-2}^{3} - \frac{1}{3}\left[y^3\right]_{-2}^{3} + 6[y]_{-2}^{3} \\
&= \frac{1}{2}(9-4) - \frac{1}{3}(27+8) + 6(3+2) \\
&= \frac{5}{2} - \frac{35}{3} + 30 \\
&= \frac{15-70+180}{6} \\
&= \frac{125}{6}
\end{aligned}
$$

**Answer: (D)**

**105.** The area of the bounded region enclosed between the parabola $y^2 = -x$ and the line $y = x + 6$ is

(A) $\dfrac{125}{3}$ (B) $\dfrac{115}{3}$ (C) $\dfrac{115}{6}$ (D) $\dfrac{125}{6}$

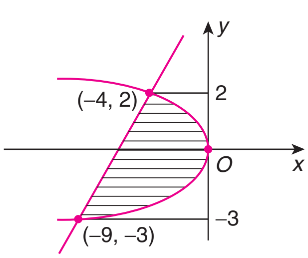



**FIGURE 5.32** Single correct choice type question 105.

**Solution:** See Fig. 5.32. The area is the shaded portion. Therefore

$$
\begin{aligned}
\text{Area} &= \int_{-3}^{2} [-y^2-(y-6)]\,dy \\
&= -\frac{1}{3}\left[y^3\right]_{-3}^{2} - \frac{1}{2}\left[y^2\right]_{-3}^{2} + 6[y]_{-3}^{2} \\
&= \frac{-1}{3}(8+27) - \frac{1}{2}(4-9) + 6(2+3) \\
&= -\frac{35}{3} + \frac{5}{2} + 30 \\
&= \frac{-70+15+180}{6} \\
&= \frac{125}{6}
\end{aligned}
$$

**Answer: (D)**

**106.** The area of the figure (Fig. 5.33) bounded by the curves $y = e^x$, $y = e^{-x}$ and the line $x = 1$ is

(A) $e + \dfrac{1}{e} - 2$ (B) $e + \dfrac{1}{e} - 1$

(C) $\dfrac{1}{2}\left(e + \dfrac{1}{e} - 2\right)$ (D) $e + \dfrac{1}{e}$

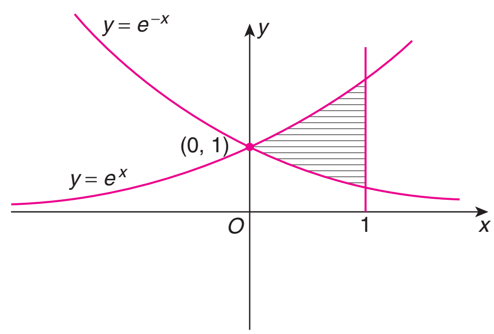



**FIGURE 5.33** Single correct choice type question 106.

**Solution:** We have

$$
\begin{aligned}
\text{Area} &= \int_{0}^{1} (e^x - e^{-x})\,dx \\
&= \left[e^x\right]_0^1 + \left[e^{-x}\right]_0^1 \\
&= (e-1) + (e^{-1}-1) \\
&= e + \frac{1}{e} - 2
\end{aligned}
$$

**Answer: (A)**

**107.** The area of the region bounded between the parabola $4y = 3x^2$ and the line $2y = 3x + 12$ is

(A) 9 (B) 18 (C) 27 (D) 36

**Solution:** The line $2y = 3x + 12$ intersects the parabola $4y = 3x^2$ in the points $(-2, 3)$ and $(4, 12)$. The required area is the shaded portion of Fig. 5.34. Therefore

$$\begin{aligned}\text{Area} &= \int_{-2}^{4}\left(\frac{3x+12}{2}-\frac{3}{4}x^2\right)dx \\ &= \frac{3}{4}\left[x^2\right]_{-2}^{4}+6\left[x\right]_{-2}^{4}-\frac{1}{4}\left[x^3\right]_{-2}^{4} \\ &= \frac{3}{4}(16-4)+6(4+2)-\frac{1}{4}(64+8) \\ &= 9+36-18 \\ &= 27\end{aligned}$$

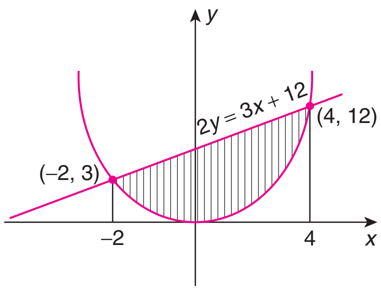



**FIGURE 5.34** Single correct choice type question 107.

**Answer: (C)**

**108.** Area enclosed between $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1. Then the value of $a$ is

(A) $\dfrac{1}{\sqrt{3}}$ (B) $\dfrac{1}{2}$ (C) 1 (D) $\dfrac{1}{3}$

**(IIT-JEE 2004)**

**Solution:** The two curves are parabolas which intersect in $(0, 0)$ and $(1/a, 1/a)$. Hence,

$$\begin{aligned}1 = \text{Area} &= \int_{0}^{1/a}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx \\ &= \frac{1}{\sqrt{a}}\times\frac{2}{3}\left[x^{3/2}\right]_0^{1/a}-\frac{a}{3}\left[x^3\right]_0^{1/a} \\ &= \frac{2}{3\sqrt{a}}\times\frac{1}{a\sqrt{a}}-\frac{a}{3}\times\frac{1}{a^3} \\ &= \frac{2}{3a^2}-\frac{1}{3a^2} \\ &= \frac{1}{3a^2}\end{aligned}$$

So

$$a^2=\frac{1}{3} \quad \text{or} \quad a=\frac{1}{\sqrt{3}}$$

**Answer: (A)**

**109.** The curve $y = a\sqrt{x} + bx$ passes through the point $(1,2)$ and the area enclosed by the curve, the $x$-axis and the line $x = 4$ is 8. Then

(A) $a = 3, b = 1$ (B) $a = 3, b = -1$

(C) $a = -3, b = 1$ (D) $a = 1, b = 3$

**Solution:** By hypothesis the curve passes through the points $(0, 0)$ and $(1, 2)$ and $x \ge 0$. Also

$$\begin{aligned}8 &= \int_0^4 (a\sqrt{x}+bx)\,dx \\ &= \frac{2a}{3}\left[x^{3/2}\right]_0^4+\frac{b}{2}\left[x^2\right]_0^4 \\ &= \frac{16a}{3}+8b\end{aligned}$$

So

$$2a + 3b = 3 \tag{5.21}$$

Since the curve passes through $(1, 2)$, we have

$$a + b = 2 \tag{5.22}$$

From Eqs. (5.21) and (5.22), $a = 3$ and $b = -1$.

**Answer: (B)**

**110.** The area of the region bounded by the curves $y = x^2$ and $y = 2/(1 + x^2)$ is

(A) $\dfrac{3\pi+2}{3}$ (B) $\dfrac{3\pi+1}{3}$

(C) $\dfrac{3\pi-1}{3}$ (D) $\dfrac{3\pi-2}{3}$

**(IIT-JEE 1992)**

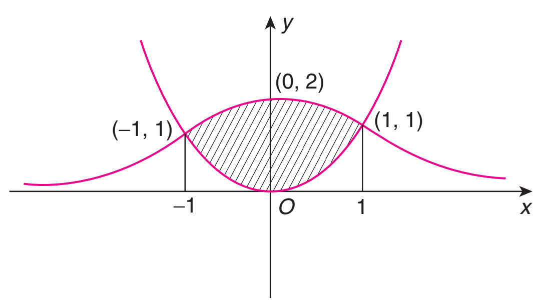



**FIGURE 5.35** Single correct choice type question 110.

**Solution:** $y = x^2$ is a parabola which we denote by $P$. Let $C$ be the curve

$$y = 2/(1 + x^2)$$

Now

**(i)** $C$ is symmetric about $y$-axis.

**(ii)** $C$ meets $y$-axis in $(0, 2)$ and cannot meet $x$-axis.

**(iii)** $x \to \pm\infty \Rightarrow y \to 0$ and $y \neq 0$.

**(iv)** $P$ and $C$ intersect in the points $(-1, 1)$ and $(1, 1)$.

Shape of $C$ is as shown in Fig. 5.35. Therefore

$$\text{Required area} = \int_{-1}^{1}\left(\frac{2}{1+x^2} - x^2\right)dx$$

$$= 2\int_{0}^{1}\left(\frac{2}{1+x^2} - x^2\right)dx$$

$$= 4\left[\operatorname{Tan}^{-1}x\right]_0^1 - \frac{2}{3}\left[x^3\right]_0^1$$

$$= 4\operatorname{Tan}^{-1}1 - \frac{2}{3}$$

$$= 4\left(\frac{\pi}{4}\right) - \frac{2}{3}$$

$$= \frac{3\pi - 2}{3}$$

**Answer: (D)**

**111.** The area bounded by the curve $|y| = 1 - x^2$ is

(A) 3 (B) $\frac{8}{3}$ (C) $\frac{7}{3}$ (D) 2

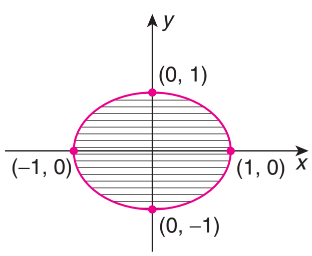



**FIGURE 5.36** Single correct choice type question 111.

**Solution:** $|y| = 1 - x^2$ represents two parabolas $y = 1 - x^2$ and $y = x^2 - 1$ with vertices at $(0, 1)$ and $(0, -1)$, respectively. Both will intersect with $x$-axis on $(-1, 0)$ and $(1, 0)$ (see Fig. 5.36). Therefore

$$\text{Area} = 4\int_{0}^{1}(1 - x^2)\,dx$$

$$= 4 - \frac{4}{3}\left[x^3\right]_0^1$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

**Answer: (B)**

**112.** The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to

(A) $\frac{5}{3}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

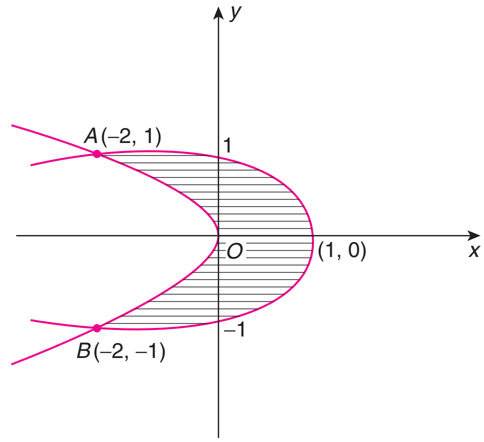



**FIGURE 5.37** Single correct choice type question 112.

**Solution:** See Fig. 5.37. The curve $x + 2y^2 = 0$ represents parabola with vertex at $(0, 0)$ and $x \le 0$. Now

$$3y^2 = 1 - x \Rightarrow y^2 = -\frac{1}{3}(x - 1)$$

This is a parabola with vertex at $(1, 0)$ and $x \le 1$. Therefore

$$\text{Required area} = 2\int_{0}^{1}(2y^2)\,dy + 2\int_{0}^{1}(1 - 3y^2)\,dy$$

$$= \frac{4}{3}\left[y^3\right]_0^1 + 2\left[y\right]_0^1 - 2\left[y^3\right]_0^1$$

$$= \frac{4}{3} + 2 - 2$$

$$= \frac{4}{3}$$

***Note:*** For the parabola $x + 3y^2 = 0, x \le 0$ so that in the first integral, we have taken $-x = 2y^2$.

**Answer: (D)**

**113.** The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If $S_1$, $S_2$, $S_3$ are, respectively, the areas of these parts numbered from top to bottom, then $S_1 : S_2 : S_3$ is

(A) 1:2:1 (B) 1:2:3

(C) 2:1:2 (D) 1:1:1

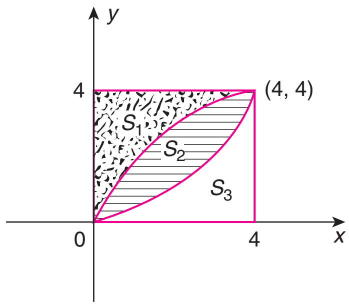



FIGURE 5.38 Single correct choice type question 113.

**Solution:** See Fig. 5.38. Now

$$S_1 = \int_0^4 \frac{y^2}{4}\,dy$$

$$= \frac{1}{12}(64) = \frac{16}{3}$$

$$S_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right)dx$$

$$= 2\times\frac{2}{3}\left[x^{3/2}\right]_0^4 - \frac{1}{12}\left[x^3\right]_0^4$$

$$= \frac{4}{3}\times 4^{3/2} - \frac{1}{12}(64)$$

$$= \frac{4}{3}\times 8 - \frac{16}{3} = \frac{16}{3}$$

$$S_3 = \int_0^4 \frac{x^2}{4}\,dx = \frac{1}{12}(64) = \frac{16}{3}$$

Therefore

$$S_1 : S_2 : S_3 = 1:1:1$$

**Answer: (D)**

**114.** The graph of $y = f(x)$ meets the $x$-axis in the points (0, 0) and (2, 0) and encloses an area of 3/4 square units with the axes. Then

$$\int_0^2 x f'(x)\,dx =$$

(A) $\frac{3}{2}$ (B) $\frac{3}{4}$ (C) $\frac{-3}{4}$ (D) 1

**Solution:** By hypothesis $f(0) = f(2) = 0$ and

$$\int_0^2 f(x)\,dx = \frac{3}{4}$$

Now

$$\int_0^2 x f'(x)\,dx = \left[x f(x)\right]_0^2 - \int_0^2 f(x)\,dx$$

$$= 2f(2) - \frac{3}{4}$$

$$= 0 - \frac{3}{4}$$

$$= -\frac{3}{4}$$

**Answer: (C)**

**115.** Let $C_1$ and $C_2$ be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$, respectively. Let $C_3$ be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point $P$ on $C_1$, let the lines through $P$, parallel to the axes, meet $C_2$ and $C_3$ at $Q$ and $R$, respectively (see Fig. 5.39). If for every position of $P$ on $C_1$, the areas of the shaded regions $OPQ$ and $ORP$ are equal, then $f(x)$ is equal to

(A) $x^2 - x$ $(0 \le x \le 1)$ (B) $x^3 - x^2$ $(0 \le x \le 1)$
(C) $x^4 - x^3$ $(0 \le x \le 1)$ (D) $x - x^2$ $(0 \le x \le 1)$

**(IIT-JEE 1998)**

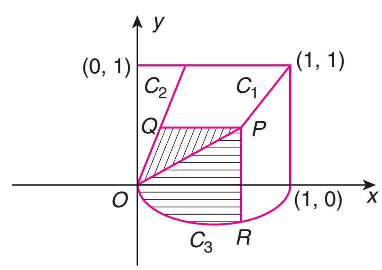



FIGURE 5.39 Single correct choice type question 115.

**Solution:** We have

$$C_1 : y = x^2\ (0 \le x \le 1)$$

$$C_2 : y = 2x\ (0 \le x \le 1)$$

$$C_3 = y = f(x)\ (0 \le x \le 1)$$

Let $P = (t, t^2)$ so that ordinate of $Q$ is $t^2$ and the abscissa of $R$ is $t$. Therefore

$$\text{Area of } OPQ = \int_0^{t^2} (x \text{ value of } C_1 - x \text{ value of } C_2)\,dy$$

$$= \int_0^{t^2} \left(\sqrt{y} - \frac{y}{2}\right)dy$$

$$= \frac{2}{3}t^3 - \frac{t^4}{4} \tag{5.23}$$

$$\text{Area of } OPR = \int_0^{t} (y \text{ value of } C_1 - y \text{ value of } C_3)\,dx$$

$$(\because y \text{ of } C_3 < 0)$$

$$= \int_0^t [x^2 - f(x)]dx$$

$$= \frac{1}{3}t^3 - \int_0^t f(x)dx \qquad (5.24)$$

Now, by hypothesis,

$$\text{Area of } OPQ = \text{Area of } OPR$$

From Eqs. (5.23) and (5.24),

$$\frac{2}{3}t^3 - \frac{t^4}{4} = \frac{1}{3}t^3 - \int_0^t f(x)dx$$

$$\Rightarrow \frac{1}{3}t^3 - \frac{t^4}{4} = -\int_0^t f(x)dx$$

Differentiating both sides w.r.t. $x$, we get

$$t^2 - t^3 = -f(t)$$

$$\Rightarrow f(t) = t^3 - t^2$$

**Answer: (B)**

*Problem 116 onwards are based on Differential Equation*

**116.** The order of the differential equation whose general solution is given by

$$y = (C_1 + C_2)\cos(x + C_3) - C_4 e^{x+C_5}$$

where $C_1$, $C_2$, $C_3$, $C_4$, and $C_5$ are arbitrary constants, is

(A) 5 (B) 4 (C) 3 (D) 2

**Solution:** The given solution can be written as

$$y = a\cos(x + C_3) - C_4 e^5 \cdot e^x$$

where $a = C_1 + C_2$. That is

$$y = a\cos(x + C_3) - be^x$$

where $a = C_1 + C_2$ and $b = C_4e^5$. Since there are only 3 arbitrary constants, the order of the differential equation is 3.

**Answer: (C)**

**117.** The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where $c$ is a positive parameter, is of

(A) Order 1, degree 2 (B) Order 1, degree 3
(C) Order 2, degree 2 (D) Order 2, degree 3

**Solution:** We have

$$y^2 = 2c(x + \sqrt{c}) \qquad (5.25)$$

Therefore

$$y\frac{dy}{dx} = c \qquad (5.26)$$

From Eqs. (5.25) and (5.26), we get

$$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{y\frac{dy}{dx}}\right)$$

or

$$\left(y^2 - 2xy\frac{dy}{dx}\right)^2 = 4y^3\left(\frac{dy}{dx}\right)^3$$

Therefore, the highest order derivative in the equation is $dy/dx$ and its power is 3. Hence, order is 1 and degree is 3.

**Answer: (B)**

**118.** The order and degree of the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$$

are, respectively

(A) 1, 2 (B) 2, 2 (C) 2, 1 (D) 2, 3

**Solution:** The highest order derivative occurring in the given equation is 2 and its power is 1. Therefore order is 2 and degree is 1.

**Answer: (C)**

**119.** The order and degree of the differential equation

$$\left[x + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = a\frac{d^2y}{dx^2}$$

are, respectively

(A) 2, 2 (B) 2, 3 (C) 2, 1 (D) 2, 4

**Solution:** The given equation can be written as

$$\left[x + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

Therefore order is 2 and degree is 2.

**Answer: (A)**

**120.** The order and degree of the differential equation of all straight lines in the $xy$-plane which are at constant distance $p$ from the origin are, respectively,

(A) 1, 1 (B) 2, 1 (C) 2, 2 (D) 1, 2

**Solution:** The equation of a straight line which is at a constant distance $p$ from the origin is

$$x\cos\theta + y\sin\theta = p \qquad (5.27)$$

Differentiating both sides w.r.t. $x$ we get

$$\cos\theta + \sin\theta\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\cot\theta$$

This implies

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+(dy/dx)^2}}$$

and

$$\cos\theta = (\sin\theta)\cot\theta = \frac{-dy/dx}{\sqrt{1+(dy/dx)^2}}$$

Substituting the values of $\cos\theta$ and $\sin\theta$ in Eq. (5.27), we have

$$x\left[\frac{-dy/dx}{\sqrt{1+(dy/dx)^2}}\right] + \frac{y}{\sqrt{1+(dy/dx)^2}} = p$$

$$\left(y - x\frac{dy}{dx}\right)^2 = p^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

Order is 1 and degree is 2.

**Answer: (D)**

**121.** The differential equation of the family of curves represented by the equation $y = Ae^{3x} + Be^{5x}$ is

(A) $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 15y = 0$

(B) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 15y = 0$

(C) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} - 15y = 0$

(D) $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} - 15y = 0$

**Solution:** We denote $dy/dx, d^2y/dx^2$, etc. by $y_1, y_2, y_3, \ldots$. Given equation is

$$y = Ae^{3x} + Be^{5x} \tag{5.28}$$

Therefore

$$y_1 = 3Ae^{3x} + 5Be^{5x} \tag{5.29}$$

Again

$$y_2 = 9Ae^{3x} + 25Be^{5x} \tag{5.30}$$

If we multiply Eq. (5.29) with 5 and subtract it from Eq. (5.30) we get

$$-6Ae^{3x} = y_2 - 5y_1$$

$$\Rightarrow Ae^{3x} = -\frac{1}{6}(y_2 - 5y_1) \tag{5.31}$$

Similarly, if we multiple Eq. (5.29) with 3 and subtract it from Eq. (5.30) we get

$$y_2 - 3y_1 = 10Be^{5x}$$

$$\Rightarrow Be^{5x} = \frac{y_2 - 3y_1}{10} \tag{5.32}$$

Substituting the values of $Ae^{3x}$ and $Be^{5x}$ as in Eqs. (5.31) and (5.32) in Eq. (5.28), we have

$$y = -\frac{1}{6}(y_2 - 5y_1) + \frac{y_2 - 3y_1}{10}$$

$$\Rightarrow 30y + 5(y_2 - 5y_1) - 3(y_2 - 3y_1) = 0$$

$$\Rightarrow 30y + 2y_2 - 16y_1 = 0$$

$$\Rightarrow y_2 - 8y_1 + 15y = 0$$

Hence

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

**Answer: (A)**

**122.** The differential equation for which $Ax^2 + By^2 = 1$ ($A$ and $B$ are arbitrary constants) is the general solution, is

(A) $\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right]^2 = y\dfrac{dy}{dx}$

(B) $x\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right] = y\dfrac{dy}{dx}$

(C) $x\left[y\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\right]^2 = y\dfrac{dy}{dx}$

(D) $y\left[x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right] = x\dfrac{dy}{dx}$

**Solution:** Given equation is

$$Ax^2 + By^2 = 1 \tag{5.33}$$

Differentiating both sides of Eq. (5.33) w.r.t. $x$ we get

$$Ax + (By)\frac{dy}{dx} = 0 \tag{5.34}$$

Again differentiating Eq. (5.34) w.r.t. $x$ we get

$$A + B\left(\frac{dy}{dx}\right)^2 + (By)\frac{d^2y}{dx^2} = 0 \tag{5.35}$$

From Eqs. (5.34) and (5.35), we have

$$\frac{y(dy/dx)}{x} = \frac{-A}{B} = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}$$

Therefore

$$y\frac{dy}{dx} = x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$$

**Answer: (B)**

**Try it out** The differential equation for which

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is the general solution, is

(A) $\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]x^2 = y\frac{dy}{dx}$

(B) $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^2 = y\frac{dy}{dx}$

(C) $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$

(D) $x\left[\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2\right] = x\frac{dy}{dx}$

***Hint:*** In Problem 122, take

$$A = \frac{1}{a^2} \quad \text{and} \quad B = -\frac{1}{b^2}$$

Therefore (C) is the correct answer. The student is advised to proceed as in Problem 122.

**123.** If the substitution $x = \tan z$ is used, then the transformed form of the equation

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = 0$$

is

(A) $\frac{d^2y}{dz^2} + 2y = 0$  (B) $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} - y = 0$

(C) $\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0$  (D) $\frac{d^2y}{dz^2} + y = 0$

**Solution:** Given that $x = \tan z$. Therefore

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{1+x^2}$$

So

$$(1+x^2)\frac{dy}{dx} = \frac{dy}{dz}$$

Again differentiating both sides w.r.t. $x$, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = \left(\frac{d^2y}{dz^2}\right)\frac{1}{1+x^2}$$

Therefore

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

Hence the given equation will be transformed to

$$\frac{d^2y}{dz^2} + y = 0$$

**Answer: (D)**

**124.** The differential equation of all circles in the $xy$-plane is

(A) $y_3(1+y_1^2) = 3y_1y_2^2$  (B) $y_3(1+y_1^2) = 3y_1^2y_2$

(C) $y_3(1+y_2^2) = 3y_1y_2^2$  (D) $y_3(1+y_2^2) = 3y_1^2y_2$

where $y_1, y_2, y_3$ are the first-, second- and third-order derivatives of $y$, respectively.

**Solution:** Equation of a circle in the $xy$-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{5.36}$$

where $g, f, c$ are arbitrary constants. Differentiating w.r.t. $x$, we get

$$x + y\,y_1 + g + fy_1 = 0 \tag{5.37}$$

Again differentiating Eq. (5.37) w.r.t. $x$, we get

$$1 + y_1^2 + yy_2 + fy_2 = 0 \tag{5.38}$$

Again

$$2y_1y_2 + y_1y_2 + y\,y_3 + fy_3 = 0$$

Therefore

$$\frac{3y_1y_2 + yy_3}{y_3} = -f \tag{5.39}$$

Substituting the value of $f$ [obtained in Eq. (5.39)] in Eq. (5.38) we have

$$1 + y_1^2 + yy_2 - \frac{y_2}{y_3}(3y_1y_2 + yy_3) = 0$$

$$\Rightarrow y_3(1+y_1^2) = 3y_1y_2^2$$

**Answer: (A)**

**125.** The differential equation whose general solution is

$$y = A \sin x + B \cos x + x \sin x$$

is

(A) $\dfrac{d^2y}{dx^2} + y = \cos x$ (B) $\dfrac{d^2y}{dx^2} + y = 2\cos x$

(C) $\dfrac{d^2y}{dx^2} - y = 2\sin x$ (D) $\dfrac{d^2y}{dx^2} - y = 2\cos x$

**Solution:** We have

$$y = A \sin x + B \cos x + x \sin x \qquad (5.40)$$

Differentiating w.r.t. $x$, we get

$$y_1 = A \cos x - B \sin x + \sin x + x \cos x \qquad (5.41)$$

Again, differentiating w.r.t. $x$, we get

$$\begin{aligned} y_2 &= -A\sin x - B\cos x + \cos x + \cos x - x\sin x \\ y_2 &= -(A\sin x + B\cos x + x\sin x) + 2\cos x \\ &= -y + 2\cos x \end{aligned}$$

Therefore

$$y_2 + y = 2 \cos x$$

**Answer: (B)**

**126.** If $y = y(x)$ and

$$\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x,\ y(0) = 1$$

then $y(\pi/2)$ equals

(A) $\dfrac{1}{3}$ (B) $\dfrac{2}{3}$ (C) $-\dfrac{1}{3}$ (D) 1

**Solution:** Given equation is

$$\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$$

Therefore

$$\frac{dy}{y+1} = \frac{-\cos x}{2+\sin x}dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{dy}{y+1} + \int \frac{\cos x}{2+\sin x}\,dx = c$$

$$\Rightarrow \log(y+1)(2+\sin x) = c$$

$$\Rightarrow (y+1)(2+\sin x) = k$$

Now

$$y(0) = 1 \Rightarrow (1+1)(2+0) = k$$

$$\Rightarrow k = 4$$

Therefore

$$(y+1)(2+\sin x) = 4$$

$$\Rightarrow y = \frac{4}{2+\sin x} - 1$$

Hence

$$y\left(\frac{\pi}{2}\right) = \frac{4}{2+1} - 1 = \frac{1}{3}$$

**Answer: (A)**

**127.** Solution of the equation

$$\frac{dy}{dx} = \sin(x+y)$$

is

(A) $\dfrac{1}{1+\tan\left(\dfrac{x+y}{2}\right)} = x + c$

(B) $1+\tan\left(\dfrac{x+y}{2}\right) = \dfrac{-2}{x+c}$

(C) $\tan\left(\dfrac{x+y}{2}\right) = \dfrac{-2}{x+c}$

(D) $\tan(x+y) = \dfrac{2}{x+c} - 1$

**Solution:** We have

$$\frac{dy}{dx} = \sin(x+y)$$

Put $x + y = z$. Therefore

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

This implies

$$\frac{dz}{dx} - 1 = \sin(x+y) = \sin z$$

$$\Rightarrow \frac{dz}{dx} = 1 + \sin z$$

$$\Rightarrow \frac{dz}{1+\sin z} = dx$$

Integrating we get

$$\int \frac{dz}{1+\sin z} = x + c$$

$$\Rightarrow \int \frac{dz}{[\sin(z/2)+\cos(z/2)]^2} = x+c$$

$$\Rightarrow \int \frac{\sec^2(z/2)}{[1+\tan(z/2)]^2}dz = x+c$$

$$\Rightarrow \frac{-2}{1+\tan(z/2)} = x+c$$

$$\Rightarrow \frac{-2}{x+c} = 1+\tan\left(\frac{x+y}{2}\right) \quad (\because z = x+y)$$

**Answer: (B)**

**128.** The differential equation

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$$

determines family of circles with

(A) Variable radii and fixed centre at (0, 1)
(B) Variable radii and fixed centre at (0, –1)
(C) Fixed radius 1 and variable centres along $x$-axis
(D) Fixed radius 1 and variable centres along $y$-axis

**Solution:** Given equation can be written as

$$\frac{y}{\sqrt{1-y^2}}dy = dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{y}{\sqrt{1-y^2}}dy = \int dx + c$$

$$\Rightarrow -\frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}} = x+c$$

$$\Rightarrow -\frac{1}{2}\times 2\sqrt{1-y^2} = x+c$$

Therefore

$$1-y^2 = x^2+2cx+c^2$$

$$\Rightarrow x^2+y^2+2cx+c^2 = 1$$

$$\Rightarrow (x+c)^2+y^2 = 1$$

**Answer: (C)**

**129.** The equation of the curve passing through the point (1, 1) and satisfying the differential equation

$$\frac{dy}{dx} = e^{x-y}+x^2e^{-y}$$

is given by

(A) $e^y = e^x + \frac{x}{2}+\frac{1}{2}$ (B) $e^y = e^x + \frac{x}{3}-\frac{1}{2}$
(C) $e^y = e^x + \frac{x^3}{3}-\frac{1}{3}$ (D) $e^y = e^x - \frac{x}{3}+\frac{1}{3}$

**Solution:** We have

$$\frac{dy}{dx} = e^{x-y}+x^2e^{-y}$$

$$\Rightarrow e^y dy = (e^x + x^2)\, dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int e^y\, dy = \int (e^x+x^2)dx + c$$

$$\Rightarrow e^y = e^x + \frac{1}{3}x^3 + c$$

The curve passes through (1, 1) implies

$$e = e + \frac{1}{3} + c$$

$$\Rightarrow c = -\frac{1}{3}$$

Therefore the curve equation is

$$e^y = e^x + \frac{x^3}{3} - \frac{1}{3}$$

**Answer: (C)**

**130.** Equation of the curve passing through the point (1,1) and which is a solution of the equation

$$\frac{dy}{dx} = \frac{2y}{x},\ x>0,\ y>0$$

is

(A) $y^2 = x$ (B) $y^2 = x^3$
(C) $x^2 = y^3$ (D) $x^2 = y$

**Solution:** We have

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{y} = \frac{2}{x}dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{dy}{y} = 2\int \frac{dx}{x} + c$$

$$\Rightarrow \log y = 2\log x + c$$

$$\Rightarrow y = kx^2$$

Curve passes through $(1, 1) \Rightarrow k = 1$. Therefore the curve is $y = x^2$.

**Answer: (D)**

**131.** Solution of the equation

$$y\frac{dy}{dx} = (1+y^2)x^2$$

is

(A) $y^2 = ce^{2x^3/3} - 1$ (B) $y^2 = ce^{x^3/3} - 1$

(C) $y^3 = ce^{2x^3/3} - 1$ (D) $y^3 = ce^{2x^3/3} + 1$

**Solution:** Given that

$$y\frac{dy}{dx} = (1+y^2)x^2$$

Therefore

$$\frac{y}{1+y^2}dy = x^2\,dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{y}{1+y^2}dy = \int x^2\,dx + c$$

$$\Rightarrow \frac{1}{2}\log(1+y^2) = \frac{1}{3}x^3 + c$$

$$\Rightarrow \log(1+y^2) = \frac{2}{3}x^3 + 2c$$

$$\Rightarrow 1+y^2 = e^{2x^3/3+2c}$$

Hence

$$y^2 = k\,e^{2x^3/3} - 1 \quad \text{where } k = e^{2c}$$

**Answer: (A)**

**132.** Solution of the equation

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

is given by

(A) $y = x + \sqrt{1-x^2}\sin c$

(B) $y = x\cos c + \sqrt{1-x^2}\sin c$

(C) $xy = \sqrt{1-x^2} + c$

(D) $y = xy + \sqrt{(1-x^2)(1-y^2)} + c$

**Solution:** We have

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Therefore

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

Integrating we get

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}} + c$$

$$\Rightarrow \mathrm{Sin}^{-1}y = \mathrm{Sin}^{-1}x + c$$

So

$$y = \sin(\mathrm{Sin}^{-1}x + c)$$

$$= \sin(\mathrm{Sin}^{-1}x)\cos c + \cos(\mathrm{Sin}^{-1}x)\sin c$$

$$= x\cos c + \sqrt{1-x^2}\sin c$$

**Answer: (B)**

**133.** The equation of the curve passing through the point (1, 3) and whose slope at any point $(x, y)$ is $-[1 + (y/x)]$ is given by

(A) $xy + 4x^2 = 7$ (B) $2xy + x^2 = 7$

(C) $3xy - 2x^2 = 7$ (D) $\sqrt{xy} + 4x^2 = 7$

**Solution:** By hypothesis

$$\frac{dy}{dx} = -1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = -1$$

Put $z = y/x$. Therefore

$$z + x\frac{dz}{dx} + z = -1$$

$$\Rightarrow x\frac{dz}{dx} = -(2z+1)$$

Therefore,

$$\frac{dz}{2z+1} + \frac{dx}{x} = 0 \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{dz}{2z+1} + \int \frac{dx}{x} = c$$

$$\Rightarrow \frac{1}{2}\log|2z+1| + \log|x| = c$$

$$\Rightarrow \log|2z+1| + \log(x^2) = c$$

$$\Rightarrow |2z+1| = e^{2c-\log(x^2)} = \frac{k}{x^2}$$

$$\Rightarrow 2z+1 = \frac{k}{x^2}$$

$$\Rightarrow 2\left(\frac{y}{x}\right) + 1 = \frac{k}{x^2}$$

The curve passes through (1, 3) $\Rightarrow k = 7$. Therefore $2xy + x^2 = 7$ is the curve equation.

**Answer: (B)**

**134.** Solution of the equation

$$\frac{dy}{dx} = (x+y)^2$$

is

(A) $y = \tan(x + c) - x$ (B) $y = \tan(x + c) + x$
(C) $y = \cot(x + c) + x$ (D) $y = \tan x + cx$

**Solution:** The given equation is not directly variable separable, but it will be variable separable if we put $z = x + y$. Now

$$\frac{dz}{dx} - 1 = z^2$$

or
$$\frac{dz}{z^2+1} = \frac{dx}{x}$$

Therefore the solution is

$$\mathrm{Tan}^{-1} z = x + c$$

or
$$z = \tan(x + c)$$

Hence

$$y = \tan(x + c) - x$$

**Answer: (A)**

**135.** A curve $y = f(x)$ passes through the point $P(1, 1)$. The equation of the normal at $P(1, 1)$ to the curve

$$y = f(x) \text{ is}$$

$$(x - 1) + a(y - 1) = 0$$

and the slope of the tangent at any point on the curve is proportional to the ordinate of the point. Then the equation of the curve is

(A) $x^2 + y^2 = 1$ (B) $y^2 = x$
(C) $(y - 1)^2 = a(x - 1)$ (D) $y = e^{a(x-1)}$

**Solution:** By hypothesis,

$$\frac{dy}{dx} = ky$$

where $k \neq 0$. Therefore the slope of the normal at $(1, 1)$ is

$$-\frac{1}{k} = -\frac{1}{a} \quad \text{(By hypothesis)}$$

Therefore $k = a$. Now

$$\frac{dy}{y} = a\,dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{dy}{y} = a\int dx + c$$

$$\Rightarrow \log y = ax + c$$

$$\Rightarrow y = e^{ax+c}$$

The curve passes through the point $(1, 1)$. This implies

$$1 = e^{a+c}$$

So

$$a + c = 0 \text{ or } c = -a$$

Hence the curve equation is

$$y = e^{ax-a} = e^{a(x-1)}$$

**Answer: (D)**

**136.** $A$ and $B$ are locations and the distance $AB$ is 8 m. $O$ is the midpoint of $AB$. At the point '$O$', a 2 m long object is fired vertically upwards. The speed of the object after $t$ seconds is given by

$$\frac{ds}{dt} = (2t + 1) \text{ m/s}$$

Let $\alpha$ and $\beta$ be the angles subtended by the object at $A$ and $B$, respectively, after 1s and 2s. Then $\cos(\alpha - \beta)$ is

(A) $\frac{4}{\sqrt{26}}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{5}{\sqrt{26}}$

**Solution:** See Fig. 5.40. Let $OM_1 = 2$ be the starting position of the object. Let $M_1N_1$ and $M_2N_2$ be the positions of the object after 1s and 2s, respectively. Since '$O$' is the midpoint of AB and $ON_2$ is perpendicular to $AB$, the object subtends equal angles at $A$ and $B$.

Let $\angle OAM_1 = \theta_1$ and $\angle OAM_2 = \theta_2$. By hypothesis

$$\frac{ds}{dt} = 2t + 1$$

$$\Rightarrow ds = (2t + 1)\,dt$$

$$\Rightarrow s = t^2 + t + c$$

So

$$t = 0, s = 0 \Rightarrow c = 0$$

Now $s = t^2 + t$ is the distance travelled by the object in $t$ seconds. When $t = 1$, we have $s = 2$.

$$\tan\theta_1 = \frac{2}{4} = \frac{1}{2}$$

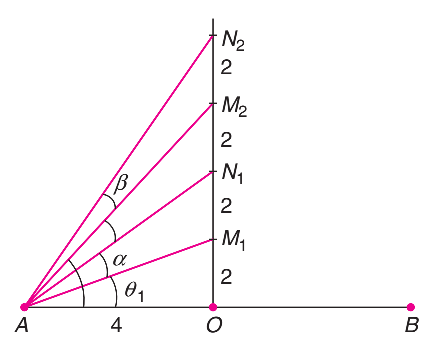



**FIGURE 5.40** Single correct choice type question 136.

and $$\tan(\alpha+\theta_1)=\frac{4}{4}=1$$

Therefore

$$\begin{aligned}\tan\alpha &= \tan(\alpha+\theta_1-\theta_1)\\ &= \frac{\tan(\alpha+\theta_1)-\tan\theta_1}{1+\tan(\alpha+\theta_1)\tan\theta_1}\\ &= \frac{1-(1/2)}{1+1(1/2)}=\frac{1}{3}\end{aligned}$$

Again

$$\tan\theta_2=\frac{6}{4}=\frac{3}{2}$$

and $$\tan(\beta+\theta_2)=\frac{ON_2}{OA}=\frac{8}{4}=2$$

Therefore

$$\begin{aligned}\tan\beta &= \tan(\beta+\theta_2-\theta_2)\\ &= \frac{2-(3/2)}{1+2(3/2)}=\frac{1}{8}\end{aligned}$$

Now, $\tan\alpha = 1/3$ and $\tan\beta = 1/8$ implies

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{(1/3)-(1/8)}{1+(1/3)(1/8)}\\ &= \frac{5}{25}=\frac{1}{5}\end{aligned}$$

Hence

$$\begin{aligned}\cos(\alpha-\beta) &= \frac{1}{\sqrt{1+\tan^2(\alpha-\beta)}}\\ &= \frac{1}{\sqrt{1+(1/25)}}=\frac{5}{\sqrt{26}}\end{aligned}$$

**Answer: (D)**

**137.** A normal is drawn at a point $P\,(x,y)$ of a curve. It meets $x$-axis in $Q$. If $PQ$ is of constant length $k$ and if the curve passes through $(0, k)$, then its equation is

(A) $xy = k^2$ (B) $x^2 + y^2 = k^2$
(C) $y^2 = kx$ (D) $x^2 = 4ky$

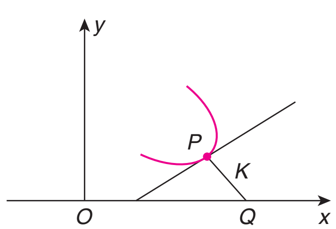



**FIGURE 5.41** Single correct choice type question 137.

**Solution:** See Fig. 5.41. The length of the normal at $P\,(x, y) = PQ$ which equals

$$\left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right| = k \quad \text{(By hypothesis)}$$

Therefore

$$\left(\frac{dy}{dx}\right)^2=\frac{k^2}{y^2}-1$$

$$\Rightarrow \frac{dy}{dx}=\pm\frac{\sqrt{k^2-y^2}}{y}$$

Now

$$\frac{y}{\sqrt{k^2-y^2}}dy=\pm dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int\frac{y}{\sqrt{k^2-y^2}}dy=\pm x+c$$

or $$-\sqrt{k^2-y^2}=\pm x+c$$

The curve passes through $(0, k) \Rightarrow c = 1$. Therefore the curve is $x^2 + y^2 = k^2$.

**Answer: (B)**

**138.** A solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}+y=0$$

is

(A) $y = 2$ (B) $y = 2x$
(C) $y = 2x - 4$ (D) $y = 2x^2 - 4$

**Solution:** We have

$$\frac{dy}{dx}=\frac{x\pm\sqrt{x^2-4y}}{2}$$

$$\Rightarrow \frac{2dy}{dx}-x=\pm\sqrt{x^2-4y} \tag{5.42}$$

Put $x^2 - 4y = z$. So

$$2x-4\frac{dy}{dx}=\frac{dz}{dx} \tag{5.43}$$

From Eqs. (5.42) and (5.43)

$$\frac{dz}{\sqrt{z}}=\mp 2\,dx$$

Integrating we get

$$\int \frac{dz}{\sqrt{z}} = \mp 2\int dx + c$$

$$\Rightarrow 2\sqrt{z} = \mp 2x + c$$

$$\Rightarrow \sqrt{z} = \mp x + c'$$

$$\Rightarrow z = x^2 \pm 2c'x + c'^2$$

$$\Rightarrow x^2 - 4y = x^2 \pm 2c'x + c'^2$$

$$\Rightarrow -4y = \pm 2c'x + c'^2$$

$$\Rightarrow 4y = \mp 2c'x - c'^2$$

If $4y = 2c'x - c'^2$ and $c' = 4$, then $y = 2x - 4$ is a solution and if $4y = -2c'x - c'^2$ and $c' = -4$ then also $y = 2x - 4$ is a solution.

**Answer: (C)**

**139.** If the length of the tangent at any point on the curve $y = f(x)$ intercepted between the point of contact and this $x$-axis is 1, then the equation of the curve is

(A) $\log\left|\dfrac{1-\sqrt{1-y^2}}{1+\sqrt{1+y^2}}\right| = \pm x + c$

(B) $\log\left|\dfrac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right| = x^2 + c$

(C) $\sqrt{1-y^2} + \dfrac{1}{2}\log\left|\dfrac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right| = \pm x + c$

(D) $\sqrt{1-y^2} - \log\left|\dfrac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right| = \pm x + c$

**Solution:** The length on the tangent is 1. This implies

$$\left|y\sqrt{1+(dx/dy)^2}\right| = 1$$

$$\Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \frac{1}{y^2}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1-y^2}{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$$

$$\Rightarrow \frac{\sqrt{1-y^2}}{y}dy = \pm dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{\sqrt{1-y^2}}{y}dy = \pm x + c$$

Put $1 - y^2 = t^2$ so that $y\,dy = -t\,dt$. Therefore

$$\int \frac{\sqrt{1-y^2}}{y^2}y\,dy = \pm x + c$$

$$\Rightarrow \int \frac{t}{1-t^2}(-t)\,dt = \pm x + c$$

$$\Rightarrow \int \frac{1-t^2-1}{1-t^2}dt = \pm x + c$$

$$\Rightarrow \int\left(1 - \frac{1}{1-t^2}\right)dt = \pm x + c$$

$$\Rightarrow t + \frac{1}{2}\log\left|\frac{1-t}{1+t}\right| = \pm x$$

So

$$\sqrt{1-y^2} + \frac{1}{2}\log\left|\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right| = \pm x + c$$

**Answer: (C)**

**140.** Let

$$\frac{dy}{dx} = \frac{6}{x+y}$$

where $y(0) = 0$. If $x + y = 6$, then the value of $y$ is equal to

(A) $\log_e 4$ (B) $2\log_e 3$
(C) $6\log_e 2$ (D) $4\log_e 2$

**(IIT-JEE 2006)**

**Solution:** Put $z = x + y$ in the given equation. We get

$$\frac{dz}{dx} - 1 = \frac{dy}{dx} = \frac{6}{z}$$

$$\Rightarrow \frac{z}{6+z}dz = dx \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{z}{6+z}dz = x + c$$

$$\Rightarrow \int\left(1 - \frac{6}{6+z}\right)dz = x + c$$

$$\Rightarrow z - 6\log_e|6+z| = x + c$$

$$\Rightarrow y - 6\log_e|6+x+y| = c \quad (\because x + y = z)$$

Now

$$y(0) = 0 \Rightarrow c = -6\log_e 6$$

Therefore

$$y - 6\log_e|6+x+y| = -6\log_e 6$$

When $x + y = 6$, then

$$y - 6\log_e 12 = -6\log_e 6$$
$$\Rightarrow y = 6\log_e 2 + 6\log_e 6 - 6\log_e 6$$
$$\Rightarrow y = 6\log_e 2$$

**Answer: (C)**

**141.** Tangent is drawn at any point $P$ of a curve which passes through (1, 1) cutting $x$- and $y$-axes in $A$ and $B$, respectively. If $AP : PB = 3 : 1$, then

(A) differential equation of the curve is
$$3x\frac{dy}{dx} + y^2 = 0$$

(B) differential equation of the curve is
$$3x\frac{dy}{dx} - y = 0$$

(c) the curve passes through the point (1/8, 2)

(d) the normal at (1, 1) is $x + 3y = 2$

**(IIT-JEE 2006)**

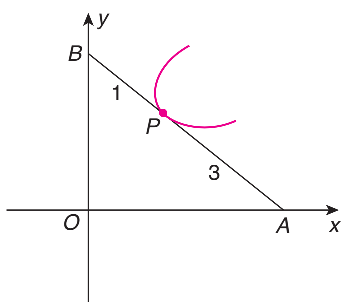



**FIGURE 5.42** Single correct choice type question 141.

**Solution:** See Fig. 5.42. Equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

where

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Now

$$y = 0 \Rightarrow x = x_1 - \frac{y_1}{m}$$

so that $A = (x_1 - y_1/m, 0)$ and

$$x = 0 \Rightarrow y = -mx_1 + y_1$$

so that $B = (0, y_1 - mx_1)$. Now $AP : PB = 3 : 1$ implies

$$x_1 = \frac{x_1 - (y_1/m)}{4}$$

and
$$y_1 = \frac{3(y_1 - mx_1)}{4}$$

Therefore

$$3x_1 + \frac{y_1}{m} = 0$$

and
$$y_1 + 3mx_1 = 0$$

So the differential equation is

$$3x\frac{dy}{dx} + y = 0 \qquad (5.44)$$
$$\Rightarrow \frac{3}{y}dy + \frac{dx}{x} = 0 \quad \text{(Variables Separable)}$$

Integrating we get

$$\int \frac{3}{y}dy + \int \frac{dx}{x} = c$$
$$\Rightarrow 3\log y + \log x = c$$
$$\Rightarrow xy^3 = k$$

Curve passes through (1, 1) $\Rightarrow k = 1$. Therefore

$$xy^3 = 1$$

Also (1/8, 2) lies on $xy^3 = 1$.

**Answer: (C)**

**142.** Let $f$ be a real-valued differentiable function on $\mathbb{R}$ such that $f(1) = 1$. If the $y$-intercept of the tangent at any point $P(x,y)$ on the curve is equal to the cube of the abscissa of $P$, then the value of $f(2)$ is

(A) 1 (B) 2 (C) –1 (D) –2

**(IIT-JEE 2010)**

**Solution:** Equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

where

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Now

$$x = 0 \Rightarrow y = y_1 - mx_1$$

By hypothesis,

$$y_1 - mx_1 = x_1^3$$

Therefore the differential equation is

$$y - x\frac{dy}{dx} = x^3$$
$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -x^2 \quad \text{(Linear equation)}$$

The integrating factor is

$$\text{I.F.} = e^{\int -dx/x} = \frac{1}{x}$$

Therefore the solution is

$$y\left(\frac{1}{x}\right) = \int (-x^2)\left(\frac{1}{x}\right)dx + c$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{2}x^2 + c$$

Now

$$f(1) = 1$$

$$\Rightarrow 1 = -\frac{1}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

Therefore

$$y = -\frac{x^3}{2} + \frac{3}{2}x$$

$$\Rightarrow f(x) = -\frac{x^3}{2} + \frac{3}{2}x$$

So

$$f(2) = -\frac{8}{2} + \frac{3}{2}(2)$$

$$= -4 + 3 = -1$$

**Answer: (C)**

**143.** A spherical raindrop evaporates at a rate proportional to its surface area at any instant $t$. The differential equation giving the rate of change of the radius of the raindrop is

(A) $\frac{dr}{dt} + k = 0 \quad (k > 0)$ (B) $2\frac{dr}{dt} + k^2 = 0$

(C) $r\,dr + t\,dt = k$ (D) $r^2\,dr - t\,dt = 0$

**Solution:** Let $r$ be the radius of the raindrop at any instant and $V$ its volume. Then

$$V = \frac{4}{3}\pi r^3$$

By hypothesis, $V$ is proportional to the surface area $S$ $(= 4\pi r^2)$. Then

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k(4\pi r^2)$$

where $k$ is the constant of proportionality. So

$$4\pi r^2 \frac{dr}{dt} = k(4\pi r^2)$$

$$\Rightarrow \frac{dr}{dt} = k$$

Since $r$ is decreasing (because $V$ is decreasing) we have

$$\frac{dr}{dt} = -k \quad \text{or} \quad \frac{dr}{dt} + k = 0$$

**Answer: (A)**

**144.** Solution of the differential equation

$$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$$

is

(A) $xy(x^2 - y^2) = c$ (B) $xy\sqrt{y^2 - x^2} = c$

(C) $xy(x^2 + y^2) = c$ (D) $x^2 + y^2 = cxy$

**Solution:** We have

$$\frac{dy}{dx} = \frac{2x^2y - y^3}{2xy^2 - x^3}$$

which is a homogeneous equation. Put $y = vx$. Then

$$v + \frac{xdv}{dx} = \frac{2v - v^3}{2v^2 - 1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v^3}{2v^2 - 1} - v = \frac{3v - 3v^3}{2v^2 - 1}$$

$$\Rightarrow \frac{2v^2 - 1}{v(1 - v^2)}dv = \frac{3}{x}dx$$

Integrating we get

$$\int\left(\frac{-1}{v} - \frac{1}{2(v-1)} - \frac{1}{2(1+v)}\right)dv = \int \frac{3}{x}dx + c$$

$$\Rightarrow \log(v\sqrt{v^2 - 1}) + \log x^3 = c$$

$$\Rightarrow x^3\left(\frac{y}{x}\right)\sqrt{\frac{y^2}{x^2} - 1} = c$$

$$\Rightarrow xy\sqrt{y^2 - x^2} = c$$

**Answer: (B)**

**145.** The normal at any point $P(x, y)$ of a curve meets $x$- and $y$-axes in $A$ and $B$, respectively. If $O$ is the origin and $OA$ and $OB$ are the algebraic intercepts of the normal at $P$ such that

$$\frac{1}{OA} + \frac{1}{OB} = 1$$

then the equation of the curve is [given that (5, 4) is a point on the curve]

(A) $x^2 - y^2 = 9$

(B) $xy = 20$

(C) $(x+1)^2+(y+1)^2=61$
(D) $(x-1)^2+(y-1)^2=25$

**Solution:** Normal at $(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

where

$$m = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$$

Now

$$y = 0 \Rightarrow x = x_1 + my_1$$

$$x = 0 \Rightarrow y = y_1 + \frac{x_1}{m}$$

By hypothesis

$$\frac{1}{x_1 + my_1} + \frac{1}{y_1 + (x_1/m)} = 1$$

$$\Rightarrow \frac{1+m}{x_1 + my_1} = 1$$

$$\Rightarrow 1 + m = x_1 + my_1$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = x_1 + y_1\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$$

Therefore, the differential equation is

$$1 + \frac{dy}{dx} = x + y\frac{dy}{dx}$$

$$\Rightarrow (1-y)\frac{dy}{dx} = x - 1$$

$$\Rightarrow (1-y)\,dy = (x-1)\,dx \quad \text{(Variables Separable)}$$

Integrating we get

$$c + \int (1-y)\,dy = \int (x-1)\,dx$$

$$\Rightarrow c - (y-1)^2 = (x-1)^2$$

$$\Rightarrow (x-1)^2 + (y-1)^2 = c$$

The curve passes through $(5, 4) \Rightarrow c = 25$. Therefore

$$(x-1)^2 + (y-1)^2 = 25$$

**Answer: (D)**

**146.** If the algebraic sub-tangent at any point of a curve is equal to half of the sum of the coordinates of the point, then the equation of the curve is

(A) $x^2 + y^2 = cy$ (B) $(x-y)^2 = cy$
(C) $(x-y)^2 = cx$ (D) $x^2 - y^2 = c^2$

**Solution:** By hypothesis

$$\frac{y}{dy/dx} = \frac{x+y}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x+y} \quad \text{(Homogeneous)}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = \frac{2v}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1+v} - v = \frac{v - v^2}{1+v}$$

$$\Rightarrow \frac{1+v}{v(1-v)}dv = \frac{dx}{x}$$

Integrating we get

$$\int \frac{1+v}{v(1-v)}dv = \int \frac{dx}{x} + c$$

$$\Rightarrow \int \left(\frac{1}{v} + \frac{2}{1-v}\right)dv = \int \frac{dx}{x} + c$$

$$\Rightarrow \log|v| - 2\log|1-v| = \log|x| + c$$

$$\Rightarrow (1-v)^2 x = cv$$

$$\Rightarrow (x-y)^2 = cy$$

**Answer: (B)**

**147.** By a suitable substitution, the equation

$$y^3\frac{dy}{dx} + x + y^2 = 0$$

can be transformed to

(A) Variables separable (B) Homogeneous
(C) Linear (D) Bernoulli's equation

**Solution:** Putting $z = x + y^2$ in the given equation, we get

$$\frac{dz}{dx} = 1 + (2y)\frac{dy}{dx}$$

$$\Rightarrow (z-x)\frac{1}{2}\left(\frac{dz}{dx} - 1\right) + z = 0$$

$$\Rightarrow (z-x)\left(\frac{dz}{dx} - 1\right) + 2z = 0$$

$$\Rightarrow (z-x)\frac{dz}{dx} - (z-x) + 2z = 0$$

$$\Rightarrow (z-x)\frac{dz}{dx} + z + x = 0$$

$$\Rightarrow \frac{dz}{dx} + \frac{z+x}{z-x} = 0$$

which is a homogeneous equation.

**Answer: (B)**

**148.** Solution of the equation

$$\frac{dy}{dx} = \sqrt{y-x}$$

is

(A) $2\sqrt{y-x} + 2\log\left|\sqrt{y-x} - 1\right| = x + c$

(B) $2\sqrt{y-x} + \log\left|\sqrt{y-x} - 1\right| = x + c$

(C) $2\sqrt{y-x} + \log|y-x-1| = x + c$

(D) $\log|y-x-1| = y - x + c$

**Solution:** Putting $y - x = z^2$ in the given equation, we get

$$\frac{dy}{dx} = 1 + 2z\frac{dz}{dx}$$

$$\Rightarrow 1 + 2z\frac{dz}{dx} = z$$

$$\Rightarrow \frac{2z}{z-1}dz = dx \quad \text{(Variables Separable)}$$

$$\Rightarrow \int \frac{2z}{z-1}dz = \int dx + c$$

Integrating we get

$$\int\left(2 + \frac{2}{z-1}\right)dz = x + c$$

$$\Rightarrow 2z + 2\log|z-1| = x + c$$

$$\Rightarrow 2\sqrt{y-x} + 2\log\left|\sqrt{y-x} - 1\right| = x + c$$

**Answer: (A)**

**149.** Solution of the differential equation

$$\frac{dy}{dx} = \frac{y^3 + y^2\sqrt{y^2 - x^2}}{x^3}$$

is

(A) $y + \sqrt{y^2 - x^2} = k\,xy$  (B) $\sqrt{y^2 - x^2} = k\,xy$

(C) $y^2 + x^2 = k\,xy$  (D) $y^2 - x^2 = k\,x^2y^2$

**Solution:** Substituting $y = vx$ in the given equation, we get

$$v + x\frac{dv}{dx} = v^3 + v^2\sqrt{v^2 - 1}$$

$$\Rightarrow x\frac{dv}{dx} = v(v^2 - 1) + v^2\sqrt{v^2 - 1}$$

$$= v\sqrt{v^2 - 1}(v + \sqrt{v^2 - 1})$$

$$\Rightarrow \frac{1}{v\sqrt{v^2 - 1}(v + \sqrt{v^2 - 1})}dv = \frac{dx}{x}$$

Integrating we get

$$\int \frac{dv}{v\sqrt{v^2 - 1}(v + \sqrt{v^2 - 1})} = \int \frac{dx}{x} + c$$

$$\Rightarrow \int \frac{v - \sqrt{v^2 - 1}}{v\sqrt{v^2 - 1}}dv = \log x + c$$

$$\Rightarrow \int\left(\frac{1}{\sqrt{v^2 - 1}} - \frac{1}{v}\right)dv = \log x + c$$

$$\Rightarrow \log(v + \sqrt{v^2 - 1}) - \log v = \log x + c$$

$$\Rightarrow v + \sqrt{v^2 - 1} = k\,y$$

$$\Rightarrow y + \sqrt{y^2 - x^2} = k\,xy$$

**Answer: (A)**

**150.** The equation of the curve passing through the point $(1, \pi/4)$ and tangent at any point of which makes angle $\mathrm{Tan}^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ with the $x$-axis, is

(A) $y = \mathrm{Tan}^{-1}\left(\log\frac{e}{x}\right)$

(B) $y = x\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) + 1$

(C) $y = x\mathrm{Tan}^{-1}(1 - \log x)$

(D) $y - x = \mathrm{Tan}^{-1}(1 - \log x)$

**Solution:** By hypothesis

$$\frac{dy}{dx} = \tan\left(\mathrm{Tan}^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)\right)$$

$$= \frac{y}{x} - \cos^2\frac{y}{x}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

$$\Rightarrow (\sec^2 v)dv + \frac{dx}{x} = 0$$

Integrating we get

$$\int(\sec^2 v)dv + \int\frac{dx}{x} = c$$

$$\Rightarrow \tan v + \log x = c$$

$$\Rightarrow \tan\left(\frac{y}{x}\right) + \log x = c$$

The curve passes through $(1, \pi/4) \Rightarrow 1 = c$. Therefore

$$\tan v + \log x = 1$$
$$\Rightarrow \tan\frac{y}{x} = 1 - \log x$$
$$\Rightarrow \frac{y}{x} = \mathrm{Tan}^{-1}(1-\log x)$$
$$\Rightarrow y = x\mathrm{Tan}^{-1}(1-\log x)$$

**Answer: (C)**

**151.** Solution of the differential equation

$$y^2dx + (x^2 - xy + y^2)\,dy = 0$$

is

(A) $y = ke^{\mathrm{Tan}^{-1}(y/x)}$ (B) $y = kxe^{\mathrm{Tan}^{-1}(y/x)}$

(C) $y = kx\mathrm{Tan}^{-1}(y/x)$ (D) $x = kye^{\mathrm{Tan}^{-1}(x/y)}$

**Solution:** We have

$y^2dx + (x^2 - xy + y^2)\,dy = 0$

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy + y^2} \quad \text{(Homogeneous)}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = \frac{-v^2}{1 - v + v^2}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2} - v = -\frac{(v+v^3)}{1-v+v^2}$$
$$\Rightarrow \frac{1-v+v^2}{v(1+v^2)}dv + \frac{dx}{x} = 0$$

Integrating we get

$$\int\left(\frac{1}{v} - \frac{1}{1+v^2}\right)dv + \log x = c$$
$$\Rightarrow \log v + \log x - \mathrm{Tan}^{-1}v = c$$
$$\Rightarrow \log(v\,x) - \mathrm{Tan}^{-1}v = c$$
$$\Rightarrow y = e^{c+\mathrm{Tan}^{-1}(y/x)} = k\,e^{\mathrm{Tan}^{-1}(y/x)}$$
$$\Rightarrow y = k\,e^{\mathrm{Tan}^{-1}(y/x)}$$

**Answer: (A)**

**152.** Solution of

$$\frac{x(dy/dx) - y}{\sqrt{x^2 - y^2}} = mx^2$$

is

(A) $\mathrm{Sin}^{-1}\left(\dfrac{y}{x}\right) = \dfrac{mx}{2} + c$

(B) $\mathrm{Sin}^{-1}\left(\dfrac{y}{x}\right) = \dfrac{mx^2}{2} + c$

(C) $\sin\left(\dfrac{y}{x}\right) = xy + c$

(D) $y = mx + \sin\left(\dfrac{y}{x}\right) + c$

**Solution:** We have

$$x\frac{dy}{dx} - y = mx^2\sqrt{x^2 - y^2}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} - v = mx\sqrt{x^2 - v^2x^2}$$
$$\Rightarrow x\frac{dv}{dx} = mx^2\sqrt{1-v^2}$$
$$\Rightarrow \frac{dv}{\sqrt{1-v^2}} = (mx)\,dx$$

Integrating we get

$$\int\frac{dv}{\sqrt{1-v^2}} = m\int x\,dx + c$$
$$\Rightarrow \mathrm{Sin}^{-1}v = \frac{mx^2}{2} + c$$
$$\Rightarrow \mathrm{Sin}^{-1}\left(\frac{y}{x}\right) = \frac{mx^2}{2} + c$$

**Answer: (B)**

**153.** Solution of the equation

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$$

is

(A) $y + \sqrt{x^2 + y^2} = k\,x^2$

(B) $y - \sqrt{x^2 + y^2} = k\,x$

(C) $\sqrt{x^2 + y^2} = k\,x^2$

(D) $x + \sqrt{x^2 + y^2} = k\,y^2$

**Solution:** We have

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$$
$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\sqrt{x^2 + y^2}}{x}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} - v = \sqrt{1 + v^2}$$
$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating we get

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} + c$$

$$\Rightarrow \log(v+\sqrt{1+v^2}) = \log x + c$$

$$\Rightarrow v+\sqrt{1+v^2} = kx$$

$$\Rightarrow y+\sqrt{x^2+y^2} = kx^2$$

**Answer: (A)**

**154.** Solution of the equation

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$$

is

(A) $\sin\left(\frac{y}{x}\right) = \log|kx|$ (B) $y = x\log|kx|$

(C) $\cos\left(\frac{y}{x}\right) = \log|kx|$ (D) $\tan\left(\frac{y}{x}\right) = \log|kx|$

**Solution:** We have

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$$

$$\frac{dy}{dx} = \frac{y\sin(y/x) - x}{x\sin(y/x)}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v = \frac{-1}{\sin v}$$

$$\Rightarrow (\sin v)\,dv + \frac{dx}{x} = 0$$

Integrating we get

$$\int(\sin v)\,dv + \int\frac{dx}{x} = c$$

$$\Rightarrow -\cos v + \log|x| = c$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \log|kx|$$

**Answer: (C)**

**155.** The general solution of the equation

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

is

(A) $\mathrm{Cos}^{-1}\left(\frac{y}{x}\right) = \log|cx|$

(B) $\sin\left(\frac{y}{x}\right) = \log|cx|$

(C) $\mathrm{Sin}^{-1}\left(\frac{y}{x}\right) = \log|cx|$

(D) $\cos\left(\frac{y}{x}\right) = \mathrm{Sin}^{-1}(\log|cx|)$

**Solution:** We have

$$\frac{dy}{dx} = \frac{y\cos(y/x) + x}{x\cos(y/x)}$$

Put $v = y/x$. Then

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v = \frac{1}{\cos v}$$

$$\Rightarrow \cos v\,dv = \frac{dx}{x}$$

The solution is

$$\sin v = \log|x| + c$$

$$\Rightarrow \sin\frac{y}{x} = \log|cx|$$

**Answer: (B)**

**156.** The general solution of the differential equation

$$2ye^{x/y}dx + (y - 2xe^{x/y})\,dy = 0$$

is

(A) $2e^{x/y} + \log|x| = c$ (B) $e^{x/y} + 2y\log|x| = c$

(C) $xe^{x/y} + \log|y| = c$ (D) $2e^{x/y} + \log|y| = c$

**Solution:** We have

$$2ye^{x/y}dx + (y - 2xe^{x/y})\,dy = 0$$

Therefore

$$\frac{dx}{dy} = -\frac{y - 2xe^{x/y}}{2ye^{x/y}}$$

$$= -\frac{1}{2}e^{-x/y} + \frac{x}{y}$$

So

$$\frac{dx}{dy} - \frac{x}{y} = -\frac{1}{2}e^{-x/y} \tag{5.45}$$

Put $x = vy$. Then

$$v + y\frac{dv}{dy} = \frac{dx}{dy}$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{1}{2}e^{-v} \quad \text{[By Eq. (5.45)]}$$

$$\Rightarrow 2e^{v}dv + \frac{dy}{y} = 0$$

Integrating we get

$$\int 2e^v dv + \int \frac{dy}{y} = c$$

$$\Rightarrow 2e^v + \log|y| = c$$

$$\Rightarrow 2e^{x/y} + \log|y| = c$$

**Answer: (D)**

**157.** General solution of the differential equation

$$\frac{ydx - xdy}{y^2} = 0$$

is

(A) $xy = c$ (B) $x^2 = cy$
(C) $y^2 = cx$ (D) $y = cx$

**Solution:** We have

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \log|y| - \log|x| = c$$

$$\Rightarrow y = kx$$

**Answer: (D)**

**158.** A curve is passing through the point $(0,1)$. If the slope of the curve at any point $(x, y)$ is equal to $x + xy$, then the equation of the curve is

(A) $y = 2e^{x^2/2}$ (B) $y + 1 = 2e^{x^2/2}$
(C) $y - 1 = e^{x^2/2}$ (D) $x + 1 = 2e^{x^2/2}$

**Solution:** By hypothesis,

$$\frac{dy}{dx} = x + xy$$

$$\Rightarrow \frac{dy}{dx} + (-x)y = x \quad \text{(Linear equation)}$$

Now $P = -x$, $Q = x$. The integrating factor is

$$\text{I.F.} = e^{\int P dx} = e^{-x^2/2}$$

Therefore, the solution is

$$y \cdot e^{-x^2/2} = \int x e^{-x^2/2} dx + c$$

$$= \int e^t(-1)\, dt + c \quad \text{where } t = \frac{-x^2}{2}$$

$$= -e^t + c$$

$$= -e^{-x^2/2} + c$$

That is

$$y = -1 + c e^{x^2/2}$$

The curve passes through the point $(0, 1)$ implies that

$$1 = -1 + c \text{ or } c = 2$$

Therefore the curve equation is

$$y = -1 + 2e^{x^2/2}$$

$$\Rightarrow y + 1 = 2e^{x^2/2}$$

**Answer: (B)**

**159.** General solution of the differential equation

$$(\cos^2 x)\frac{dy}{dx} + y = \tan x \quad \left(0 \le x < \frac{\pi}{2}\right)$$

is

(A) $y + 1 = \tan x + ce^{-\tan x}$
(B) $y + 1 = \tan x\, e^{-\tan x} + c$
(C) $(y + 1)e^{\tan x} = \tan x + c$
(D) $(y + 1) \tan x = c + e^{-\tan x}$

**Solution:** We have

$$(\cos^2 x)\frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + (\sec^2 x)y = \sec^2 x \tan x \quad \text{(Linear)}$$

So $P = \sec^2 x$ and $Q = \sec^2 \tan x$. The integrating factor is

$$\text{I.F.} = e^{\int P dx} = e^{\tan x}$$

Therefore, the solution is

$$ye^{\tan x} = \int (\sec^2 x \tan x) e^{\tan x} dx + c$$

$$= \int t e^t\, dt \quad \text{where } t = \tan x$$

So

$$ye^{\tan x} = e^t(t - 1) + c$$

$$= e^{\tan x}(\tan x - 1) + c$$

Hence

$$y + 1 = \tan x + c\, e^{-\tan x}$$

**Answer: (A)**

**160.** The general solution of the equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

is

(A) $xy = \frac{x^2}{2} + c$ (B) $xy = \frac{x^3}{3} + c$
(C) $xy = \frac{x^4}{4} + c$ (D) $xy = \frac{x^5}{5} + c$

**Solution:** We have

$$\frac{dy}{dx}+\frac{y}{x}=x^2 \quad \text{(Linear)}$$

Therefore $P=1/x$ and $Q=x^2$. The integrating factor is

$$\text{I.F.}=e^{\int P\,dx}=e^{\log x}=x$$

Therefore, the solution is

$$y(x)=\int x^2(x)\,dx+c$$
$$=\frac{x^4}{4}+c$$

This implies

$$xy=\frac{x^4}{4}+c$$

**Answer: (C)**

**161.** General solution of the differential equation

$$(x\log x)\frac{dy}{dx}+y=\frac{2}{x}\log x$$

is

(A) $xy\log x=cx-2(1+|\log(x)|)$

(B) $y\log x=cx-2(1+\log|x|)$

(C) $x\log x=cx+y-2(1+\log|x|)$

(D) $xy\log x=cx-2(1+\log|x|)$

**Solution:** Given equation is

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.}=e^{\int\frac{dx}{x\log x}}=e^{\log(\log x)}=\log x$$

Therefore, the solution is

$$y\log x=\int\frac{2}{x^2}\log x\,dx+c$$
$$=2\left[\frac{-1}{x}\log x-\int\frac{-1}{x}\cdot\frac{1}{x}\right]+c$$
$$=-\frac{2}{x}(\log x+1)+c$$

Therefore

$$xy\log x=cx-2(\log|x|+1)$$

**Answer: (D)**

**162.** Solution of

$$x\frac{dy}{dx}+y-x+xy\cot x=0 \quad (x\neq 0)$$

is

(A) $xy=\dfrac{c}{\sin x}+1-x\cot x$

(B) $xy=c\sin x+x\cot x$

(C) $xy\sin x=c-\cot x$

(D) $y\sin x=cx-x\cot x$

**Solution:** We have

$$x\frac{dy}{dx}+(1+x\cot x)y=x$$
$$\Rightarrow \frac{dy}{dx}+\frac{(1+x\cot x)}{x}y=1 \quad \text{(Linear)}$$

Here

$$P=\frac{1+x\cot x}{x},\ Q=1$$

The integrating factor is

$$\text{I.F.}=e^{\int P\,dx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log(x\sin x)}=x\sin x$$

Therefore, the solution is

$$y(x\sin x)=\int 1(x\sin x)\,dx+c$$
$$=\sin x-x\cos x+c$$

So

$$xy=1-x\cot x+\frac{c}{\sin x}$$

**Answer: (A)**

**163.** If $y=0$ when $x=1$, then a particular solution of the equation

$$(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$

is

(A) $y(1+x^2)=\text{Tan}^{-1}x+\dfrac{\pi}{3}$

(B) $y(1+x^2)=\text{Cot}^{-1}x+\dfrac{\pi}{2}$

(C) $y(1+x^2)=\dfrac{\pi}{4}-\text{Cot}^{-1}x$

(D) $y(1+x^2)=\text{Tan}^{-1}x+\dfrac{\pi}{4}$

**Solution:** Given equation is

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{1}{(1+x^2)^2} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2}dx}$$
$$= e^{\log(1+x^2)}$$
$$= 1+x^2$$

Therefore, the solution is

$$y(1+x^2) = \int \frac{1}{(1+x^2)^2}(1+x^2)\,dx + c$$
$$= \text{Tan}^{-1}x + c$$

Now $x=1 \Rightarrow y=0$ so that

$$0 = \text{Tan}^{-1}1 + c \quad \text{or} \quad c = \frac{-\pi}{4}$$

Hence the solution is

$$y(1+x^2) = \text{Tan}^{-1}x - \frac{\pi}{4}$$
$$= \left(\frac{\pi}{2} - \text{Cot}^{-1}x\right) - \frac{\pi}{4}$$

So

$$y(1+x^2) = \frac{\pi}{4} - \text{Cot}^{-1}x$$

**Answer: (C)**

**164.** General solution of the equation

$$(x\cos x)\frac{dy}{dx} + y(x\sin x + \cos x) = 1$$

is

(A) $y \sec x = \tan x + c$ (B) $xy \sec x = \tan x + c$

(C) $xy = \sec x \tan x + c$ (D) $y \sec x = x \tan x + c$

**Solution:** Given equation is

$$\frac{dy}{dx} + y\frac{(x\sin x + \cos x)}{x\cos x} = \frac{1}{x}\sec x$$

So

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right)y = \frac{1}{x}\sec x \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int\left(\tan x + \frac{1}{x}\right)dx}$$
$$= e^{\log(x\sec x)}$$
$$= x\sec x$$

Hence the solution is

$$y(x\sec x) = \int\left(\frac{1}{x}\sec x\right)(x\sec x)\,dx + c$$
$$= \int \sec^2 x\,dx + c$$
$$= \tan x + c$$

Therefore

$$xy \sec x = \tan x + c$$

**Answer: (B)**

**165.** General solution of the equation

$$(1+x)\frac{dy}{dx} - xy = 1 - x$$

is

(A) $y(1+x) = c\,e^x$ (B) $y(1+x) = x + c\,e^x$

(C) $x(y+1) = x + c\,e^x$ (D) $x(y+1) = c\,e^x$

**Solution:** Given equation is

$$\frac{dy}{dx} - \left(\frac{x}{1+x}\right)y = \frac{1-x}{1+x} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{-x}{1+x}dx}$$
$$= e^{\int\left(-1+\frac{1}{1+x}\right)dx}$$
$$= e^{-x+\log(1+x)}$$
$$= e^{-x}(1+x)$$

Hence the solution is

$$y[e^{-x}(1+x)] = \int\left(\frac{1-x}{1+x}\right)e^{-x}(1+x)\,dx$$
$$= \int e^{-x}(1-x)\,dx + c$$
$$= -e^{-x} - [-xe^{-x} - \textstyle\int -e^{-x}(1)\,dx] + c$$
$$= -e^{-x} + xe^{-x} + e^{-x} + c$$
$$= xe^{-x} + c$$

Therefore

$$y(1+x) = x + c\,e^x$$

**Answer: (B)**

**166.** General solution of the equation

$$(1+x^2)\frac{dy}{dx} + y = e^{\text{Tan}^{-1}x}$$

is

(A) $y = e^{\text{Tan}^{-1}x} + c$

(B) $ye^{\text{Tan}^{-1}x} = ce^{2\text{Tan}^{-1}x}$

(C) $2y = e^{\text{Tan}^{-1}x} + ce^{-\text{Tan}^{-1}}x$

(D) $2y = e^{2\text{Tan}^{-1}x} + c$

**Solution:** The given equation is

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{e^{\mathrm{Tan}^{-1}x}}{1+x^2} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.}=e^{\int\frac{dx}{1+x^2}}=e^{\mathrm{Tan}^{-1}x}$$

Therefore the solution is

$$\begin{aligned} ye^{\mathrm{Tan}^{-1}x} &= \int\frac{e^{\mathrm{Tan}^{-1}x}}{1+x^2}\cdot e^{\mathrm{Tan}^{-1}x}dx+c \\ &= \frac{1}{2}(e^{\mathrm{Tan}^{-1}x})^2+c \\ &= \frac{1}{2}e^{2\mathrm{Tan}^{-1}x}+c \end{aligned}$$

Hence

$$2y=e^{\mathrm{Tan}^{-1}x}+2ce^{-\mathrm{Tan}^{-1}x}$$

Take $2c$ as $c$.

**Answer: (C)**

**167.** If $y(t)$ is a solution of

$$(1+t)\frac{dy}{dt}-ty=1$$

and $y(0)=1$, then $y(1)$ is equal to

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) 1 (D) $-1$

**(IIT-JEE 2003)**

**Solution:** Given equation is

$$\frac{dy}{dt}-\frac{t}{1+t}y=\frac{1}{1+t} \quad \text{(Linear)}$$

The integrating factor is

$$\begin{aligned} \text{I.F.} &= e^{\int\frac{-t}{1+t}dt} \\ &= e^{-\int\frac{1+t-1}{1+t}dt} \\ &= e^{-t+\log(1+t)} \\ &= (1+t)e^{-t} \end{aligned}$$

The solution is

$$\begin{aligned} y(1+t)e^{-t} &= \int\left(\frac{1}{1+t}\right)e^{-t}(1+t)\,dt+c \\ &= -e^{-t}+c \end{aligned}$$

Therefore

$$y(1+t)=-1+ce^{t}$$

Now

$$\begin{aligned} & y(0)=-1 \\ & \Rightarrow -1=-1+c \\ & \Rightarrow c=0 \end{aligned}$$

Hence

$$y(1+t)=-1 \quad \text{or} \quad y=\frac{-1}{1+t}$$

Therefore

$$y(1)=-\frac{1}{2}$$

**Answer: (A)**

**168.** If $y=y(x)$ is a function of $x$ and $y(0)=0$, then a solution of the equation

$$\frac{dy}{dx}-y\tan x=\sec x$$

is

(A) $y=x\sec x$ (B) $y=x\operatorname{cosec} x$

(C) $x=y\operatorname{cosec} x$ (D) $x=y\sec x$

**Solution:** The given equation is

$$\frac{dy}{dx}-y\tan x=\sec x$$

The integrating factor is

$$\text{I.F.}=e^{-\int\tan dx}=e^{\log(\cos x)}=\cos x$$

Therefore the general solution is

$$y(\cos x)=\int\sec x\cos x\,dx+c=x+c$$

Now

$$y(0)=0\Rightarrow c=0$$

So the solution is $y\cos x=x$ or $y=x\sec x$.

**Answer: (A)**

**169.** If $y=y(x)$ is a function of $x$ and $y(1)=0$, then the solution of the equation

$$x\frac{dy}{dx}-\frac{y}{x+1}=x$$

is

(A) $y=\frac{x}{x+1}(x+\log|x|)$

(B) $y=\frac{1}{x+1}(x-1+\log|x|)$

(C) $y = \frac{x+1}{x}(x - 1 + \log|x|)$

(D) $y = \frac{x}{x+1}(x - 1 + \log|x|)$

**Solution:** Given equation is

$$\frac{dy}{dx} - \frac{y}{x(x+1)} = 1$$

The integrating factor is

$$\text{I.F.} = e^{-\int \frac{dx}{x(x+1)}}$$
$$= e^{-\int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx}$$
$$= e^{\log\left(\frac{x+1}{x}\right)}$$
$$= \frac{x+1}{x}$$

The solution is

$$y\left(\frac{x+1}{x}\right) = \int \frac{x+1}{x} dx + c$$
$$= x + \log|x| + c$$

Now

$$y(1) = 0 \Rightarrow 0 = 1 + 0 + c$$
$$\Rightarrow c = -1$$

Therefore

$$y\left(\frac{x+1}{x}\right) = x + \log|x| - 1$$
$$\Rightarrow y = \frac{x}{x+1}(x - 1 + \log|x|)$$

**Answer: (D)**

**170.** Tangent at any point $P(x, y)$ of a curve $y = f(x)$ meets the $y$-axis in $B$. If the algebraic value of $OB$ is equal to $x + 2$, then the equation of the curve is

(A) $y + 2 = c\,x - x \log|x|$
(B) $y = c\,x - x \log|x|$
(C) $y - 2 = c\,x - x \log|x|$
(D) $x + 2 = c\,y - x \log|x|$

**Solution:** Tangent at $(x_1, y_1)$ is

$$y - y_1 = m\,(x - x_1)$$

where

$$m = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$$

Now

$$x = 0 \Rightarrow y = y_1 - mx_1$$

Therefore

$$OB = y_1 - mx_1 = x_1 - 2 \quad \text{(By hypothesis)}$$

Therefore differential equation is

$$y - x\frac{dy}{dx} = x - 2$$
$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{2-x}{x} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int -(1/x)dx} = \frac{1}{x}$$

The solution is

$$y\left(\frac{1}{x}\right) = \int\left(\frac{2-x}{x}\right)\frac{1}{x}dx + c$$
$$= \int\left(\frac{2}{x^2} - \frac{1}{x}\right)dx + c$$
$$= -\frac{2}{x} - \log|x| + c$$

Therefore

$$y = -2 - x\log|x| + cx$$
$$\Rightarrow y + 2 = cx - x\log|x|$$

**Answer: (A)**

**171.** The tangent at any point $P$ on the curve $y = f(x)$ meets the $y$-axis in $B$. $N$ is the foot of the perpendicular drawn from $P$ onto the $x$-axis. If the rectangle with $OB$ and $ON$ ($O$ is the origin) as adjacent sides is of constant area and equals $a^2$, then the equation of the curve is

(A) $y = cx \pm \frac{a^2}{2}$ (B) $y = cx^2 \pm \frac{a^2}{x}$

(C) $y = cx \pm \frac{a^2}{x}$ (D) $y = cx \pm \frac{a^2}{2x}$

**Solution:** Tangent at $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

where

$$m = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$$

$B = (0, y_1 - mx_1), N = (x_1, 0)$. The differential equation is

Now

$$\left|x\left(y - x\frac{dy}{dx}\right)\right| = a^2$$

$$\Rightarrow xy - x^2\frac{dy}{dx} = \pm a^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \mp\frac{a^2}{x^2}$$

**(i)** $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{a^2}{x^2}$ (Linear)

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$$

Therefore

$$y\left(\frac{1}{x}\right) = \int \frac{a^2}{x^2}\cdot\frac{1}{x} + c$$

$$= \frac{-a^2}{2x^2} + c$$

This implies

$$y = cx - \frac{a^2}{2x}$$

**(ii)** If

$$\frac{dy}{dx} - \frac{y}{x} = -\frac{a^2}{x^2}$$

then

$$y = cx + \frac{a^2}{2x}$$

Therefore the solution is

$$y = cx \pm \frac{a^2}{2x}$$

**Answer: (D)**

**172.** Tangent at a point $P$ on the curve meets the $x$-axis in $A$ and $O$ is the origin. If the area of the triangle $OAP$ is constant $(= a^2)$, then the equation of the curve is

(A) $x = cy \pm \dfrac{a^2}{y}$ (B) $y = cx \pm \dfrac{a^2}{x}$

(C) $xy = c(x+y)$ (D) $cxy = a^2$

**Solution:** See Fig. 5.43. The equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

where

$$m\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

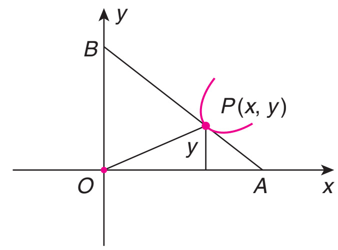



**FIGURE 5.43** Single correct choice question 172.

The co-ordinates of $A$ are $(x_1 - (y_1/m), 0)$. Now

$$\text{Area of } \Delta OAP = \left|y_1\left(x_1 - \frac{y_1}{m}\right)\right| = 2a^2$$

Differential equation is

$$xy - y^2\frac{dx}{dy} = \pm 2a^2$$

**(i)** We have

$$xy - y^2\frac{dx}{dy} = 2a^2$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = -\frac{2a^2}{y^2}$$

The integrating factor is

$$\text{I.F.} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

The solution is

$$x\left(\frac{1}{y}\right) = \int \frac{-2a^2}{y^2}\cdot\frac{1}{y}dy + c$$

$$= -2a^2\left(\frac{y^{-3+1}}{-3+1}\right) + c$$

$$= \frac{a^2}{y^2} + c$$

Therefore

$$x = \frac{a^2}{y} + cy$$

**(ii)** If

$$xy - y^2\frac{dx}{dy} = -2a^2$$

then

$$\frac{dx}{dy} - \frac{x}{y} = \frac{2a^2}{y^2}$$

In this case

$$x = cy - \frac{a^2}{y}$$

Therefore the solution is

$$x = cy \pm \frac{a^2}{y}$$

**Answer: (A)**

**173.** If $y(0) = 1$, then the solution of the differential equation

$$x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$$

is given by

(A) $x^3 = y^3 \sin x$ (B) $x^3 = 3y^3 \sin x$
(C) $3x^3 = y^3 \sin x$ (D) $y^3 = 3x^3 \sin x$

**Solution:** Dividing the given equation with $x^3 y^4$, we have

$$y^{-4} \frac{dy}{dx} - \frac{y^{-3}}{x} = \frac{-\cos x}{x^3}$$

Put $y^{-3} = z$ so that

$$-3y^{-4} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow -\frac{1}{3} \frac{dz}{dx} - \frac{z}{x} = -\frac{\cos x}{x^3}$$

$$\Rightarrow \frac{dz}{dx} + \frac{3z}{x} = \frac{3\cos x}{x^3}$$

The integrating factor is

$$\text{I.F.} = e^{\int (3/x)\,dx} = x^3$$

Therefore the solution is

$$zx^3 = \int \frac{3\cos x}{x^3}(x^3)\,dx + c$$
$$= 3\sin x + c$$

This implies

$$\frac{x^3}{y^3} = 3\sin x + c$$

Now $y(0) = 1 \Rightarrow c = 0$. Therefore

$$x^3 = 3y^3 \sin x$$

**Answer: (B)**

**174.** General solution of the equation

$$\frac{dy}{dx} + 2y\tan x = y^2$$

is

(A) $\cos^2 x + y\left(\frac{x}{2} + \sin 2x\right) = cy$

(B) $2\cos^2 x + y\left(x + \frac{1}{2}\sin 2x\right) = cy$

(C) $\cos^2 x + y\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right) = cy$

(D) $\cos^2 x + \frac{y}{4}\sin 2x = cy$

**Solution:** The given equation is

$$\frac{dy}{dx} + 2y\tan x = y^2 \quad \text{(Bernoulli)}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{2}{y}\tan x = 1$$

Put $1/y = z$. Therefore

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow -\frac{dz}{dx} + 2z\tan x = 1$$

$$\Rightarrow \frac{dz}{dx} - (2\tan x)z = -1 \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int -2\tan x\,dx}$$
$$= e^{\log(\cos^2 x)}$$
$$= \cos^2 x$$

Therefore the solution is

$$z\cos^2 x = \int -\cos^2 x\,dx + c$$
$$= -\frac{1}{2}\int (1 + \cos 2x)\,dx + c$$
$$= -\frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$$
$$= -\frac{x}{2} - \frac{\sin 2x}{4} + c$$

But $z = 1/y$. Therefore

$$\cos^2 x = -y\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) + cy$$

$$\Rightarrow \cos^2 x + y\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) = cy$$

**Answer: (C)**

**175.** The curve passing through the point $(0, \pi/4)$ satisfying the differential equation

$$\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$$

is

(A) $\tan y = \frac{1}{2}(x^2 - 1) + \frac{3}{2}e^{-x^2/2}$

(B) $\tan y = \frac{1}{2}(x^2 + 1) + \frac{3}{2}e^{-x^2/2}$

(C) $\tan y = \frac{1}{2}(x^2 - 1) + \frac{3}{2}e^{-x^2}$

(D) $\tan y = \frac{1}{2}(x^2 + 1) + \frac{3}{2}e^{-x^2}$

**Solution:** The given equation is

$$\sec^2 y \frac{dy}{dx} + x(2\tan y) = x^3$$

Put $\tan y = z$. Therefore

$$\frac{dz}{dx} + (2x)z = x^3 \quad \text{(Linear in } z\text{)}$$

The integrating factor is

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Therefore

$$\begin{aligned} ze^{x^2} &= \int x^3 e^{x^2}\,dx + c \\ &= \int x^2 e^{x^2} x\,dx + c \\ &= \frac{1}{2}\int x^2 e^{x^2}(2x)\,dx + c \\ &= \frac{1}{2}\int te^t\,dt + c \quad \text{where } t = x^2 \\ &= \frac{1}{2}e^t(t-1) + c \\ &= \frac{1}{2}e^{x^2}(x^2 - 1) + c \end{aligned}$$

So

$$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

The curve passes through $(0, \pi/4)$. This implies

$$1 = -\frac{1}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

Therefore

$$\tan y = \frac{1}{2}(x^2 - 1) + \frac{3}{2}e^{-x^2}$$

**Answer: (C)**

**176.** General solution of the equation

$$(1 + x^2)\frac{dy}{dx} + xy = x^3 y^3$$

is

(A) $\frac{1}{x^2+1}\left(\frac{1}{y^2} + \frac{1}{2x}\right) = -\frac{1}{2}\log(1 + x^2) + c$

(B) $\frac{1}{x^2+1}\left(\frac{1}{y^2} + 1\right) = -1\log(1 + x^2) + c$

(C) $\frac{1}{y^2(x^2+1)} = -\frac{1}{2}\log(1 + x^2) + c$

(D) $\frac{1}{y^2(x^2+1)} = -\frac{1}{2(x^2+1)} + c$

**Solution:** The given equation is

$$\frac{1}{y^3}\frac{dy}{dx} + \left(\frac{x}{1+x^2}\right)\frac{1}{y^2} = \frac{2x^3}{1+x^2}$$

Put $1/y^2 = z$ so that

$$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{-1}{2}\frac{dz}{dx} + \left(\frac{x}{1+x^2}\right)z = \frac{x^3}{1+x^2}$$

$$\Rightarrow \frac{dz}{dx} - \left(\frac{2x}{1+x^2}\right)z = -\frac{x^3}{1+x^2} \quad \text{(Linear)}$$

The integrating factor is

$$\begin{aligned} \text{I.F.} &= e^{-\int [2x/(1+x^2)]dx} \\ &= e^{-\log(1+x^2)} \\ &= \frac{1}{1+x^2} \end{aligned}$$

Therefore

$$\begin{aligned} z\left(\frac{1}{1+x^2}\right) &= \int \frac{-2x^3}{1+x^2}\cdot\frac{1}{1+x^2}\,dx + c \\ &= -\int \frac{2x^3}{(1+x^2)^2}\,dx + c \\ &= -\int \frac{t-1}{t^2}\,dt + c \quad \text{where } t = 1 + x^2 \\ &= \int\left(\frac{1}{t^2} - \frac{1}{t}\right)dt + c \end{aligned}$$

$$= -\left(\frac{1}{t} + \log t\right) + c$$

$$= -\left[\frac{1}{1+x^2} + \log(1+x^2)\right] + c$$

So

$$\frac{1}{y^2(1+x^2)} + \frac{1}{(1+x^2)} = -1\log(1+x^2) + c$$

$$\Rightarrow \frac{1}{x^2+1}\left(\frac{1}{y^2} + 1\right) = -1\log(1+x^2) + c$$

**Answer: (B)**

**177.** Solution of the equation

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y)$$

is

(A) $e^y = (e^x - 1) + ce^{-x}$

(B) $e^y = (e^x - 1) + c(e^{e^x})^{-1}$

(C) $e^{x+y} = (e^x - 1) + ce^{-e^x}$

(D) $e^{x+y} = e^x \cdot e^{e^x} + c$

**Solution:** The given equation is

$$e^y \frac{dy}{dx} + e^x \cdot e^y = e^{2x}$$

$$\Rightarrow \frac{dz}{dx} + e^x z = e^{2x} \quad \text{where } z = e^y$$

The integrating factor is

$$\text{I.F.} = e^{\int e^x\,dx} = e^{e^x}$$

Therefore

$$ze^{e^x} = \int e^{2x} \cdot e^{e^x}\,dx + c$$

$$= \int e^x \cdot e^{e^x} e^x\,dx + c$$

$$= \int te^t\,dt + c \quad \text{where } t = e^x$$

$$= e^t(t-1) + c$$

$$= e^{e^x}(e^x - 1) + c$$

This implies

$$e^y = (e^x - 1) + c(e^{e^x})^{-1}$$

**Answer: (B)**

**178.** General solution of the equation

$$\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$$

is

(A) $2x = e^y(2cx^2 + 1)$ (B) $2y = -e^x(2cx^2 - 1)$

(C) $2x = e^y(cx^2 - 1)$ (D) $2y = e^x(cx^2 - 1)$

**Solution:** We have

$$\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$$

$$\Rightarrow e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2}$$

Put $e^{-y} = z$ so that

$$-e^{-y}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow -\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{-\int \frac{dx}{x}}$$

$$= e^{-\log x}$$

$$= \frac{1}{x}$$

Therefore

$$z\left(\frac{1}{x}\right) = \int -\frac{1}{x^2}\cdot\frac{1}{x}\,dx + c$$

$$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + c$$

$$\Rightarrow 2x = e^y(2cx^2 + 1)$$

**Answer: (A)**

## *Multiple Correct Choice Type Questions*

**1.** If

$$\int_0^{16} \frac{x^{1/4}}{1+\sqrt{x}}\,dx = a + b\,\text{Tan}^{-1}(k)$$

then

(A) $a = \frac{8}{3}$ (B) $b = 2$

(C) $k = 2$ (D) $b = 4$

**Solution:** Let

$$I = \int_0^{16} \frac{x^{1/4}}{1+\sqrt{x}}\,dx$$

Put $t = x^{1/4}$ so that $dx = 4t^3 dt$. Also

$$x = 0 \Rightarrow t = 0$$

and $$x = 16 \Rightarrow t = 2$$

Therefore

$$I = \int_0^2 \frac{t}{1+t^2}(4t^3)\,dt$$
$$= 4\int_0^2 \frac{t^4 - 1 + 1}{1+t^2}\,dt$$
$$= 4\int_0^2 \left(t^2 - 1 + \frac{1}{1+t^2}\right)dt$$
$$= 4\left[\frac{1}{3}(8-0) - (2-0) + \left[\operatorname{Tan}^{-1} t\right]_0^2\right]$$
$$= 4\left(\frac{8}{3} - 2\right) + 4\operatorname{Tan}^{-1} 2$$
$$= \frac{8}{3} + 4\operatorname{Tan}^{-1} 2$$

Therefore $a = 8/3$, $b = 4$ and $k = 2$.

**Answers: (A), (C), (D)**

**2.** Which of the following statements are true?

(A) $$\int_{-\pi/2}^{\pi/2} \frac{\pi^{\sin x}}{1+\pi^{\sin x}}\,dx = \frac{\pi}{2}$$

(B) $$\int_0^{\pi} \frac{dx}{1+2^{\tan x}} = \frac{\pi}{2}$$

(C) $$\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(3\pi + x)]dx = \frac{\pi}{2}$$

(D) $$\lim_{x\to\pi/4} \frac{\int_2^{\sec^2 x} f(t)dt}{x^2 - (\pi^2/16)} = \frac{2}{\pi} f(2)$$

**Solution** (A) We have

$$I = \int_{-\pi/2}^{\pi/2} \frac{\pi^{\sin x}}{1+\pi^{\sin x}}\,dx$$
$$= \int_{-\pi/2}^{\pi/2} \frac{\pi^{\sin(\pi/2-\pi/2-x)}}{1+\pi^{\sin(\pi/2-\pi/2-x)}}\,dx$$
$$= \int_{-\pi/2}^{\pi/2} \frac{\pi^{-\sin x}}{1+\pi^{-\sin x}}\,dx$$
$$= \int_{-\pi/2}^{\pi/2} \frac{dx}{1+\pi^{\sin x}} = I$$

Therefore

$$2I = \int_{-\pi/2}^{\pi/2} 1\,dx = \pi \Rightarrow I = \frac{\pi}{2}$$

So (A) is true.

(B) We have

$$I = \int_0^{\pi} \frac{dx}{1+2^{\tan x}}dx$$
$$= \int_0^{\pi} \frac{dx}{1+2^{\tan(\pi - x)}}$$
$$= \int_0^{\pi} \frac{dx}{1+2^{-\tan x}}$$
$$= \int_0^{\pi} \frac{2^{\tan x}}{1+2^{\tan x}}\,dx$$
$$= I$$

Therefore

$$2I = \int_0^{\pi} 1\,dx = \pi \Rightarrow I = \frac{\pi}{2}$$

So (B) is true.

(C) We have

$$I = \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(3\pi + x)]dx$$

Put $t = x + \pi$ so that

$$x = -\frac{3\pi}{2} \Rightarrow t = -\frac{\pi}{2}$$

and $$x = \frac{-\pi}{2} \Rightarrow t = \frac{\pi}{2}$$

Therefore

$$I = \int_{-\pi/2}^{\pi/2} (t^3 + \cos^2 t)dt \quad (\because 3\pi + x = 2\pi + \pi + x)$$
$$= 0 + \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2}dt \quad (\because t^3 \text{ is an odd function})$$
$$= 2\int_0^{\pi/2} \frac{1+\cos 2t}{2}\,dt$$
$$= \frac{\pi}{2} + \left[\frac{\sin 2t}{2}\right]_0^{\pi/2}$$
$$= \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Therefore (C) is true.

(D) We have

$$\lim_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{f(\sec^2 x) 2\sec^2 x \tan x}{2x}$$

$$= \frac{f(2)2(2)(1)}{2(\pi/4)}$$

$$= \frac{8}{\pi} f(2)$$

So (D) is not true.

**Answers: (A), (B), (C)**

**3.** If $I_n = \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx$, then

(A) $I_1, I_2, I_3, \ldots$ are in AP (B) $I_n = \frac{\pi}{2}$

(C) $(2012)I_n = (2012)\pi$ (D) $(2012)I_n = (606)\pi$

**Solution:** We have

$$I_{n+1} - I_n = \int_0^{\pi/2} \frac{\sin(2n+3) - \sin(2n+1)}{\sin x} dx$$

$$= \int_0^{\pi/2} \frac{2\cos 2(n+1)x \sin x}{\sin x} dx$$

$$= \frac{2}{2(n+1)} [\sin 2(n+1)x]_0^{\pi/2} = 0$$

Therefore

$$I_{n+1} = I_n$$

$$\Rightarrow I_1 = I_2 = I_3 = \cdots$$

So (A) is true.

Now

$$I_n = I_1$$

$$= \int_0^{\pi/2} \frac{\sin 3x}{\sin x} dx$$

$$= \int_0^{\pi/2} \left(\frac{3\sin x - 4\sin^3 x}{\sin x}\right) dx$$

$$= \int_0^{\pi/2} 3dx - 2\int_0^{\pi/2} (1 - \cos 2x)\, dx$$

$$= 3\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{2}\right) + 2\cdot\frac{1}{2}[\sin 2x]_0^{\pi/2}$$

$$= \frac{3\pi}{2} - \pi + 0 = \frac{\pi}{2}$$

So (B) is true.

Also

$$(2012)I_n = (2012)\left(\frac{\pi}{2}\right) = (606)\pi$$

Therefore (D) is true.

**Answers: (A), (B), (D)**

**4.** Let

$$J_n = \int_0^{\pi/2} \left(\frac{\sin nx}{\sin x}\right)^2 dx$$

and

$$I_n = \int_0^{\pi/2} \frac{\sin(2n+1)}{\sin x} x\, dx$$

Then

(A) $J_{n+1} - J_n = I_n$ (B) $J_n = I_{n+1} - I_n$

(C) $J_n = \frac{n\pi}{2}$ (D) $J_n = \frac{\pi}{2}$

**Solution:** We have

$$J_{n+1} - J_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x \sin x}{\sin^2 x} dx$$

$[\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx = I_n$$

Therefore (A) is true. So

$$J_{n+1} - J_n = I_n = \frac{\pi}{2} \quad \text{(See Problem 3)}$$

Hence $J_1, J_2, J_3, \ldots$ are in AP with common difference $\pi/2$. So

$$J_n = J_1 + (n-1)\frac{\pi}{2}$$

$$= \frac{\pi}{2} + (n-1)\frac{\pi}{2}$$

$$= \frac{n\pi}{2}$$

**Answers: (A), (C)**

**5.** $f(x)$ is a differentiable function such that

$$f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$$

Then

(A) $f(x) = \dfrac{x^3}{3} + x$  (B) $f(x) = \dfrac{x^3}{3} + x^2$

(C) $f'(x) = x^2 + 2x$  (D) $\displaystyle\int_{-2}^{2} (f(x) - x^2)\, dx = 0$

**Solution:** We have

$$f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$$
$$= x^2 + \int_0^x e^{-(x-t)} f(x-(x-t))\, dt$$
$$= x^2 + e^{-x} \int_0^x e^t f(t)\ dt$$

Therefore

$$e^x f(x) = x^2 e^x + \int_0^x e^t f(t)\, dt$$

Differentiating both sides w.r.t. $x$ we get

$$e^x f(x) + e^x f'(x) = x^2 e^x + 2x\, e^x + e^x f(x)$$

So

$$f'(x) = x^2 + 2x$$
$$\Rightarrow f(x) = \int (x^2 + 2x)\, dx$$
$$= \frac{x^3}{3} + x^2 + c$$

Also, $f(0) = 0 \Rightarrow c = 0$. So

$$f(x) = \frac{x^3}{3} + x^2$$

Hence (B), (C), (D) are true.

**Answers: (B), (C), (D)**

**6.** Let $f$ be a real-valued function defined on the interval $(0, \infty)$ by

$$f(x) = \int_0^x \sqrt{1+\sin t}\ \ dt + \log_e x$$

Then which of the following statement(s) is(are) true?

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and $f'$ is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (0, \infty)$

(D) There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

**(IIT-JEE 2010)**

**Solution:** We have

$$f(x) = \int_0^x \sqrt{1+\sin t}\, dt + \log_e x$$

By Leibnitz Theorem,

$$f'(x) = \sqrt{1+\sin x} + \frac{1}{x}$$

and $$f''(x) = \frac{\cos x}{2\sqrt{1+\sin x}} - \frac{1}{x^2}$$

When $\sin x + 1 = 0$, $f''(x)$ does not exist. Therefore (A) is not true. Hence, $f'(x)$ exists for all $x > 0$, is continuous but not differentiable when $\sin x = -1$ and hence (B) is correct.

Since $\lim\limits_{x\to\infty} f(x) = \infty$, $f'(x)$ is bounded and both $f(x)$ and $f'(x)$ are positive in $(1, \infty)$, there exists $\alpha > 1$ such that

$$f'(x) < f(x)\ \forall\ x > 0$$

Hence (C) is correct.

**Answers: (B), (C)**

**7.** Let

$$S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$$

and $$T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$$

for $n = 1, 2, 3, \ldots$. Then

(A) $S_n < \dfrac{\pi}{3\sqrt{3}}$  (B) $S_n > \dfrac{\pi}{3\sqrt{3}}$

(C) $T_n > \dfrac{\pi}{3\sqrt{3}}$  (D) $T_n < \dfrac{\pi}{3\sqrt{3}}$

**Solution:** We can see that $\{S_n\}$ is an increasing sequence and $\{T_n\}$ is a decreasing sequence. Also

$$\lim_{n\to\infty} S_n = \int_0^1 \frac{dx}{1+x+x^2} = \lim_{n\to\infty} T_n$$

$$\Rightarrow S_n < \int_0^1 \frac{dx}{1+x+x^2}$$

$$= \int_0^1 \frac{dx}{\left(\frac{1}{2}+x\right)^2 + \frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}} \text{Tan}^{-1}\left[\left(x+\frac{1}{2}\right)\frac{2}{\sqrt{3}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\text{Tan}^{-1}\left(\frac{3}{2}\times\frac{2}{\sqrt{3}}\right) - \text{Tan}^{-1}\left(\frac{1}{2}\times\frac{2}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{2}{\sqrt{3}}\times\frac{\pi}{6}$$

$$= \frac{\pi}{3\sqrt{3}}$$

Also $\{T_n\}$ is decreasing. This implies

$$T_n > \int_0^1 \frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}$$

**Answers: (A), (C)**

**8.** The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are)

(A) $\frac{22}{7}-\pi$  (B) $\frac{2}{105}$

(C) 0  (D) $\frac{71}{15}-\frac{3\pi}{2}$

**(IIT-JEE 2010)**

**Solution:** Let

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}[x^8 - 4x^7 + 6x^6 - 4x^5 + x^4]\,dx$$

$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$$

$$= \left[\frac{x^7}{7} - \frac{4x^6}{6} + x^5 - \frac{4x^3}{3} + 4x - 4\,\text{Tan}^{-1}x\right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4 - 4\,\text{Tan}^{-1}1$$

$$= \frac{22}{7} - \pi$$

So only (A) is correct.

**Answer: (A)**

**8.** If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx,\ n = 0, 1, 2, \ldots$ then

(A) $I_n = I_{n+2}$  (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$  (D) $I_n = I_{n+1}$

**(IIT-JEE 2009)**

**Solution:** We have

$$I_n = \int_{-\pi}^{0} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx + \int_0^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$$

$$= \int_0^{\pi} \frac{\sin n(-x)}{(1+\pi^{-x})\sin(-x)}\,dx + \int_0^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$$

$$= \int_0^{\pi} \frac{\pi^x \sin nx}{(\pi^x+1)\sin x}dx + \int_0^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$$

$$= \int_0^{\pi} \frac{\sin nx(\pi^x+1)}{(\pi^x+1)\sin x}dx$$

$$= \int_0^{\pi} \frac{\sin nx}{\sin x}dx \tag{5.46}$$

Now

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}dx$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x\sin x}{\sin x}\,dx$$

$$= \frac{2}{n+1}[\sin(n+1)x]_0^{\pi}$$

$$= \frac{2}{n+1}(0) = 0$$

Therefore $I_n = I_{n+2}$. So (A) is true.
Now, from Eq. (5.46)

$$I_1 = \int_0^{\pi} \frac{\sin x}{\sin x}dx = \int_0^{\pi} 1\,dx = \pi$$

and $$I_2 = \int_0^{\pi} \frac{\sin 2x}{\sin x}dx = \int_0^{\pi} 2\cos x dx$$

$$= 2[\sin x]_0^{\pi}$$
$$= 2(\sin\pi - \sin 0) = 0$$

Therefore

$$I_1 = I_3 = I_5 = I_7 = \cdots = I_{21} = \pi$$

This implies

$$\sum_{m=1}^{10} I_{2m+1} = 10\pi$$

Also

$$I_2 = I_4 = I_6 = \cdots = I_{20} = 0$$

This implies

$$\sum_{m=1}^{10} I_{2m} = 0$$

**Answers: (A), (B), (C)**

**9.** Let

$$f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 \le x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$$

and $g(x) = \int_0^x f(t)dt$ for $1 \le x \le 3$

Then $g(x)$ has

(A) local maximum at $x = 1 + \log 2$

(B) local maximum at $x = 1$ and local minimum at $x = 2$

(C) no local maxima

(D) local minima at $x = e$

**(IIT-JEE 2005)**

**Solution:**

**(i)** For $0 \le x \le 1$,

$$g(x) = \int_0^x f(t)dt = \int_0^x e^t dt = e^x - 1 \qquad (5.47)$$

**(ii)** For $1 \le x \le 2$,

$$g(x) = \int_0^1 f(t)dt + \int_1^x f(t)dt$$
$$= \int_0^1 e^t dt + \int_1^x (2 - e^{t-1})dt$$
$$= e - 1 + (2x - 2) - (e^{x-1} - 1)$$
$$= 2x - e^{x-1} + e - 2$$

Therefore

$$g(x) = 2x - e^{x-1} + e - 2 \text{ for } 1 \le x \le 2 \qquad (5.48)$$

**(iii)** For $2 < x \le 3$,

$$g(x) = \int_0^1 f(t)\,dt + \int_1^2 f(t)dt + \int_2^x f(t)dt$$
$$= (e-1) + \int_1^2 (2 - e^{t-1})dt + \int_2^x (t-e)dt$$
$$= e - 1 + 2 - (e-1) + \frac{1}{2}(x^2 - 4) - e(x-2)$$
$$= \frac{x^2}{2} - ex + 2e$$

Therefore

$$g(x) = \frac{x^2}{2} - ex + 2e \text{ for } 2 < x \le 3 \qquad (5.49)$$

From Eqs. (5.47), (5.48) and (5.49), we have

$$g(x) = \begin{cases} e^x - 1 & \text{for } 0 \le x \le 1 \\ 2x - e^{x-1} + e - 2 & \text{for } 1 < x \le 2 \\ \dfrac{x^2}{2} - ex + 2e & \text{for } 2 < x \le 3 \end{cases}$$

Clearly $g$ is continuous at $x = 1, 2$. Also $g'(x) = f(x)$ (by Leibniz rule) and $f$ is not differentiable at $x = 1$, because $f$ is discontinuous at $x = 1$. Even though $f$ is continuous at $x = 2$, it is not differentiable at $x = 2$.

Further $g'(x) = 0$ only when $x = 1 + \log_e 2$ or $x = e$. Also

$$g''(x) = -e^{x-1} \quad \text{for} \quad 1 < x \le 2 \Rightarrow g''(1 + \log_e 2) < 0$$

and $\quad g''(x) = 1 \quad \text{for} \quad 2 < x \le 3 \Rightarrow g''(e) > 0$

Therefore, $g$ has local maximum at $x = 1 + \log_e 2$ and local minimum at $x = e$.

**Answers: (A), (D)**

***Note:*** Actually computation of $g(x)$ is not necessary because we know that $g'(x) = f(x)$ by Leibniz Rule. Finding $g(x)$ is only for academic interest.

**10.** The function

$$f(x) = \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$$

has a local minima at $x$ equals

(A) 0 (B) 1 (C) 2 (D) 3

**(IIT-JEE 1999)**

**Solution:** By Leibnitz Rule,

$$f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$$

Therefore

$$f'(x) = 0 \Leftrightarrow x = 0, 1, 2 \text{ and } 3$$

The critical points of $f$ are $0, 1, 2$ and $3$.

**(i)** Clearly $f'(x)$ does not change its sign at $x = 0$; so $f$ has no local extrema at $x = 0$.

**(ii)** $x < 1 \Rightarrow f'(x) < 0$ and $x > 1 \Rightarrow f'(x) > 0$. Hence $f$ has local minima at $x = 1$. Also $f$ has local minima at $x = 3$.

**(iii)** $x < 2 \Rightarrow f'(x) > 0$ and $x > 2 \Rightarrow f'(x) < 0$. So $f$ has maxima at $x = 2$.

**Answers: (B), (D)**

**11.** For $x > 0$, let

$$f(x) = \int_1^x \frac{\log_e t}{1+t}\, dt$$

Then

(A) $f(x) + f\left(\dfrac{1}{x}\right) = \dfrac{1}{2}(\log_e x)^2$

(B) $f(e) + f\left(\dfrac{1}{e}\right) = \dfrac{1}{2}$

(C) $f(x) + f\left(\dfrac{1}{x}\right) = (\log_e x)^2$

(D) $f(1) = 0$

**Solution:** We have

$$f(x) = \int_1^x \frac{\log_e t}{1+t}$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log_e t}{1+t}\, dt$$

Put $t = 1/u$ so that

$$dt = \frac{-1}{u^2}\, du$$

So

$$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e(1/u)}{1+(1/u)}\left(\frac{-1}{u^2}\right) du = \int_1^x \frac{\log_e x}{x(1+x)}\, dx$$

Now

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left(\frac{\log_e t}{1+t} + \frac{\log_e t}{t(1+t)}\right) dt$$

$$= \int_1^x \frac{\log_e(t)(1+t)}{t(1+t)}\, dt$$

$$= \int_1^x \frac{\log_e t}{t}\, dt$$

$$= \frac{1}{2}(\log_e x)^2$$

Hence

$$f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2} \quad \text{and} \quad f(1) = 0$$

So (A), (B) and (D) are true.

**Answers: (A), (B), (D)**

**12.** Let

$$I = \int_0^1 \mathrm{Tan}^{-1}\left(\frac{1}{1-x+x^2}\right) dx$$

and $$J = \int_0^1 \mathrm{Tan}^{-1}(1-x+x^2)\, dx$$

Then

(A) $I = \displaystyle\int_0^1 \mathrm{Tan}\, x\, dx$ (B) $I = 2\displaystyle\int_0^1 \mathrm{Tan}^{-1} x\, dx$

(C) $J = \log_e 2$ (D) $I + J = \dfrac{\pi}{2}$

**Solution:** We have

$$I = \int_0^1 \mathrm{Tan}^{-1}\left(\frac{1}{1-x+x^2}\right) dx$$

$$= \int_0^1 \mathrm{Tan}^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right) dx$$

$$= \int_0^1 (\mathrm{Tan}^{-1} x - \mathrm{Tan}^{-1}(x-1))\, dx$$

$$= \int_0^1 \mathrm{Tan}^{-1} x\, dx - \int_0^1 \mathrm{Tan}^{-1}(x-1)\, dx$$

$$= \int_0^1 \mathrm{Tan}^{-1} x - \int_0^1 \mathrm{Tan}^{-1}(1-x-1)\, dx$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= 2\int_0^1 \mathrm{Tan}^{-1} x\, dx$$

Now

$$
\begin{aligned}
J &= \int_0^1 \operatorname{Tan}^{-1}(1-x+x^2)\,dx \\
&= \int_0^1 \left(\frac{\pi}{2} - \operatorname{Cot}^{-1}(1-x+x^2)\right) dx \\
&= \frac{\pi}{2} - \int_0^1 \operatorname{Tan}^{-1}\left(\frac{1}{1-x+x^2}\right) dx \\
&= \frac{\pi}{2} - I
\end{aligned}
$$

Therefore

$$J + I = \pi/2 \tag{5.50}$$

Also

$$
\begin{aligned}
I &= 2\int_0^1 \operatorname{Tan}^{-1} x\,dx \\
&= 2\left[\left[x\operatorname{Tan}^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\right] \\
&= 2\left[\frac{\pi}{4} - 0 - \frac{1}{2}\left[\log(1+x^2)\right]_0^1\right] \\
&= 2\left(\frac{\pi}{4} - \frac{1}{2}\log_e 2\right) \\
&= \frac{\pi}{2} - \log_e 2
\end{aligned}
$$

From Eq. (5.50), we have

$$
\begin{aligned}
J &= \frac{\pi}{2} - I \\
&= \frac{\pi}{2} - \left(\frac{\pi}{2} - \log_e 2\right) \\
&= \log_e 2
\end{aligned}
$$

**Answers: (B), (C), (D)**

**13.** The function

$$f(x) = \int_1^x \left[2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\right] dt$$

(A) has three critical points
(B) has maxima at $x = 1$
(C) has minima at $x = \dfrac{7}{5}$
(D) $f''(2)$ is equal to zero

**Solution:** By Leibnitz Rule,

$$
\begin{aligned}
f'(x) &= 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2 \\
&= (x-1)(x-2)^2[2(x-2) + 3(x-1)] \\
&= (x-1)(x-2)^2(5x-7)
\end{aligned}
$$

Therefore

$$f'(x) = 0 \Leftrightarrow x = 1, 2, \frac{7}{5}$$

Again

$$
\begin{aligned}
f''(x) = {} & (x-2)^2(5x-7) + 2(x-1)(x-2)(5x-7) \\
& + (x-1)(x-2)^2(5)
\end{aligned}
$$

Now $f''(1) = -2 < 0 \Rightarrow f$ has maximum at $x = 1$. Also

$$f''(2) = 0$$

and $$f''\left(\frac{7}{5}\right) = 5\left(\frac{7}{5} - 1\right)\left(\frac{7}{5} - 2\right)^2 > 0$$

Therefore $f$ is minima at $x = 7/5$. Hence (A), (B), (C) and (D) all are true.

**Answers: (A), (B), (C), (D)**

**14.** Area of the region bounded by the curve $y = e^x$ and the lines $x = 0$ and $y = e$ is

(A) $e - 1$ (B) $\displaystyle\int_1^e \log_e(e+1-y)\,dy$

(C) $e - \displaystyle\int_0^1 e^x\,dx$ (D) $\displaystyle\int_1^e \log_e y\,dy$

**(IIT-JEE 2009)**

**Solution:** The line $y = e$ meets the curve $y = e^x$ in $P(1, e)$ and the $y$-axis in $(0, 1)$. See Fig. 5.44. Then

$$
\begin{aligned}
\text{Area (shaded part)} &= \int_0^1 (e - e^x)\,dx \\
&= e(1) - (e-1) = 1
\end{aligned}
$$

So (C) is true. Also,

$$
\begin{aligned}
\text{Area} &= \int_1^e \log_e y\,dy \quad (\because y = e^x \Rightarrow x = \log_e y) \\
&= \int_1^e \log(e+1-y)\,dy \\
&\quad \left(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right)
\end{aligned}
$$

Hence (B), (D) are also correct.

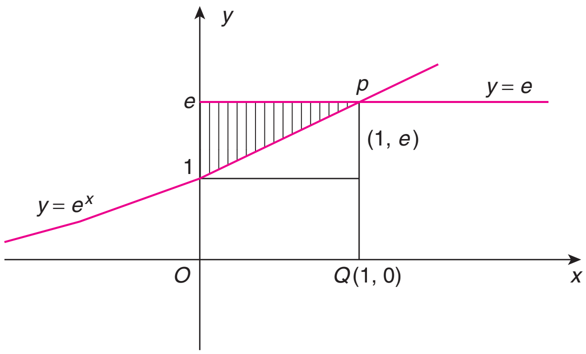



**FIGURE 5.44** Multiple correct choice type question 14.

**Answers: (B), (C), (D)**

**15.** The area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the line $y = 3x - 15$ is divided into three regions $A_1$, $A_2$ and $A_3$ (see Fig. 5.45). Then

(A) $A_1 = \int_1^5 (-x^2 + 6x - 5)\,dx - \int_1^3 (-x^2 + 4x + 3)\,dx$

(B) $A_2 = \int_3^4 (x^2 - 4x - 3)\,dx$

(C) $A_3 = -\int_4^5 (3x - 15)\,dx$

(D) $A_1 + A_2 + A_3 = \dfrac{73}{6}$

**Solution:** Let

$$C_1 : -x^2 + 6x - 5 = y$$
$$\Rightarrow (x-3)^2 = -(y-4)$$

Therefore $C_1$ represents a parabola (downward) with vertex at $(3, 4)$ meeting the $x$-axis in $A(1, 0)$ and $D(5, 0)$.

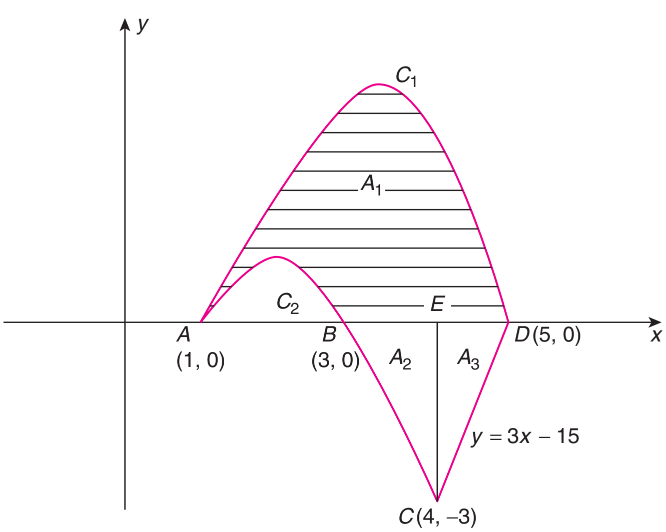



**FIGURE 5.45** Multiple correct choice type question 15.

Let $C_2$ represent

$$y = -x^2 + 4x - 3$$
$$= -(x-2)^2 + 1$$
$$\Rightarrow (x-2)^2 = -(y-1)$$

Therefore $C_2$ is also a downward parabola with vertex at $(2, 1)$ and meeting the $x$-axis in $A(1, 0)$ and $B(3, 0)$. The line $y = 3x - 15$ intersects the curve $C_2$ in $(4, -3)$ and the curve $C_1$ in $(5, 0)$. The area bounded by the curves $C_1$, $C_2$ and the line $y = 3x - 15$ is $A_1 + A_2 + A_3$. Now

$$A_1 = \int_1^5 (-x^2 + 6x - 5)\,dx - \int_1^3 (-x^2 + 4x - 3)\,dx$$
$$= -\frac{1}{3}\left[x^3\right]_1^5 + 3\left[x^2\right]_1^5 - 5[x]_1^5$$
$$-\left\{-\frac{1}{3}\left[x^3\right]_1^3 + 2\left[x^2\right]_1^3 - 3[x]_1^3\right\}$$
$$= -\frac{1}{3}[125-1] + 3[25-1] - 5(5-1)$$
$$-\left\{-\frac{1}{3}(27-1) + 2(9-1) - 3(3-1)\right\}$$
$$= -\frac{124}{3} + 72 - 20 + \frac{26}{3} - 16 + 6$$
$$= -\frac{124}{3} + 42 + \frac{26}{3}$$
$$= \frac{-124 + 152}{3}$$
$$= \frac{28}{3}$$

Now

$$A_2 = \int_3^4 (x^2 - 4x + 3)\,dx$$
$$= \frac{1}{3}\left[x^3\right]_3^4 - 2\left[x^2\right]_3^4 + 3[x]_3^4$$
$$= \frac{1}{3}(64-27) - 2[16-9] + 3\,(4-3)$$
$$= \frac{37}{3} - 14 + 3 = \frac{4}{3}$$

Finally

$$A_3 = -\int_4^5 (3x - 15)\,dx$$
$$= -\frac{3}{2}[x^2]_4^5 + 15[x]_4^5$$
$$= -\frac{3}{2}(25-16) + 15(5-4)$$
$$= -\frac{3}{2} \times 9 + 15 = \frac{3}{2}$$

or $A_3 = \frac{1}{2}(ED)(EC) = \frac{1}{2}\times 1\times 3 = \frac{3}{2}$

Therefore

$$\begin{aligned}\text{Area } &= A_1 + A_2 + A_3\\ &= \frac{28}{3}+\frac{4}{3}+\frac{3}{2}\\ &= \frac{32}{3}+\frac{3}{2}\\ &= \frac{64+9}{6}\\ &= \frac{73}{6}\end{aligned}$$

So (A), (B), (C), (D) all are correct.

**Answers: (A), (B), (C), (D)**

**16.** Function $f(x)$ satisfies the relation

$$f(x) = e^x + \int_0^1 e^x f(t)dt$$

Then

(A) $f(0) < 0$ (B) $f(x)$ is increasing

(C) $f(x)$ is decreasing (D) $\int_0^1 f(x)\,dx > 0$

**Solution:** We have

$$\begin{aligned}f(x) &= e^x + \int_0^1 e^x f(t)\,dt\\ &= e^x + e^x\int_0^1 f(t)\,dt\\ &= e^x + \lambda e^x \quad \text{where } \lambda = \int_0^1 f(t)\,dt \qquad (5.51)\end{aligned}$$

Now

$$\begin{aligned}\lambda &= \int_0^1 f(t)\,dt\\ &= \int_0^1 (e^t + \lambda e^t)\,dt \quad [\text{from Eq. (5.51)}]\\ &= e - 1 + \lambda(e-1)\end{aligned}$$

Therefore

$$\lambda(2-e) = e-1$$

$$\Rightarrow \lambda = \frac{e-1}{2-e}$$

Hence

$$\begin{aligned}f(x) &= e^x + \left(\frac{e-1}{2-e}\right)e^x\\ &= e^x\left(1+\frac{e-1}{2-e}\right)\\ &= \frac{e^x}{2-e}\end{aligned}$$

So

$$f(0) = \frac{1}{2-e} < 0$$

Hence (A) is true. Now

$$f'(x) = \frac{e^x}{2-e} < 0 \quad \forall\, x \in \mathbb{R}$$

Thus $f(x)$ is decreasing and so (C) is true. Again

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 \frac{e^x}{2-e}\,dx\\ &= \frac{1}{2-e}(e-1)\\ &< 0 \;(\because 2 < e)\end{aligned}$$

So (D) is not true.

**Answers: (A), (C)**

**17.** $f(x) = x\int_1^x \frac{e^t}{t}\,dt - e^x$ for $x \ge 1$. Then

(A) $f$ is increasing in $[1, \infty)$

(B) $\lim\limits_{x\to\infty} f(x) = \infty$

(C) $f$ is decreasing in $[1, \infty)$

(D) $f'$ has maximum at $x = e$

**Solution:** We have

$$\begin{aligned}f'(x) &= \int_1^x \frac{e^t}{t}\,dt + x\left(\frac{e^x}{x}\right) - e^x\\ &= \int_1^x \frac{e^t}{t}\,dt \ge 0\end{aligned}$$

Therefore $f$ is increasing in $[1, \infty)$. So (A) is true. Now

$$f''(x) = \frac{e^x}{x} > 0 \text{ for } x \ge 1$$

So $f'$ is strictly increasing. Since $f$ is increasing,

$$\lim_{x\to\infty} f(x) = \infty$$

Hence (B) is true.

**Answers: (A), (B)**

**18.** Consider the equation

$$(\sin y)\frac{dy}{dx} = \cos y\,(1 - x\cos y)$$

Then

(A) the equation can be reduced to linear equation

(B) the equation can be reduced to homogeneous using the substitution $\cos y = z$

(C) the equation can be reduced to variable separable, using the substitution $\cos y = z$

(D) the general solution of the equation is given by $\sec y = (1 + x) + ce^x$

**Solution:** We have

$$(\sin y)\frac{dy}{dx} = \cos y(1 - x\cos y)$$

Put $\cos y = z$. Therefore

$$-\sin y\frac{dy}{dx} = \frac{dz}{dx}$$

So the given equation is

$$\frac{dz}{dx} = -z(1 - xz)$$

$$\Rightarrow \frac{dz}{dx} + z = z^2 x$$

$$\Rightarrow \frac{1}{z^2}\frac{dz}{dx} + \frac{1}{z} = x$$

Put $1/z = v$, so that

$$\frac{-1}{z^2}\frac{dz}{dx} = \frac{dv}{dx}$$

Therefore

$$\frac{dv}{dx} - v = -x \quad \text{(Linear)}$$

So (A) is true. The integrating factor is

$$\text{I.F.} = e^{\int -dx} = e^{-x}$$

So the solution is

$$v(e^{-x}) = \int -xe^{-x}\,dx + c$$
$$= xe^{-x} + e^{-x} + c$$

So

$$v = (x + 1) + ce^x$$

$$\Rightarrow \frac{1}{z} = (x + 1) + ce^x$$

$$\Rightarrow \sec y = x + 1 + ce^x \quad (\because z = \cos y)$$

So (D) is true.

**Answers: (A), (D)**

**19.** Consider the differential equation

$$\frac{dy}{dx} + 1 + x(x + y) = x^3(x + y)^3$$

Then the given equation

(A) can be reduced to Bernoulli's (extended form) linear equation

(B) can be reduced to homogeneous form using the substitution $x + y = u$

(C) can be reduced to variable separable using the substitution $\dfrac{1}{x + y} = v$

(D) has solution $\dfrac{1}{(x + y)^2} = ce^{x^2} + x^2 + 1$

**Solution:** We have

$$\frac{dy}{dx} + 1 + x(x + y) = x^3(x + y)^3$$

Put $x + y = u$. Then

$$\frac{dy}{dx} + 1 = \frac{du}{dx}$$

Therefore, given equation is

$$\frac{du}{dx} + ux = x^3u^3 \text{ (Bernoulli)} \tag{5.52}$$

So (A) is true. Also from Eq. (5.52)

$$u^{-3}\frac{du}{dx} + u^{-2}x = x^3 \tag{5.53}$$

Put $u^{-2} = z$ so that

$$u^{-3}\frac{du}{dx} = \frac{-1}{2}\frac{dz}{dx}$$

Therefore, Eq. (5.53) transforms into

$$\frac{dz}{dx} - (2x)z = -2x^3 \quad \text{(Linear)}$$

Now the integrating factor is

$$\text{I.F.} = e^{\int -2x\,dx} = e^{-x^2}$$

so that the general solution is

$$ze^{-x^2} = \int -2x^3 e^{-x^2}\,dx + c$$
$$= (x^2 + 1)e^{-x^2} + c$$

So

$$z = x^2 + 1 + ce^{x^2}$$

$$\Rightarrow u^{-2} = x^2 + 1 + ce^{x^2} \quad (\because z = u^{-2})$$

$$\Rightarrow \frac{1}{(x+y)^2} = x^2 + 1 + ce^{x^2} \quad (\because u = x + y)$$

So (D) is true.

**Answers: (A), (D)**

**20.** The family of curves for which the length of the normal at any point equals the length of the radius vector joining the point with the origin is (are)

(A) circles with centre at origin
(B) ellipses with centre at origin
(C) rectangular hyperbolas
(D) all lines passing through origin

**Solution:** Let $y = f(x)$ be the curve. Length of the normal at $P(x, y)$ is $OP$. That is

$$\left| y\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} \right| = \sqrt{x^2 + y^2}$$

$$\Rightarrow y^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = x^2 + y^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 = x^2$$

$$\Rightarrow y\frac{dy}{dx} = \pm x$$

**Case I:** $y\dfrac{dy}{dx} = x$. Then

$$y\,dy = x\,dx \quad \text{(Variables Separable)}$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

$$\Rightarrow y^2 - x^2 = 2c$$

which is a rectangular hyperbola.

**Case II:** $y\dfrac{dy}{dx} = -x$. Then

$$ydy + xdx = 0$$

$$\Rightarrow y^2 + x^2 = c$$

which is a circle.

**Answers: (A), (C)**

## *Matrix-Match Type Questions*

**1.** Match the items of Column I to those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\tan^3 x} =$ | (p) $\dfrac{\pi}{4}$ |
| (B) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}} =$ | (q) $\dfrac{\pi}{2}$ |
| | (r) $\pi$ |
| (C) $\displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx =$ | (s) $\dfrac{\pi}{3}$ |
| (D) $\displaystyle\int_0^{\pi} \frac{f(x)}{f(x)+f(\pi-x)}dx =$ | (t) $\dfrac{2\pi}{3}$ |

**Solution:**

(A) We have

$$\int_0^{\pi/2} \frac{dx}{1+\tan^3 x} = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}dx$$

$$= \frac{\pi}{4}$$

**Answer: (A) → (p)**

(B) We have

$$\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$= \frac{\pi}{4}$$

**Answer: (B) → (p)**

(C) We have

$$I = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$= 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx \quad [(f(2a-x) = f(x) \text{ where } a = \pi/2]$$

$$= 2\int_0^{1} \frac{dt}{1+t^2} \quad \text{where } t = \cos x$$

$$= 2\,\mathrm{Tan}^{-1} 1 = \frac{\pi}{2}$$

**Answer: (C) → (q)**

(D) We have

$$I = \int_0^{\pi} \frac{f(x)}{f(x)+f(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{f(\pi - x)}{f(\pi - x)+f(x)}\,dx$$

Therefore

$$2I = \int_0^{\pi} 1dx = \pi$$

$$\Rightarrow I = \frac{\pi}{2}$$

**Answer: (D) → (q)**

**2.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int_{-1}^{1} \frac{dt}{1+t^2}$ | (p) $\frac{\pi}{3}$ |
| (B) $\int_0^1 \frac{dt}{\sqrt{1-x^2}}$ | (q) $2\log_e\left(\frac{2}{3}\right)$ |
| (C) $\int_2^3 \frac{dx}{1-x^2}$ | (r) $\frac{1}{2}\log_e\left(\frac{2}{3}\right)$ |
| (D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ | (s) $\frac{\pi}{2}$ |

**Solution:**

(A) We have

$$\int_{-1}^{1} \frac{dt}{1+t^2} = 2\int_0^1 \frac{dt}{1+t^2} = 2\mathrm{Tan}^{-1}1 = \frac{\pi}{2}$$

**Answer: (A) → (s)**

(B) We have

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t \to 1-0} \int_0^t \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{t \to 1-0} (\mathrm{Sin}^{-1}t - \mathrm{Sin}^{-1}0)$$

$$= \mathrm{Sin}^{-1}1 = \frac{\pi}{2}$$

**Answer: (B) → (s)**

(C) We have

$$\int_2^3 \frac{dx}{1-x^2} = \frac{1}{2}\int_2^3 \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$$

$$= \frac{1}{2}\left[\log\left|\frac{1+x}{1-x}\right|\right]_2^3$$

$$= \frac{1}{2}\left[\log\left(\frac{4}{2}\right) - \log\left(\frac{3}{1}\right)\right]$$

$$= \frac{1}{2}\log\frac{2}{3}$$

**Answer: (C) → (r)**

(D) We have

$$\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \left[\mathrm{Sec}^{-1}x\right]_1^2$$

$$= \mathrm{Sec}^{-1}2 - \mathrm{Sec}^{-1}1$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

**Answer: (D) → (p)**

**3.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \sqrt{\frac{n+r}{n-r}} =$ | (p) $\pi/3$ |
| (B) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)\sqrt{r(2n+r)}} =$ | (q) $\pi/2$ |
| (C) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2rn-r^2}} =$ | (r) $\frac{\pi+2}{2}$ |
| (D) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2} =$ | (s) $\frac{\pi}{4}$ |

**Solution:**

(A) We have

$$T_r = r\text{th term} = \frac{1}{n}\sqrt{\frac{n+r}{n-r}}$$

Put $n = 1/h$ so that

$$`T_r = h\sqrt{\frac{1+rh}{1-rh}}$$

Since there are $n$ terms in the sum, the required limit is

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^{\pi/2} (1+\sin\theta)\, d\theta \quad \text{where } x = \sin\theta$$

$$= \frac{\pi}{2} - [\cos\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2} - (0-1)$$

$$= \frac{\pi}{2} + 1$$

**Answer: (A) → (r)**

(B) We have

$$T_r = r\text{th term} = \frac{n}{(n+r)\sqrt{r(2n+r)}}$$

Put $n = 1/h$ so that

$$T_r = \frac{\sqrt{h}}{(1+rh)\sqrt{r(2+rh)}}$$

$$= \frac{h}{(1+rh)\sqrt{rh(2+rh)}}$$

Therefore the required sum is

$$\int_0^1 \frac{dx}{(1+x)\sqrt{x(2+x)}} = \int_0^1 \frac{dx}{(1+x)\sqrt{(1+x)^2-1}}$$

$$= \left[\mathrm{Sec}^{-1}(1+x)\right]_0^1$$

$$= \mathrm{Sec}^{-1}2 - \mathrm{Sec}^{-1}1$$

$$= \frac{\pi}{3} - 0$$

$$= \frac{\pi}{3}$$

**Answer: (B) → (p)**

(C) We have

$$T_r = r\text{th term} = \frac{1}{\sqrt{2rn-r^2}}$$

$$= \frac{\sqrt{h}}{\sqrt{2r-r^2h}} \quad \text{where } h = \frac{1}{n}$$

$$= \frac{h}{\sqrt{2rh-r^2h^2}}$$

Therefore the required limit is

$$\int_0^1 \frac{dx}{\sqrt{2x-x^2}} = \int_0^1 \frac{dx}{\sqrt{1-(1-x)^2}}$$

$$= -\left[\mathrm{Sin}^{-1}(1-x)\right]_0^1$$

$$= -\mathrm{Sin}^{-1}0 + \mathrm{Sin}^{-1}1$$

$$= \frac{\pi}{2}$$

**Answer: (C) → (q)**

(D) We have

$$T_r = r\text{th term} = \frac{n}{n^2+r^2}$$

Put $n = 1/h$ so that

$$T_r = \frac{h}{1+(rh)^2}$$

Therefore the required limit is

$$\int_0^1 \frac{dx}{1+x^2} = \left[\mathrm{Tan}^{-1}x\right]_0^1 = \mathrm{Tan}^{-1}1 = \frac{\pi}{4}$$

**Answer: (D) → (s)**

**4.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} =$ | (p) $\frac{2\pi}{3}$ |
| (B) $\int_0^\infty \frac{x}{(1+x)(1+x^2)}\, dx =$ | (q) $\frac{\pi^2}{2\sqrt{2}}$ |
| (C) $\int_0^\pi x\sin^3 x\, dx =$ | (r) $\frac{\pi}{2\sqrt{2}}$ |
| (D) $\int_0^\pi \frac{x}{1+\cos^2 x}\, dx =$ | (s) $\pi/4$ |

**Solution:**

(A) We have

$$I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

Put $x = a\sin\theta$. Therefore

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\, d\theta = \frac{\pi}{4}$$

**Answer: (A) → (s)**

(B) We have

$$I = \int_0^\infty \frac{x}{(1+x)(1+x^2)}\, dx$$

Put $x = \tan\theta$ so that $x \to \infty \Rightarrow \theta \to \pi/2$. Therefore

$$I=\int_0^{\pi/2}\frac{\tan\theta}{1+\tan\theta}\,d\theta$$
$$=\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$$
$$=\frac{\pi}{4}$$

**Answer: (B) → (s)**

(C) We have

$$I=\int_0^{\pi}x\sin^3 x\,dx$$
$$=\int_0^{\pi}(\pi-x)\sin^3(\pi-x)\,dx$$
$$=\pi\int_0^{\pi}\sin^3 x\,dx-I$$

Therefore

$$2I=\pi\int_0^{\pi}\sin^3 x\,dx$$
$$=2\pi\int_0^{\pi/2}\sin^3 x\,dx$$
$$=2\pi\frac{(3-1)}{3}$$
$$=\frac{4\pi}{3}$$

So

$$I=\frac{2\pi}{3}$$

**Answer: (C) → (p)**

(D) We have

$$I=\int_0^{\pi}\frac{x}{1+\cos^2 x}\,dx$$
$$=\int_0^{\pi}\frac{\pi-x}{1+\cos^2(\pi-x)}\,dx$$
$$=\pi\int_0^{\pi}\frac{dx}{1+\cos^2 x}-I$$

Therefore

$$2I=\pi\int_0^{\pi}\frac{dx}{1+\cos^2 x}$$
$$=\pi\times 2\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}$$

So

$$I=\pi\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}$$
$$=\pi\int_0^{\pi/2}\frac{\sec^2 x}{\sec^2 x+1}\,dx$$
$$=\pi\int_0^{\pi/2}\frac{\sec^2 x}{2+\tan^2 x}\,dx$$
$$=\pi\int_0^{\infty}\frac{dt}{2+t^2}\quad\text{where } t=\tan x$$
$$=\frac{\pi}{\sqrt{2}}\left[\mathrm{Tan}^{-1}\frac{t}{2}\right]_0^{\infty}$$
$$=\frac{\pi}{\sqrt{2}}\left[\lim_{t\to\infty}\mathrm{Tan}^{-1}\frac{t}{2}-0\right]$$
$$=\frac{\pi}{\sqrt{2}}\times\frac{\pi}{2}$$
$$=\frac{\pi^2}{2\sqrt{2}}$$

**Answer: (D) → (q)**

**5.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\int_0^4\{\lvert x-1\rvert+\lvert x-3\rvert\}\,dx$ is | (p) 5 |
| (B) The value of $\int_{1/e}^{e^2}\left\lvert\frac{\log_e x}{x}\right\rvert dx$ is | (q) 1 |
| (C) $\int_0^9\{\sqrt{x}\}\,dx$ where $\{t\}$ denotes the fractional part of $t$ is | (r) 5/2 |
| (D) $\int_0^2\lvert 1-x\rvert\,dx$ equals | (s) 10 |

**Solution:**

(A) We have

$$I=\int_0^4\{\lvert x-1\rvert+\lvert x-3\rvert\}\,dx$$
$$=\int_0^1(4-2x)\,dx+\int_1^3((x-1)+(3-x))\,dx+\int_3^4(2x-4)\,dx$$

$$= 4 - (1 - 0) + 2\,(3 - 1) + (16 - 9) - 4\,(4 - 3)$$
$$= 4 + 4 + 7 - 5 = 10$$

**Answer: (A) → (s)**

(B) We have

$$I = \int_{1/e}^{e^2} \left|\frac{\log_e x}{x}\right| dx$$
$$= \int_{1/e}^{1} \frac{-\log_e x}{x}\,dx + \int_{1}^{e^2} \frac{\log_e x}{x}\,dx$$
$$= -\frac{1}{2}\Big[(\log_e x)^2\Big]_{1/e}^{1} + \frac{1}{2}\Big[(\log_e x)^2\Big]_{1}^{e^2}$$
$$= -\frac{1}{2}\left[0 - \left(\log_e \frac{1}{e}\right)^2\right] + \frac{1}{2}\left[\left(\log_e e^2\right)^2 - 0\right]$$
$$= \frac{1}{2} + \frac{1}{2}(2^2 - 0)$$
$$= \frac{1}{2} + 2$$
$$= 5/2$$

**Answer: (B) → (r)**

(C) We have

$$I = \int_0^9 \{\sqrt{x}\}\,dx \quad (\{t\} \text{ is the fractional part of } t)$$
$$= \int_0^9 (\sqrt{x} - [\sqrt{x}])\,dx \quad ([t] \text{ is the integer part of } t)$$
$$= \int_0^9 \sqrt{x}dx - \left(\int_0^1 [\sqrt{x}]dx + \int_1^4 [\sqrt{x}]dx + \int_4^9 [\sqrt{x}]dx\right)$$
$$= \frac{2}{3}\Big[x^{3/2}\Big]_0^9 - \left(0 + \int_1^4 1dx + \int_4^9 2dx\right)$$
$$= \frac{2}{3}(3^3) - (3 + 2\times 5)$$
$$= 18 - 13 = 5$$

**Answer: (C) → (p)**

(D) We have

$$\int_0^2 |1-x|\,dx = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$$
$$= 1 - \frac{1}{2} + \frac{1}{2}(2^2 - 1) - (2 - 1)$$
$$= 1 - \frac{1}{2} + \frac{3}{2} - 1$$
$$= 1$$

**Answer: (D) → (q)**

**6.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x} =$ | (p) $-\dfrac{1}{2}$ |
| (B) $\displaystyle\lim_{n\to\infty}\left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + \frac{3}{1-n^2} + \cdots + \frac{n}{1-n^2}\right]$ is | (q) 1 |
| (C) $\displaystyle\int_0^{\pi/2} \frac{f(x)}{f(x) + f\left(\frac{\pi}{2} - x\right)}\,dx = \frac{\pi}{k}$ where $k$ is | (r) 4 |
| (D) The area common to the curves $y^2 = x$ and $x^2 = y$ is | (s) 1/3 |
| | (t) 2/3 |

**Solution:**

(A) We have

$$I = \int_0^{\pi/2} \frac{dx}{1+\sin x}$$
$$= \int_0^{\pi/2} \frac{dx}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$$
$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{\left(1 + \tan\frac{x}{2}\right)^2}\,dx$$
$$= 2\int_0^1 \frac{dt}{(1+t)^2}$$
$$= -2\left[\frac{1}{1+t}\right]_0^1$$
$$= -2\left[\frac{1}{2} - 1\right]$$
$$= 1$$

**Answer: (A) → (q)**

(B) We have

$$\frac{1}{1-n^2} + \frac{2}{1-n^2} + \frac{3}{1-n^2} + \cdots + \frac{n}{1-n^2} = \frac{1+2+3+\cdots+n}{1-n^2}$$
$$= \frac{n(n+1)}{2(1-n^2)}$$
$$= \frac{n}{2(1-n)}$$

$$= \frac{1}{2\left(\frac{1}{n}-1\right)}$$

So

$$\lim_{n\to\infty}\left(\frac{1}{1-n^2}+\frac{2}{1-n^2}+\frac{3}{1-n^2}+\cdots+\frac{n}{1-n^2}\right)$$
$$= \lim_{n\to\infty}\frac{1}{2\left(\frac{1}{n}-1\right)} = -\frac{1}{2}$$

**Answer: (B) → (p)**

(C) We have

$$I = \int_0^{\pi/2} \frac{f(x)}{f(x)+f\left(\frac{\pi}{2}-x\right)}$$
$$= \int_0^{\pi/2} \frac{f\left(\frac{\pi}{2}-x\right)}{f\left(\frac{\pi}{2}-x\right)+f(x)}\,dx$$

Now

$$2I = \int_0^{\pi/2} 1dx = \frac{\pi}{2}$$

so that $I = \pi/4$. Hence $k = 4$.

**Answer: (C) → (r)**

(D) The required area (see Fig. 5.46) is given by

$$\int_0^1 (\sqrt{x}-x^2)\,dx = \frac{2}{3}-\frac{1}{3} = \frac{1}{3}$$

**Answer: (D) → (s)**

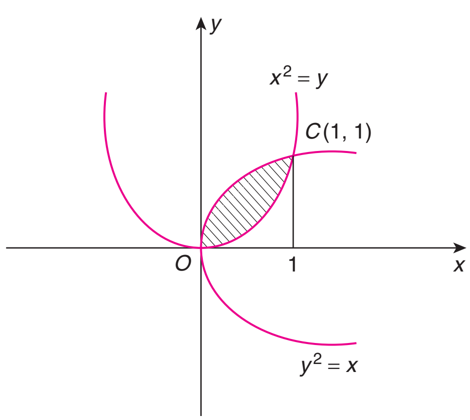



**FIGURE 5.46** Matrix-match type question 6.

**7.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) The area bounded by the curves $y=(x-1)^2$, $y=(x+1)^2$ and the line $y=\frac{1}{4}$ is | (p) 9 |
| (B) The area bounded by the curves $y=\sqrt{x}$, $2y+3=x$ and $x$-axis in the first quadrant is | (q) 5/6 |
| (C) The slope of the tangent to a curve $y=f(x)$ at $(x, f(x))$ is $2x+1$. If the curve passes through the point (1, 2), then the area bounded by the curve, the $x$-axis and the line $x = 1$ is | (r) 4 |
| (D) The area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$ is | (s) 1/3 |
| | (t) 2/3 |

**Solution:**

(A) See Fig. 5.47. The two curves $y = (x-1)^2$ and $y = (x+1)^2$ intersect on $y$-axis at (0, 1).

The line $y = 1/4$ cuts $y = (x-1)^2$ in (1/2, 1/4) and (3/2, 1/4).

The curve $y = (x+1)^2$ in (–3/2, 1/4) and (–1/2, 1/4).

Therefore the required area is

$$2\int_0^{1/2}\left((x-1)^2-\frac{1}{4}\right)dx = 2\int_0^{1/2}\left(x^2-2x+\frac{3}{4}\right)dx$$
$$= 2\left[\frac{1}{24}-\frac{1}{4}+\frac{3}{8}\right]$$
$$= \frac{1}{3}$$

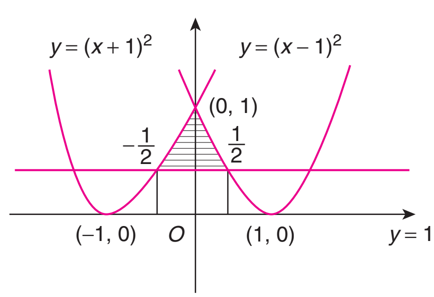



**FIGURE 5.47** Matrix-match type question 7, part (A).

**Answer: (A) → (s)**

(B) The line $x = 2y + 3$ meets the curve in (9, 3) (see Fig. 5.48). Then the required area (shaded portion) is

$$\int_0^3 \sqrt{x}\,dx + \int_3^9\left(\sqrt{x}-\frac{x-3}{2}\right)dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^3 + \frac{2}{3}\left[x^{3/2}\right]_3^9 - \frac{1}{4}\left[x^2\right]_3^9 + \frac{3}{2}(9-3)$$
$$= \frac{2}{3}\cdot 3^{3/2} + \frac{2}{3}(27 - 3^{3/2}) - \frac{1}{4}(81-9) + \frac{3}{2}(6)$$
$$= 18 - 18 + 9 = 9$$

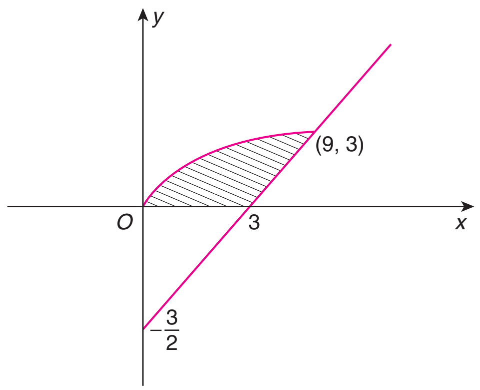



FIGURE 5.48 Matrix-match type question 7, part (B).

**Answer: (B) → (p)**

(C) We have

$$\frac{dy}{dx} = 2x + 1$$
$$\Rightarrow y = x^2 + x + c$$

The curve passes through the point (1, 2). This implies

$$2 = 1 + 1 + c \Rightarrow c = 0$$

Therefore the curve is

$$y = x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$$
$$\Rightarrow \left(x + \frac{1}{2}\right)^2 = y + \frac{1}{4}$$

(See Fig. 5.49.) Therefore

$$\text{Area} = \int_0^1 (x^2 + x)\, dx = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

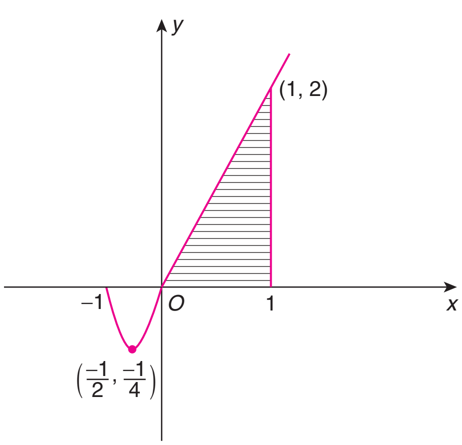



FIGURE 5.49 Matrix-match type question 7, part (C).

**Answer: (C) → (q)**

(D) See the graph of $y = \cos x$ in $[0, 2\pi]$ shown in Fig. 5.50. Then the required area (shaded part) is

$$\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx + \int_{3\pi/2}^{2\pi} \cos x\, dx$$
$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$
$$= (1 - 0) - [-1 - 1] + [0 - (-1)]$$
$$= 1 + 2 + 1 = 4$$

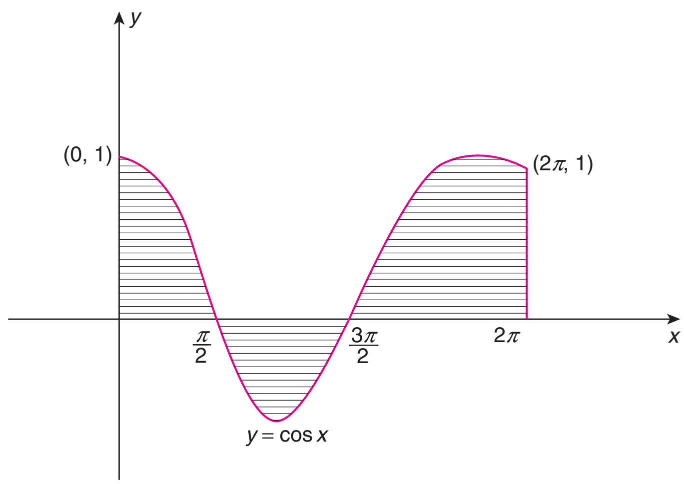



FIGURE 5.50 Matrix-match type question 7, part (D).

**Answer: (D) → (r)**

**8.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) The area bounded by the curves $y = x^2$, $y = -x^2$ and $y^2 = 4x - 3$ is | (p) $\log_e\left(\frac{3}{2}\right)$ |
| (B) The area bounded by the curve $y = x(x-1)^2$, the $y$-axis and the line $y = 2$ where $0 \le x \le 2$, is | (q) $\log_e\left(\frac{9}{2}\right)$ |
| (C) The area of the region bounded by the curves $y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$, $y = \cot x, \frac{\pi}{6} \le x \le \frac{\pi}{2}$ and the $x$-axis is | (r) $\frac{1}{3}$ |
| (D) $\int_0^{\log_e 3} \left[\frac{3}{e^x}\right] dx$ (where [·] denotes the integer part) is | (s) $\frac{10}{3}$ |

**Solution:**

(A) The curves $y = x^2$, $y = -x^2$ are touched by $y^2 = 4x - 3$ at the points (1, 1) and (1,–1), respectively (see Fig. 5.51). The curve $y^2 = 4x - 3$ cuts the $x$-axis at (3/4, 0). The required area (shaded portion ) is given by

$$2\left[\int_0^1 x^2dx - \int_{3/4}^1 \sqrt{4x-3}\,dx\right] = \frac{2}{3} - \frac{2}{4}\times\frac{2}{3}\left[(4x-3)^{3/2}\right]_{3/4}^1$$
$$= \frac{2}{3} - \frac{1}{3}(1-0) = \frac{1}{3}$$

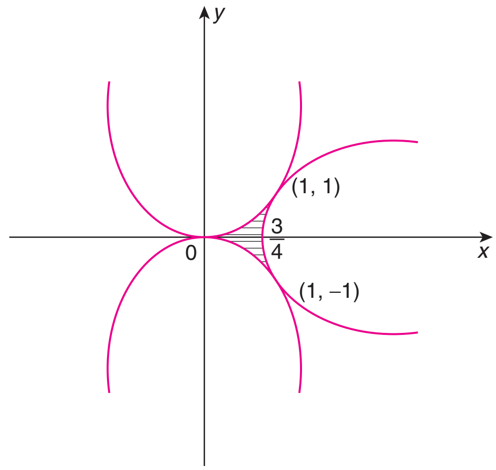



**FIGURE 5.51** Matrix-match type question 8, part (A).

**Answer: (A) → (r)**

(B) The curve $y = x(x-1)^2$ meets $x$-axis at (0,0) and (1, 0) and $y \to +\infty$ as $x \to +\infty$. Also $y \to -\infty$ as $x \to -\infty$. Therefore the required area (dotted portion, Fig. 5.52) is equal to

$$(2\times 2)\ (= \text{area of the square})\ -\int_0^2 (x-1)^2 x\,dx$$
$$= 4 - \left[\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right]_0^2$$
$$= 4 - \left[\frac{16}{4} - \frac{16}{3} + \frac{4}{2}\right]$$
$$= 4 - \left[\frac{48-64+24}{12}\right]$$

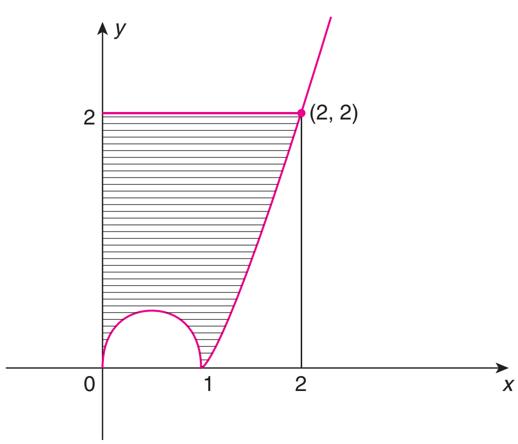



**FIGURE 5.52** Matrix-match type question 8, part (B).

$$= 4 - \frac{8}{12}$$
$$= 4 - \frac{2}{3}$$
$$= \frac{10}{3}$$

**Answer: (B) → (s)**

(C) $y = \tan x$ and $y = \cot x$ intersect in $A(\pi/4, 1)$. The required area (shaded part, Fig. 5.53) is

$$\int_{\pi/6}^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/3} \cot x\,dx = [\log_e \sec x]_{\pi/6}^{\pi/4} + [\log_e \sin x]_{\pi/4}^{\pi/3}$$
$$= \left(\log_e \sqrt{2} - \log_e \frac{2}{\sqrt{3}}\right) + \left(\log_e \frac{\sqrt{3}}{2} - \log_e \frac{1}{\sqrt{2}}\right)$$
$$= 2\log_e \sqrt{2} + 2\log_e \frac{\sqrt{3}}{2}$$
$$= \log_e 2 + \log_e \frac{3}{4} = \log \frac{3}{2}$$

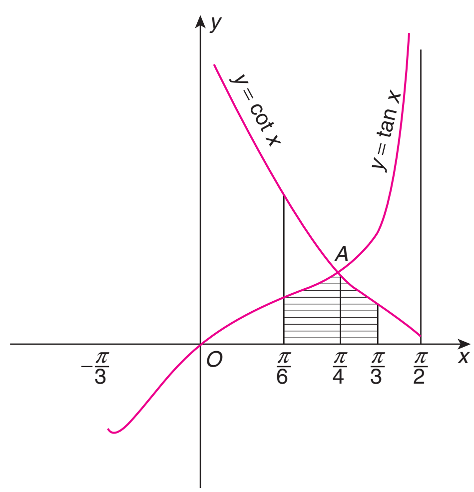



**FIGURE 5.53** Matrix-match type question 8, part (C).

**Answer: (C) → (p)**

(D) We have

**(i)** $3e^{-x} = 3$ if $x = 0$

**(ii)** $3e^{-x} = 2$ if $x = \log_e \frac{3}{2}$

**(iii)** $3e^{-x} = 1$ if $x = \log_e 3$

**(iv)** $[3e^{-x}] = 0$ if $x > \log_e 3$

Therefore

$$\int_0^{\log_e 3} [3e^{-x}]\,dx = \int_0^{\log_e 3/2} 2\,dx + \int_{\log_e 3/2}^{\log_e 3} 1\,dx$$
$$= 2\left(\log_e \frac{3}{2}\right) + \left(\log_e 3 - \log_e \frac{3}{2}\right)$$

$$= \log_e \frac{9}{4} + \log_e 2$$
$$= \log_e \frac{9}{2}$$

**Answer: (D) → (q)**

**9.** In Column I, differential equations are given and in Column II their solutions are given. Match them.

| *Column I* | *Column II* |
|---|---|
| (A) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ | (p) $e^y = e^x + \frac{x^3}{3} + c$ |
| (B) $\frac{dy}{dx} = e^{x+y}$ | (q) $y(cx + \log_e ex) = 1$ |
| (C) $(y \log_e x - 1) y\,dx = x dy$ | (r) $xe^y = \tan y + c$ |
| (D) $\frac{dx}{dy} + x = e^{-y} \sec^2 y$ | (s) $e^x + e^{-y} = c$ |

**Solution**

(A) We have

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$
$$\Rightarrow e^y dy = (e^x + x^2)\, dx \quad \text{(Variables Separable)}$$
$$\Rightarrow \int e^y dy = \int (e^x + x^2)\, dx + c$$
$$\Rightarrow e^y = e^x + \frac{x^3}{3} + c$$

**Answer: (A) → (p)**

(B) We have

$$\frac{dy}{dx} = e^{x+y}$$
$$\Rightarrow e^{-y} dy = e^x dx$$

The solution is

$$-e^{-y} = e^x + c$$
$$\Rightarrow e^x + e^{-y} = c$$

**Answer: (B) → (s)**

(C) We have

$$(y \log x - 1) y\, dx = x dy$$
$$\Rightarrow \frac{dy}{dx} = (y \log x - 1) \frac{y}{x}$$
$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = y^2 \left( \frac{\log x}{x} \right) \quad \text{(Bernoulli)}$$
$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{xy} = \left( \frac{\log x}{x} \right)$$
$$\Rightarrow -\frac{dz}{dx} + \frac{z}{x} = \left( \frac{\log x}{x} \right) \quad \text{where } z = \frac{1}{y}$$
$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = \frac{-\log x}{x}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{-1}{x} dx} = e^{\log \frac{1}{x}} = \frac{1}{x}$$

The solution is

$$z\left( \frac{1}{x} \right) = \int \frac{-\log x}{x} \left( \frac{1}{x} \right) dx + c$$
$$= -\int \frac{\log x}{x^2}\, dx + c$$
$$= -\left[ \frac{-1}{x} \log x - \int \frac{-1}{x} \cdot \frac{1}{x} dx \right] + c$$
$$= \frac{1}{x} \log x + \frac{1}{x} + c$$

Therefore

$$z = \log x + 1 + cx = \log (ex) + cx$$
$$\Rightarrow \frac{1}{y} = \log (ex) + cx$$
$$\Rightarrow 1 = y[\log (ex) + cx]$$

**Answer: (C) → (q)**

(D) We have

$$\frac{dx}{dy} + x = e^{-y} \sec^2 y$$
$$\Rightarrow e^y \frac{dx}{dy} + xe^y = \sec^2 y$$
$$\Rightarrow \frac{d}{dy}(xe^y) = \sec^2 y$$
$$\Rightarrow xe^y = \tan y + c$$

**Answer: (D) → (r)**

**10.** Match the items of Column I with those of Column II.

| *Column I* | *Column II* |
|---|---|
| (A) A normal $PG$ to a curve meets the $x$-axis in $G$. If the distance $OG$ ('$O$' is origin) is twice the abscissa of $P$, then the curve is a | (p) hyperbola |
| (B) A normal is drawn at a point $P(x, y)$ of a curve. It meets the $x$-axis in $G$. If $PG$ is of constant length $k$ and the curve passes through $(0, k)$, then the curve is a | (q) parabola |

*(Continued)*

| *Column I* | *Column II* |
|---|---|
| (C) The normal at every point of a curve passes through a fixed point. Then the curve is a | (r) rectangular hyperbola |
| (D) The curve in which the sub-tangent at every point is bisected at the origin is | (s) circle |
| | (t) ellipse |

## Solution

(A) See Fig. 5.54. Normal at $(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

where $m = (dy/dx)_{(x_1,y_1)}$. Therefore $G = (x_1 + my_1, 0)$.

By hypothesis

$$x_1 + my_1 = 2x_1$$

$$\Rightarrow y_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = x_1$$

The differential equation is

$$y\frac{dy}{dx} = x \Rightarrow ydy = xdx$$

The solution is

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = 2c$$

which is a rectangular hyperbola.

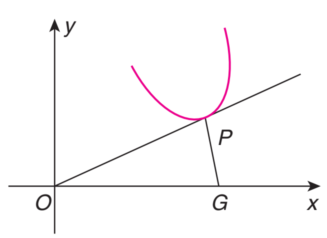



**FIGURE 5.54** Matrix-match type question 10, part (A).

**Answer: (A) → (p), (r)**

(B) In Fig. 5.54, $PG = k$ (constant). This implies

$$\left| y\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} \right| = k$$

$$\Rightarrow y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = k^2$$

$$\Rightarrow y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

$$\Rightarrow \frac{y}{\sqrt{k^2 - y^2}}dy = \pm dx \text{ (Variables Separable)}$$

$$\Rightarrow \int \frac{y}{\sqrt{k^2 - y^2}}dy = \pm\int dx + c$$

$$\Rightarrow \sqrt{k^2 - y^2} = \pm x + c$$

So the curve passes through $(0, k) \Rightarrow c = 0$. Therefore the curve is

$$k^2 - y^2 = x^2 \text{ or } x^2 + y^2 = k^2$$

**Answer: (B) → (s)**

(C) Equation of the normal at $(x_1, y_1)$ is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

where $m = (dy/dx)_{(x_1,y_1)}$. This passes through a fixed point $(\alpha, \beta)$. This implies

$$\beta - y_1 = -\frac{1}{m}(\alpha - x_1)$$

Therefore, the differential equation is

$$(\beta - y)dy + (\alpha - x)dx = 0$$

The solution is

$$\int(\beta - y)dy + \int(\alpha - x)dx = c$$

$$\Rightarrow \frac{(y-\beta)^2}{2} + \frac{(x-\alpha)^2}{2} = c$$

$$\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = 2c$$

which is a circle and $(\alpha, \beta)$ is its centre.

**Answer: (C) → (s)**

(D) In Fig. 5.55, $PG$ is drawn perpendicular to $x$-axis and $T$ is the point on the $x$-axis where the tangent at $P$ meets $x$-axis. By definition, $TG$ is the sub-tangent. The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

where $m = (dy/dx)_{(x_1,y_1)}$. Therefore

$$T = \left(x_1 - \frac{y_1}{m}, 0\right) \text{ and } G = (x_1, 0)$$

Since $(0, 0)$ is the mid-point of $TG$, we have

$$x_1 - \frac{y_1}{m} + x_1 = 0$$

Therefore the differential equation

$$2x\frac{dy}{dx}-y=0$$

$$\Rightarrow \frac{2}{y}dy=\frac{dx}{x} \text{ (Variables Separable)}$$

$$\Rightarrow \int \frac{2}{y}dy=\int \frac{dx}{x}+c$$

$$\Rightarrow 2\log y=\log x+c$$

$$\Rightarrow y^2=kx$$

which is a parabola.

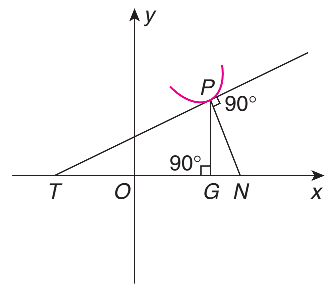



**FIGURE 5.55** Matrix-match type question 10, part (D).

**Answer: (D) → (q)**

## *Comprehension-Type Questions*

1. **Passage:** $f$ is continuous for all $x \ge 0$. Then answer the following questions.

**(i)** If $g$ : $[0, a]$ is continuous and $f(x) = f(a-x)$, $g(x)+g(a-x)=2$, then

$$\int_0^a f(x)g(x)dx =$$

(A) $\int_0^a g(x)\,dx$ (B) $\int_0^a f(x)\,dx$

(C) $2\int_0^a f(x)\,dx$ (D) 0

**(ii)** $\int_0^\pi e^{|\cos x|}\left(\left[2\sin\left(\frac{1}{2}\cos x\right)dx\right]\right)=$

(A) $\pi$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi^2}{2}$ (D) 0

**(iii)** $\int_{\pi/4}^{3\pi/4}\frac{x}{1+\sin x}dx=$

(A) $\pi(\sqrt{2}-1)$ (B) $\pi(\sqrt{2}+1)$

(C) $2\pi\sqrt{2}$ (D) $\pi\sqrt{2}$

**Solution:**

**(i)** Since $f$ and $g$ are continuous for $x \ge 0$ it follows that $\int_0^a f(x)\,dx$, $\int_0^a g(x)dx$ and $\int_0^a f(x)g(x)dx$ exist. Now

$$\int_0^a f(x)g(x)dx=\int_0^a f(a-x)g(a-x)dx$$

$$=\int_0^a f(x)[2-g(x)]dx$$

$$[\because f(a-x)=f(x) \text{ and } g(x)+g(a-x)=2]$$

Therefore

$$2\int_0^a f(x)g(x)dx=2\int_0^a f(x)dx$$

$$\Rightarrow \int_0^a f(x)g(x)dx=\int_0^a f(x)\,dx$$

**Answer: (B)**

**(ii)** Let $f(x)=e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)\right)$

Since $f$ is continuous for all real $x$, it follows that $\int_0^a f(x)\,dx$ exists. Also

$$f(\pi-x)=e^{|\cos(\pi-x)|}\left(2\sin\left(\frac{1}{2}\cos(\pi-x)\right)\right)$$

$$=e^{|\cos x|}\left(2\sin\left(-\frac{1}{2}\cos x\right)\right)$$

$$=-e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)\right)$$

$$=-f(x)$$

Therefore

$$\int_0^\pi f(x)\,dx=\int_0^\pi f(\pi-x)\,dx$$

$$= -\int_0^{\pi} f(x)\,dx$$

Now

$$\int_0^{\pi} f(x)\,dx = 0$$

**Answer: (D)**

**(iii)** Let

$$I = \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x}\,dx$$

It exists because $x/(1 + \sin x)$ is continuous on $[\pi/4, 3\pi/4]$. Now

$$I = \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x}\,dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\frac{\pi}{4}+\frac{3\pi}{4}-x}{1+\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}-x\right)}\,dx$$

$$\left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1+\sin x}\,dx$$

$$= \pi \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\sin x} - I$$

So

$$2I = \pi \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\sin x}$$

$$\Rightarrow I = \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\sin x}$$

$$= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} (\sec^2 x - \sec x \tan x)\,dx$$

$$= \frac{\pi}{2}\left\{[\tan x]_{\pi/4}^{3\pi/4} - [\sec x]_{\pi/4}^{3\pi/4}\right\}$$

$$= \frac{\pi}{2}[(-1-1)-(-\sqrt{2}-\sqrt{2})]$$

$$= \frac{\pi}{2}(-2+2\sqrt{2})$$

$$= \pi(\sqrt{2}-1)$$

**Answer: (A)**

**2. Passage:** If $f:[a,b] \to \mathbb{R}$, $g:[a,b] \to \mathbb{R}$ are continuous and $0 \le f(x) \le g(x)\ \forall\ x \in [a,b]$, then the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a, x = b$ is given by $\int_a^b [g(x) - f(x)]dx$. Using this information, answer the following questions.

**(i)** The area of the region between the curves $y = x^2$ and $y = x^3$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$

(C) $\frac{1}{6}$ (D) $\frac{1}{12}$

**(ii)** The area of the region bounded between the parabola $y = x^2 - x - 6$ and the line $y = -4$ is

(A) $\frac{7}{2}$ (B) $\frac{9}{2}$

(C) $\frac{5}{2}$ (D) 4

**(iii)** The area of the region bounded by the curves $y = ex\log_e x$ and $y = \frac{\log_e x}{ex}$ is

(A) $\frac{e^2-5}{4e}$ (B) $\frac{e^2+1}{2e}$

(C) $\frac{e^2+1}{3e}$ (D) $\frac{e^2+1}{e}$

**Solution:**

**(i)** The curves $y = x^2$ and $y = x^3$ intersect in $(0, 0)$ and $(1, 1)$ only. Also $0 \le x \le 1 \Rightarrow x^3 \le x^2$. Hence the required area is

$$\int_0^1 (x^2 - x^3)\,dx = \frac{1}{3}\left[x^3\right]_0^1 - \frac{1}{4}\left[x^4\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

**Answer: (D)**

**(ii)** We have

$$y = x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 = \left(y + \frac{25}{4}\right)$$

The line $y=-4$ intersects the parabola in $(-1,-4)$ and $(2,-4)$ (see Fig. 5.56). Hence the required area is

$$\int_{-1}^{2}[-4-(x^2-x-6)]\,dx=\int_{-1}^{2}(2-x^2+x)\,dx$$
$$=2[x]_{-1}^{2}-\frac{1}{3}\left[x^3\right]_{-1}^{2}+\frac{1}{2}\left[x^2\right]_{-1}^{2}$$
$$=2(2+1)-\frac{1}{3}(8+1)+\frac{1}{2}(4-1)$$
$$=6-3+\frac{3}{2}=\frac{9}{2}$$

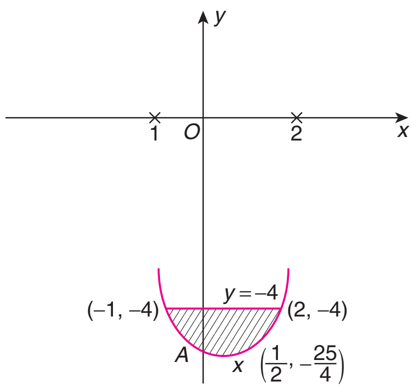



**FIGURE 5.56** Comprehension-type question 2, part (ii).

**Answer: (B)**

**(iii)** The two curves intersect in $(1, 0)$ and $(1/e, -1)$. Since

$$\lim_{x\to 0+}\frac{1}{x}=+\infty$$
$$\lim_{x\to 0+}\log_e x=-\infty$$

and
$$\lim_{x\to 0+} ex\log_e x=\lim_{x\to 0+}\frac{e\log_e x}{\left(\frac{1}{x}\right)}\left(\frac{-\infty}{\infty}\right)$$
$$=e\lim_{x\to 0+}\frac{1/x}{-1/x^2}$$
$$=e\lim_{x\to 0+}(-x)$$
$$=0$$

Now see Fig. 5.57.

$$\text{Required area }=\int_{1/e}^{1}\left(\frac{\log_e x}{ex}-ex\log_e x\right)dx$$
$$=\frac{1}{2e}\left[(\log_e x)^2\right]_{1/e}^{1}-e\left[\left[\frac{x^2}{2}\log x\right]_{1/e}^{1}-\int_{1/e}^{1}\frac{x^2}{2}\cdot\frac{1}{x}dx\right]$$
$$=\frac{1}{2e}(0-1)-e\left[\frac{1}{2}\left(0+\frac{1}{e^2}\right)-\frac{1}{4}\left(1-\frac{1}{e^2}\right)\right]$$
$$=\frac{-1}{2e}-\frac{1}{2e}+\frac{e}{4}-\frac{1}{4e}$$
$$=\frac{e}{4}-\frac{5}{4e}$$
$$=\frac{e^2-5}{4e}$$

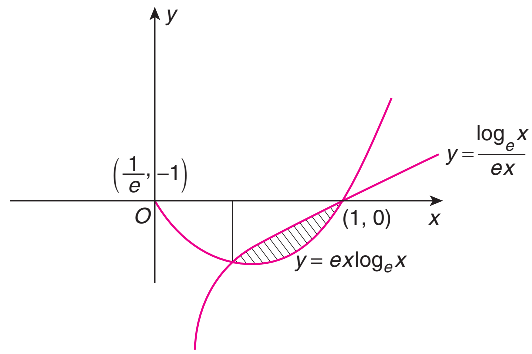



**FIGURE 5.57** Comprehension-type question 2, part (iii).

**Answer: (A)**

**3. Passage:** General solution of the differential equation $f(x)dx+g(y)dy=0$ is $\int f(x)\,dx+\int g(y)\,dy=c$. Answer the following questions.

**(i)** General solution of the equation $\dfrac{dy}{dx}=\sin(x+y)+\cos(x+y)$ is

(A) $\tan(x+y)=x+c$

(B) $1+\tan\left(\dfrac{x+y}{2}\right)=ce^x$

(C) $\tan(x+y)=ke^x$

(D) $\tan\left(\dfrac{x+y}{2}\right)=ce^{x+y}$

**(ii)** Solution of $\dfrac{dy}{dx}=\dfrac{x(2\log x+1)}{\sin y+y\cos y}$ is

(A) $\sin y=x^2\log x+c$

(B) $y=x^2\log x+c$

(C) $y\sin y=x\log x+c$

(D) $y\sin y=x^2\log x+c$

**(iii)** Solution of $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{1}{(1+\log x+\log y)^2}$ is

(A) $xy\left[1+(\log xy)^2\right]=\dfrac{x^2}{2}+c$

(B) $1+(\log xy)^2=\dfrac{x^2}{2}+y+c$

(C) $xy[1+\log(xy)] = \dfrac{x^2}{2} + c$

(D) $xy(1+\log x + \log y) = \dfrac{x}{2} + c$

## Solution

**(i)** The given equation is

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

Put $x + y = u$. Then

$$\frac{du}{dx} = 1 + \sin u + \cos u$$
$$= 2\cos^2\frac{u}{2} + 2\sin\frac{u}{2}\cos\frac{u}{2}$$

Therefore

$$\frac{1}{2\cos^2\frac{u}{2} + 2\sin\frac{u}{2}\cos\frac{u}{2}} du = dx$$

$$\Rightarrow \frac{\frac{1}{2}\sec^2\frac{u}{2}}{1+\tan\frac{u}{2}} du = dx$$

Integrating we get

$$\int \frac{\frac{1}{2}\sec^2\frac{u}{2}}{1+\tan\frac{u}{2}} du = x + c$$

$$\Rightarrow \log\left(1+\tan\frac{u}{2}\right) = x + c$$

$$\Rightarrow 1+\tan\left(\frac{x+y}{2}\right) = ce^x$$

**Answer: (B)**

**(ii)** The given equation is

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

$$\Rightarrow (\sin y + y\cos y)dy = (2x\log x + x)dx$$

Integrating we get

$$\int(\sin y + y\cos y)dy = \int(2x\log x + x)dx + c$$

$$-\cos y + \cos y + y\sin y = x^2\log x - \int x^2 \cdot \frac{1}{x}dx + \int x dx + c$$
$$= x^2 \log x + c$$

Therefore

$$y\sin y = x^2\log x + c$$

**Answer: (D)**

**(iii)** We have

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{(1+\log x + \log y)^2} = \frac{1}{(1+\log xy)^2}$$

Put $xy = u$ so that

$$\frac{du}{dx} = \frac{x}{(1+\log u)^2}$$

Therefore

$$(1+\log u)^2 du = x dx$$

$$\Rightarrow \int(1+\log u)^2 du = \int x dx + c$$

$$\Rightarrow u(1+\log u)^2 - \int \frac{2(1+\log u)}{u}\cdot u\, du = \frac{x^2}{2} + c$$

$$\Rightarrow u(1+\log u)^2 - 2u - 2\left[u\log u - \int 1 du\right] = \frac{x^2}{2} + c$$

$$\Rightarrow u(1+\log u)^2 - 2u - 2u\log u + 2u = \frac{x^2}{2} + c$$

$$\Rightarrow u\left[1 + 2\log u + (\log u)^2\right] - 2u\log u = \frac{x^2}{2} + c$$

$$\Rightarrow u[1+(\log u)^2] = \frac{x^2}{2} + c$$

$$\Rightarrow xy[1+(\log(xy))^2] = \frac{x^2}{2} + c$$

**Answer: (A)**

**4. Passage:** The differential equation

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where $f$ and $g$ are homogeneous functions of same degree can be solved by using the substitution $y = vx$. Answer the following questions.

**(i)** Solution of the differential equation

$$\left(x\frac{dy}{dx} - y\right)\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = x$$

is

(A) $\sqrt{x^2+y^2} = ke^{\frac{y}{x}\mathrm{Tan}^{-1}\frac{y}{x}}$

(B) $(x^2+y^2)y = kxe^{\mathrm{Tan}^{-1}\frac{y}{x}}$

(C) $xy = ke^{\frac{y}{x}\mathrm{Tan}^{-1}\frac{y}{x}}$

(D) $xy(x^2+y^2) = ke^{\frac{y}{x}}$

**(ii)** Solution of the equation $(x^3 - 2y^3)\,dx + 3xy^2dy = 0$ is

(A) $x^3 + y^3 = cx^2$ (B) $x^2 + y^2 = cx$

(C) $x^3y^3 = c\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$ (D) $\frac{1}{x^3}+\frac{1}{y^3} = cx^2$

**(iii)** Solution of the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ is

(A) $x^2+y^2+2\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = c$

(B) $\log(x^2+y^2)+2\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = c$

(C) $\tan(y^2+x^2) = \log\left(\frac{y}{x}\right)+c$

(D) $x^2+y^2 = x^2y^2\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)+c$

**Solution:**

**(i)** Put $y = vx$ so that

$$v+x\frac{dv}{dx} = \frac{dy}{dx}$$

Therefore

$$\left[x\left(v+x\frac{dv}{dx}\right)-vx\right]\mathrm{Tan}^{-1}v = x$$

$$\Rightarrow \left(v+x\frac{dv}{dx}-v\right)\mathrm{Tan}^{-1}v = 1$$

$$\Rightarrow (\mathrm{Tan}^{-1}v)dv = \frac{dx}{x}$$

Integrating we get

$$\int(\mathrm{Tan}^{-1}v)dv = \int\frac{dx}{x}+c$$

$$\Rightarrow v\ \mathrm{Tan}^{-1}v - \int\frac{v}{1+v^2}\,dv = \log x + c$$

$$\Rightarrow v\,\mathrm{Tan}^{-1}v - \frac{1}{2}\log(1+v^2) = \log x + c$$

$$\Rightarrow \frac{y}{x}\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log(x^2+y^2)+\log x = \log x + c$$

$$\Rightarrow \frac{1}{2}\log(x^2+y^2) = \frac{y}{x}\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)-c$$

$$\Rightarrow \sqrt{x^2+y^2} = ke^{\frac{y}{x}\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)}$$

**Answer: (A)**

**(ii)** We have

$$(x^3-2y^3)dx+3xy^2dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^3-x^3}{3xy^2} \quad \text{(Homogeneous)}$$

Put $y = vx$. Then

$$v+x\frac{dv}{dx} = \frac{2v^3-1}{3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v^3-1}{3v^2}-v = \frac{-(1+v^3)}{3v^2}$$

$$\Rightarrow \left(\frac{3v^2}{1+v^3}\right)dv+\frac{dx}{x} = 0$$

The solution is

$$\int\frac{3v^2}{1+v^3}\,dv+\int\frac{dx}{x} = c$$

$$\Rightarrow \log(1+v^3)+\log x = c$$

$$\Rightarrow \log(1+v^3)x = c$$

$$\Rightarrow (1+v^3)x = c$$

$$\Rightarrow x^3+y^3 = cx^2$$

**Answer: (A)**

**(iii)** We have

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

Put $y = vx$. Then

$$v+x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1}-v = \frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \left(\frac{1+v}{1+v^2}\right)dv+\frac{dx}{x} = 0$$

$$\Rightarrow \int\left(\frac{1+v}{1+v^2}\right)dv+\int\frac{dx}{x} = c$$

$$\Rightarrow \mathrm{Tan}^{-1}v+\frac{1}{2}\log(1+v^2)+\log x = c$$

$$\Rightarrow \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}\log(x^2+y^2)-\log x+\log x = c$$

$$\Rightarrow 2\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)+\log(x^2+y^2) = c$$

**Answer: (B)**

**6. Passage:** The general solution of the differential equation

$$\frac{dy}{dx}+Py = Q$$

where $P$ and $Q$ are functions of $x$ alone is given by

$$y\,e^{\int P dx} = \int Q\,e^{\int P dx}dx + c$$

Answer the following questions.

**(i)** Solution of the differential equation

$$x\left(\frac{dy}{dx}-y\right) = (1+x^2)e^x$$

is

(A) $ye^{-x} = \log|x| + \frac{x^2}{2} + c$

(B) $ye^{x} = \log|x| + \frac{x^2}{2} + c$

(C) $y = e^{x^2}\left(\log|x| + \frac{x^2}{2}\right) + ce^{x^2}$

(D) $ye^{-x^2} = \log|x| + \frac{x^2}{2} + c$

**(ii)** Solution of the equation $x(1 + x^2)dy = (y + yx^2 - x^2)dx$ is

(A) $y = x\tan x + c$ (B) $y = x\text{Tan}^{-1} x + c$

(C) $y = -x\text{Tan}^{-1} x + cx$ (D) $xy = \text{Tan}^{-1}x + \frac{c}{x}$

**(iii)** General solution of the differential equation

$$(x+1)\frac{dy}{dx} - ny = e^x(x+1)^{n+1}$$

is

(A) $y = e^x(x+1)^n + c$

(B) $y(1+x)^n = e^x + c$

(C) $y = (c + e^x)(1+x)^n$

(D) $y(c + e^x) = (1+x)^n$

## Solution

**(i)** We have

$$x\left(\frac{dy}{dx} - y\right) = (1+x^2)e^x$$

$$\Rightarrow \frac{dy}{dx} - y = \frac{(1+x^2)e^x}{x} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int -1dx} = e^{-x}$$

The solution is

$$ye^{-x} = \int \frac{(1+x^2)e^x}{x} \cdot e^{-x}dx + c$$

$$= \int \frac{1+x^2}{x}dx + c$$

$$= \log|x| + \frac{x^2}{2} + c$$

**Answer: (A)**

**(ii)** We have

$$x(1+x^2)dy = (y + yx^2 - x^2)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(1+x^2) - x^2}{x(1+x^2)}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{-x}{1+x^2} \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

The solution is

$$y\left(\frac{1}{x}\right) = \int \frac{-x}{1+x^2} \cdot \frac{1}{x}dx + c = -\text{Tan}^{-1}x + c$$

$$\Rightarrow y = -x\text{Tan}^{-1}x + cx$$

**Answer: (C)**

**(iii)** We have

$$(x+1)\frac{dy}{dx} - ny = e^x(x+1)^{n+1}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{n}{x+1}\right)y = e^x(x+1)^n \quad \text{(Linear)}$$

The integrating factor is

$$\text{I.F.} = e^{\int \frac{-n}{x+1}dx} = e^{-n\log(x+1)} = \frac{1}{(x+1)^n}$$

The solution is

$$y\left(\frac{1}{(x+1)^n}\right) = \int \frac{e^x(x+1)^n}{(x+1)^n}dx = e^x + c$$

$$\Rightarrow y = (e^x + c)(1+x)^n$$

**Answer: (C)**

## *Integer Answer Type Questions*

**1.** The value of $\int_0^1 xe^x dx$ is _______.

**Solution:** We have

$$\int_0^1 xe^x dx = \left[e^x(x-1)\right]_0^1$$

$$= 0 - e^0(0-1)$$

$$= 1$$

**Answer: 1**

**2.** $3\int_0^{\pi/2} \sin^3 x\, dx$ equals _______.

**Solution:** We have

$$\int_0^{\pi/2} \sin^3 x dx = \int_0^{\pi/2} \left(\frac{3\sin x - \sin 3x}{4}\right) dx$$
$$= \frac{1}{4}\left[-3[\cos x]_0^{\pi/2} + \frac{1}{3}[\cos 3x]_0^{\pi/2}\right]$$
$$= \frac{1}{4}\left[-3(0-1) + \frac{1}{3}(0-1)\right]$$
$$= \frac{1}{4}\left(3 - \frac{1}{3}\right) = \frac{8}{12} = \frac{2}{3}$$

Therefore

$$3\int_0^{\pi/2} \sin^3 x\, dx = 2$$

**Answer: 2**

**Try it out** If we use Walli's formula, then

$$\int_0^{\pi/2} \sin^3 xdx = \frac{(3-1)}{3(3-2)} = \frac{2}{3}$$

**3.** $\int_0^1 \mathrm{Tan}^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is equal to ______.

**Solution:** We have

$$\int_0^1 \mathrm{Tan}^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx = \int_0^1 \mathrm{Tan}^{-1}\left(\frac{x+x-1}{1-x(x-1)}\right) dx$$
$$= \int_0^1 \mathrm{Tan}^{-1} xdx + \int_0^1 \mathrm{Tan}^{-1}(x-1)dx$$
$$= \int_0^1 \mathrm{Tan}^{-1} xdx + \int_0^1 \mathrm{Tan}^{-1}(1-x-1)dx$$
$$= \int_0^1 \mathrm{Tan}^{-1} xdx - \int_0^1 \mathrm{Tan}^{-1} xdx$$
$$= 0$$

**Answer: 0**

**4.** The area bounded by the curves $y = \log_e(x+e)$, $y = e^{-x}$ and the $x$-axis is _______.

**Solution:** The two given curves intersect in $(0, 1)$ (see Fig. 5.58). So

$$\log_e(x+e) \to +\infty \text{ as } x \to +\infty$$

and $$y = e^{-x} \to 0 \text{ as } x \to \infty$$

Now $y = \log_e(x+e)$ meets $x$-axis in $(1-e, 0)$. Therefore the required area (shaded part) is

$$\int_{1-e}^{0} \log_e(x+e)\, dx + \int_0^{\infty} e^{-x} dx = [x \log_e(x+e)]_{1-e}^{0}$$
$$- \int_{1-e}^{0} \frac{x}{x+e} dx - [e^{-x}]_0^{\infty}$$
$$= -\int_{1-e}^{0}\left(1 - \frac{e}{x+e}\right) dx - (\lim_{x\to\infty} e^{-x} - 1)$$
$$= -[x]_{1-e}^{0} + e[\log(x+e)]_{1-e}^{0} - (0-1)$$
$$= -[0-(1-e)] + e[1-0] + 1$$
$$= 1 - e + e + 1$$
$$= 2$$

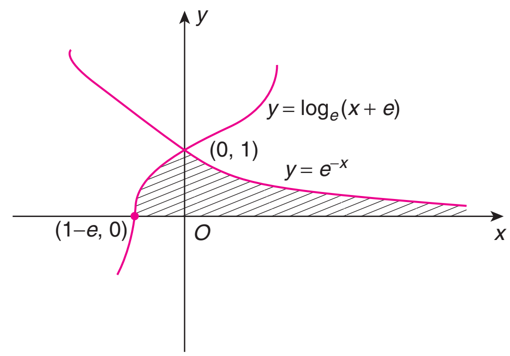



**FIGURE 5.58** Integer answer type question 4.

**Answer: 2**

**5.** If $A$ is the area bounded between the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$, then the integer part of $A$ is ______.

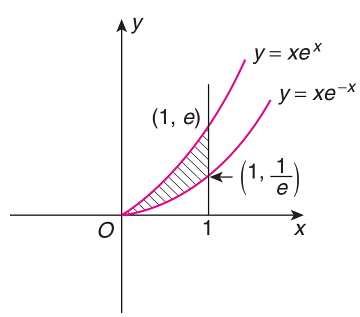



**FIGURE 5.59** Integer answer type question 5.

**Solution:** The two curves meet at $(0, 0)$ and are intersected by the line $x = 1$ in $(1, e)$ and $(1, 1/e)$, respectively. Hence the required area is

$$\int_0^1 (xe^x - xe^{-x})dx = \int_0^1 xe^x dx - \int_0^1 xe^{-x}\, dx$$
$$= [e^x(x-1)]_0^1 - \left\{[x(-e^{-x})]_0^1 - \int_0^1 -e^{-x} dx\right\}$$

$$= [0-(-1)] - \left\{ -\left[ xe^{-x} \right]_0^1 - \left[ e^{-x} \right]_0^1 \right\}$$

$$= 1 + \left\{ \left( \frac{1}{e} - 0 \right) + \left( \frac{1}{e} - 1 \right) \right\}$$

$$= 1 + \frac{2}{e} - 1$$

$$= \frac{2}{e}$$

Therefore, $A = 2/e$, which implies that the integer part of $A$ is zero.

**Answer: 0**

**6.** If $[x]$ denotes the greatest integer not exceeding $x$, then

$$\int_{-1}^{3} (|x-2| + [x])\, dx =$$

**Solution: Let**

$$I = \int_{-1}^{3} (|x-2| + [x])\, dx$$

$$I_1 = \int_{-1}^{3} |x-2|\, dx$$

$$I_2 = \int_{-1}^{3} [x]\, dx$$

Now

$$I_1 = \int_{-1}^{2} (2-x)\, dx + \int_{2}^{3} (x-2) dx$$

$$= 2[x]_{-1}^{2} - \frac{1}{2}\left[ x^2 \right]_{-1}^{2} + \frac{1}{2}\left[ x^2 \right]_{2}^{3} - [2x]_2^3$$

$$= 2(2+1) - \frac{1}{2}(4-1) + \frac{1}{2}(9-4) - (6-4)$$

$$= 6 - \frac{3}{2} + \frac{5}{2} - 2$$

$$= \frac{17}{2} - \frac{7}{2} = 5$$

$$I_2 = \int_{-1}^{3} [x]\, dx$$

$$= \int_{-1}^{0} (-1)\, dx + \int_{0}^{1} 0 dx + \int_{1}^{2} 1 dx + \int_{2}^{3} 2 dx$$

$$= -(0+1) + 0 + (2-1) + 2(3-2)$$

$$= -1 + 1 + 2 = 2$$

Therefore

$$I = I_1 + I_2 = 5 + 2 = 7$$

**Answer: 7**

**7.** Let $f(x) = \begin{cases} x^2 & \text{for} \quad 0 \le x < 1 \\ \sqrt{x} & \text{for} \quad 1 \le x \le 2 \end{cases}$

If $K = \int_0^2 f(x)\, dx$, then $[K+1]$ (where $[t]$ denotes the integer part of $t$) is ______.

**Solution:** We have

$$\int_0^2 f(x) dx = \int_0^1 x^2 dx + \int_1^2 \sqrt{x} dx$$

$$= \frac{1}{3}\left[ x^3 \right]_0^1 + \frac{2}{3}\left[ x^{3/2} \right]_1^2$$

$$= \frac{1}{3} + \frac{2}{3}(2\sqrt{2} - 1)$$

$$= \frac{4\sqrt{2} - 1}{3}$$

Therefore

$$K + 1 = \frac{4\sqrt{2} + 2}{3}$$

$$\Rightarrow [K+1] = 2$$

**Answer: 2**

**8.** If $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x dx}$ , then the value of $3I$ is ______.

**Solution:** We have

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x} dx$$

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x - \cos^3 x} \quad (\because \cos\, x - \cos^3 x \text{ is an even function})$$

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x} |\sin x| dx$$

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x} \sin x \quad \left( \because \text{in} \left( 0, \frac{\pi}{2} \right), \sin x \ge 0 \right)$$

$$= 2\left( -\frac{2}{3} \right) \left[ (\cos x)^{3/2} \right]_0^{\pi/2}$$

$$= -\frac{4}{3}[0-1] = \frac{4}{3}$$

This implies $3I = 4$.

**Answer: 4**

**9.** If $[t]$ denotes the integer part of $t$, then

$$\left[\int_0^1 \mathrm{Sin}^{-1} x\,dx\right] =$$

**Solution:** We have

$$\int_0^1 \mathrm{Sin}^{-1} x\,dx = \left[x\,\mathrm{Sin}^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$
$$= \mathrm{Sin}^{-1} 1 + \left[\sqrt{1-x^2}\right]_0^1$$
$$= \frac{\pi}{2} + (0-1)$$
$$= \frac{\pi}{2} - 1$$

Therefore

$$\left[\int_0^1 \mathrm{Sin}^{-1} x\,dx\right] = 0$$

**Answer: 0**

**10.** If $K = \int_{-1}^{1} 5x^4\sqrt{x^5+1}\;dx$, then the integral part of $K$ is ______.

**Solution:** We have

$$K = \int_{-1}^{1} 5x^4\sqrt{x^5+1}\;dx$$
$$= \frac{2}{3}\left[(x^5+1)^{3/2}\right]_{-1}^{1}$$
$$= \frac{2}{3}\left[2\sqrt{2}-0\right]$$
$$= \frac{4\sqrt{2}}{3}$$

Therefore

$$[K] = \left[\frac{4\sqrt{2}}{3}\right] = 1$$

**Answer: 1**

**11.** Let $f(x) = \mathrm{Min}\{x^2+1, x+1\}$ for $0 \le x \le 2$. If $A$ is the area bounded by $y = f(x)$, the $x$-axis in the interval $[0, 2]$, then the integral part of $A$ is ______.

**Solution:** The line $y = x + 1$ and the curve $y = x^2 + 1$ cut each other in the points $(0, 1)$ and $(1, 2)$ (see Fig. 5.60). Now

$$0 \le x \le 1 \Rightarrow x^2 + 1 < x + 1$$

and for $x > 1$, $x + 1 < x^2 + 1$. Therefore

$$f(x) = \begin{cases} x^2+1 & \text{for } 0 \le x \le 1 \\ x+1 & \text{for } 1 < x \le 2 \end{cases}$$

So

$$A = \text{Required area (shaded portion)}$$
$$= \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx$$
$$= \frac{1}{3}\left[x^3\right]_0^1 + [x]_0^1 + \frac{1}{2}\left[x^2\right]_1^2 + [x]_1^2$$
$$= \frac{1}{3} + 1 + \frac{1}{2}(4-1) + (2-1)$$
$$= \frac{23}{6}$$

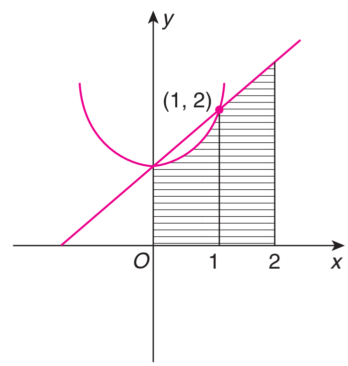



**FIGURE 5.60** Integer answer type question 11.

**Answer: 3**

**12.** The cut off from the parabola $4y = 3x^2$ by the straight line $2y = 3x + 12$ is ______.

**Solution:** The line intersects the parabola $4y = 3x^2$ in the points $(-2, 3)$ and $(4, 12)$ (see Fig. 5.61). Therefore

$$\text{Required area} = \int_{-2}^{4}\left(\frac{3x+12}{2} - \frac{3}{4}x^2\right)dx$$
$$= \frac{3}{4}\left[x^2\right]_{-2}^{4} + 6[x]_{-2}^{4} - \frac{1}{4}\left[x^3\right]_{-2}^{4}$$
$$= \frac{3}{4}(16-4) + 6(4+2) - \frac{1}{4}(64+8)$$

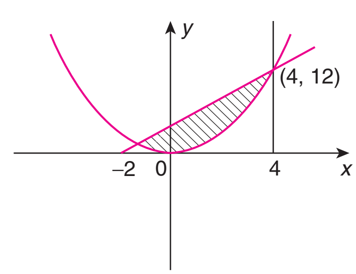



**FIGURE 5.61** Integer answer type question 12.

$$= 9 + 36 - 18$$
$$= 27$$

**Answer: 27**

**13.** The area bounded by the curves $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant is ______.

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**Solution:**

The line and the curve intersect at $(9, 3)$ in the first quadrant (see Fig. 5.62). Therefore

$$\text{Required area} = \int_0^9 \sqrt{x}\, dx - \text{Area of the triangle with vertices } (3,0), (9,0) \text{ and } (9,3)$$
$$= \frac{2}{3}\left[x^{3/2}\right]_0^9 - \frac{1}{2} \times 6 \times 3$$
$$= \frac{2}{3}(3^3) - 9$$
$$= 9$$

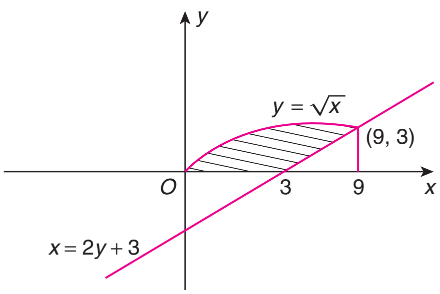


**FIGURE 5.62** Integer answer type question 13.

**Answer: 9**

**14.** The area bounded by the curve $3y^2 = x^2(3 - x^2)$ is ______.

**Solution:** We know that

**(i)** The curve is symmetric about both axes.
**(ii)** It passes through origin.
**(iii)** It meets $x$-axis in $(-\sqrt{3}, 0)$, $(0, 0)$ and $(\sqrt{3}, 0)$.
**(iv)** $-\sqrt{3} \le x \le \sqrt{3}$

Therefore shape of the curve is as shown in Fig. 5.63. Therefore

$$\text{Required area} = 4\int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\sqrt{3 - x^2}\, dx$$
$$= \frac{4}{\sqrt{3}}\left(\frac{-1}{2}\right)\int_0^{\sqrt{3}} (-2x)\sqrt{3 - x^2}\, dx$$
$$= \frac{-2}{\sqrt{3}} \times \frac{2}{3}\left[(3 - x^2)^{3/2}\right]_0^{\sqrt{3}}$$
$$= \frac{-4}{3\sqrt{3}}(0 - 3\sqrt{3})$$
$$= 4$$

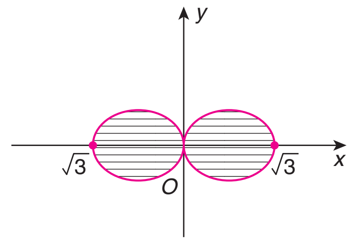


**FIGURE 5.63** Integer answer type question 14.

**Answer: 4**

**15.** If $A$ is the area bounded by the curves $y = x \log_e x$, $y = 2x - 2x^2$, then $12A$ is _____.

**Solution:** We have

$$y = 2x - 2x^2 = 2\left[\frac{1}{4} - \left(x - \frac{1}{2}\right)^2\right]$$

Then

$$\left(x - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{y}{2} = -\frac{1}{2}\left(y - \frac{1}{2}\right)$$

which is a downward parabola with vertex at (1/2, 1/2) meeting $x$-axis in $(0, 0)$ and $(1, 0)$. The curve $y = x \log_e x$ meets the $x$-axis in $(1, 0)$. Further

$$\lim_{x \to +\infty} x \log_e x = +\infty$$

and
$$\lim_{x \to 0+} x \log x = 0$$

Now the required area is (see Fig. 5.64)

$$A = \int_0^1 (2x - 2x^2)\, dx + \int_0^1 -x \log_e x\, dx \quad (\because 0 < x < 1 \Rightarrow \log_e x < 0)$$

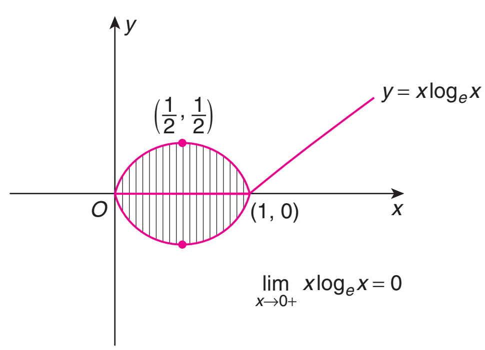


**FIGURE 5.64** Integer answer type question 15.

$$=\left[x^2\right]_0^1-\frac{2}{3}\left[x^3\right]_0^1-\left\{\left[\frac{x^2}{2}\log_e x\right]_0^1-\left[\frac{x^2}{4}\right]_0^1\right\}$$

$$=1-\frac{2}{3}-\left\{(0-0)-\frac{1}{4}\right\}$$

$$=\frac{1}{3}+\frac{1}{4}$$

$$=\frac{7}{12}$$

Therefore $12A = 7$.

**Answer: 7**

**16.** The value of

$$\int_0^{\pi/2}(\sin x)^{\cos x}[\cos x\cot x-\log(\sin x)^{\sin x}]dx$$

is ______. **(IIT-JEE 2006)**

**Solution:** Let $f(x)=(\sin x)^{\cos x}$, $0\le x\le \pi/2$ so that

$$f'(x)=(\sin x)^{\cos x}[\cos x\cot x-\sin x\log\sin x]\ \text{(check it)}$$

Therefore

$$\int_0^{\pi/2} f'(x)dx=[f(x)]_0^{\pi/2}=1^0-0^1=1$$

**Answer: 1**

**17.** The value of $(5050)\dfrac{\int_0^1(1-x^{50})^{100}dx}{\int_0^1(1-x^{50})^{101}dx}$ is ______.

**Solution: Let**

$$I=(5050)\frac{\int_0^1(1-x^{50})^{100}dx}{\int_0^1(1-x^{50})^{101}dx}$$

Let

$$I_1=\int_0^1(1-x^{50})^{100}dx$$

$$I_2=\int_0^1(1-x^{50})^{101}dx$$

Therefore

$$I_2=\int_0^1(1-x^{50})(1-x^{50})^{100}dx$$

$$=I_1-\int_0^1 x^{50}(1-x^{50})^{100}dx$$

$$=I_1+\int_0^1 x(1-x^{50})^{100}(-x^{49})dx$$

$$=I_1+\left\{\left[\frac{x(1-x^{50})^{101}}{101\times 50}\right]_0^1-\int_0^1\frac{(1-x^{50})^{101}}{101\times 50}dx\right\}$$

$$=I_1+0-\frac{1}{5050}I_2$$

Hence

$$\left(\frac{5051}{5050}\right)I_2=I_1$$

So

$$I=\frac{5050\ I_1}{I_2}=5051$$

**Answer: 5051**

**18.** Let

$$y(x)=\int_{\pi^2/16}^{x^2}\frac{\cos x\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta$$

then

$$\left(\frac{dy}{dx}\right)_{x=\pi}=K\pi$$

where the value of $K$ is ______.

**Solution:** By Leibnitz Rule,

$$\frac{dy}{dx}=-\sin x\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta+\cos x\left(\frac{\cos x}{1+\sin^2 x}\times 2x\right)$$

Therefore

$$\left(\frac{dy}{dx}\right)_{x=\pi}=0+(-1)\left(\frac{-1}{1}\times 2\pi\right)=2\pi$$

So $K=2$.

**Answer: 2**

**19.** Let $f(x)=\int_1^x\sqrt{2-t^2}\,dt$. Then, the number of real roots of the equation $x^2-f'(x)=0$ is ______.

**(IIT-JEE 2002)**

**Solution:** We have

$$f'(x) = \sqrt{2 - x^2} \quad \text{(By Leibnitz Rule)}$$

Now

$$\begin{aligned} & x^2 - f'(x) = 0 \\ \Rightarrow\; & x^4 = 2 - x^2 \\ \Rightarrow\; & x^4 + x^2 - 2 = 0 \\ \Rightarrow\; & (x^2+2)(x^2-1) = 0 \\ \Rightarrow\; & x = \pm 1 \text{ are the real values of } x \end{aligned}$$

Therefore the number of real roots of the equation $x^2 - f'(x) = 0$ is 2.

**Answer: 2**

**20.** Let $[x]$ denote the largest integer not exceeding $x$ and $\{x\} = x - [x]$. Then the value of

$$\int_0^{2012} \frac{e^{\cos(\pi\{x\})}}{e^{\cos(\pi\{x\})} + e^{-\cos(\pi\{x\})}}\,dx$$

is ______.

**Solution:** Let

$$f(x) = \frac{e^{\cos(\pi\{x\})}}{e^{\cos(\pi\{x\})} + e^{-\cos(\pi\{x\})}}$$

Since the period of $\{x\}$ is 1, we have that the period of $f(x)$ is also 1. Therefore

$$\int_0^{2012} f(x)dx = (2012)\int_0^1 f(x)dx \quad (\text{See } P_4) \qquad (5.54)$$

Now, let

$$\begin{aligned} I &= \int_0^1 f(x)dx \\ &= \int_0^1 \frac{e^{\cos(\pi(x-[x])}}{e^{\cos\pi(x-[x])} + e^{-\cos(x-[x])\pi}}\,dx \\ &= \int_0^1 \frac{e^{\cos\pi x}}{e^{\cos\pi x} + e^{-\cos\pi x}}\,dx \\ &= \int_0^1 \frac{e^{\cos\pi(1-x)}}{e^{\cos\pi(1-x)} + e^{-\cos\pi(1-x)}}\,dx \\ &= \int_0^1 \frac{e^{-\cos\pi x}}{e^{-\cos\pi x} + e^{\cos\pi x}}\,dx \end{aligned}$$

Therefore

$$2I = \int_0^1 \frac{e^{\cos\pi x} + e^{-\cos\pi x}}{e^{\cos\pi x} + e^{-\cos\pi x}}\,dx = \int_0^1 dx = 1$$

So

$$I = \frac{1}{2}$$

From Eq. (5.54), we get

$$\int_0^{2012} f(x)dx = 2012\left(\frac{1}{2}\right) = 1006$$

**Answer: 1006**

**21.** If $f$ is differentiable on $[0, 1]$, $f(0) = 0$ and $f(1) = 1$, then the minimum value of $\int_0^1 [f'(x)]^2\,dx$ is equal to ______.

**Solution:** We have

$$\begin{aligned} & [f'(x)-1]^2 \geq 0 \\ \Rightarrow\; & 0 \leq \int_0^1 [f'(x)-1]^2\,dx \\ & = \int_0^1 [\{f'(x)\}^2 - 2f'(x) + 1]dx \\ & = \int_0^1 [f'(x)]^2 dx - 2\int_0^1 f'(x)dx + \int_0^1 1dx \\ & = \int_0^1 [f'(x)]^2 dx - 2[f(1) - f(0)] + 1 \\ & = \int_0^1 [f'(x)]^2 dx - 2(1-0) + 1 \end{aligned}$$

Therefore

$$\int_0^1 [f'(x)]^2 dx \geq 1$$

Hence the minimum value of $\int_0^1 [f'(x)]^2 dx$ is 1 and the minimum value will be attained when $f(x) = x \;\forall\; x \in [0, 1]$

**Answer: 1**

## EXERCISES

**1.** Evaluate $\int_0^{\pi} \|\sin x| - |\cos x\|\, dx$.

**Hint:** Draw the graphs of both $\sin x$ and $\cos x$ in $[0, \pi]$.

**2.** $f$ is a real-valued function defined for all $x \geq 1$ such that

$$f(1) = 1 \text{ and } f'(x) = \frac{1}{x^2 + [f(x)]^2}$$

Show that $\lim\limits_{x\to\infty} f(x)$ exists and is less than $1 + \pi/4$.

**Hint:** Observe that $f$ is strictly increasing on $[1, \infty)$ and $\int_1^t \frac{dx}{x^2 + [f(x)]^2} < \int_1^t \frac{dx}{1+x^2}$

**3.** Let $f$ be differentiable function such that

$$f'(x) = f(x) + \int_0^2 f(x)dx$$

and $$f(0) = \frac{4 - e^2}{3}$$

Determine $f(x)$.

**4.** Prove that

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2K-1)x = \frac{\sin^2 Kx}{\sin x}$$

where $K$ is a positive integer and hence show that

$$\int_0^{\pi/2} \frac{\sin^2 Kx}{\sin x} dx = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2K-1}$$

**5.** Evaluate $\int_0^{n/2} e^{2x - [2x]}\, dx$ where $[x]$ is the integral part of $x$ and $n$ is a positive integer.

**6.** Evaluate $\int_0^{100\pi} \sqrt{1 - \cos 2x}\, dx$.

**7.** Evaluate $\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2} dx$.

**8.** Evaluate $\int_0^3 \frac{x\,dx}{\sqrt{x+1} + \sqrt{5x+1}}$.

**9.** Show that $\int_0^{\pi/4} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$ $(a > 0, b > 0)$ is equal to $\frac{1}{ab}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right)$.

**10.** Show that $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8}\log 2$.

**Hint:** Put $x = \tan\theta$.

**11.** Evaluate $\int_3^{29} \frac{\sqrt[3]{(x-2)^2}}{3 + \sqrt[3]{(x-2)^2}} dx$.

**12.** Show that $\int_0^{\pi} \frac{\sin 2Kx}{\sin x} dx = 0$, if $K$ is an integer.

**13.** Prove that $\int_0^{\pi} f(\sin x) = 2\int_0^{\pi/2} f(\sin x) dx$.

**14.** Show that $\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$.

**15.** Evaluate $\int_{-\pi/8}^{\pi/8} x^8 \sin^9 x\, dx$.

**16.** Compute $\int_0^{\pi^2/4} \sin\sqrt{x}\, dx$.

**Hint:** Put $t = \sqrt{x}$.

**17.** If $f''(x)$ is continuous on $[a, b]$, then show that

$$\int_a^b x f''(x) dx = [b f'(b) - a f'(a)] + [f(a) - f(b)]$$

**18.** Let $f$ be an odd function in the interval $[-T/2, T/2]$ where $T$ is a period of $f$. Prove that $\int_a^x f(t)dt$ is also a periodic function with the same period.

**Hint:**

$$F(x) = \int_a^x f(t)dt$$

$$\Rightarrow F(x+T) = F(x) + \int_x^{T/2} f(t)\,dt + \int_{T/2}^{x+T} f(t)\, dt$$

**19.** If $n$ is a positive integer and $C_K$ denotes the Binomial coefficient ${}^nC_K$, then show that

$$C_0 - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + \cdots + \frac{(-1)^n C_n}{2n+1} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}$$

**Hint:** Compute $\int_0^1 (1-x^2)^n\,dx$ using Binomial expansion and also using the substitution $x = \sin\theta$.

**20.** On the interval [–1, 1] find the greatest and least values of the function

$$F(x) = \int_0^x \frac{2t+1}{t^2-2t+2}\,dt$$

**Hint:** $F$ is greatest at $x = 1$ and least at $x = -1/2$.

**21.** Solve the equation $\displaystyle\int_{\log_e 2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$.

**22.** Show that $\displaystyle\int_0^2 \frac{dx}{x\sqrt{x^2+5x+1}} = \log_e\left(\frac{7+2\sqrt{7}}{9}\right)$.

**23.** Show that $\displaystyle\int_1^{16} \mathrm{Tan}^{-1}\sqrt{\sqrt{x}-1}\;dx = \frac{16}{3}\pi - 2\sqrt{3}$.

**24.** Show that $\displaystyle\int_0^{\pi/2} \frac{dx}{3+2\cos x} = \frac{2}{\sqrt{5}}\mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{5}}\right)$.

**25.** If $f$ is an odd function, then show that $\displaystyle\int_a^x f(t)\,dt$ is an even function. Will $\displaystyle\int_a^x f(t)\;dt$ be odd, if $f$ is even?

**Hint:** For the latter part, consider $f(x) = \cos x$ and $a \neq 0$, so that the conclusion is not true. If $a = 0$, the conclusion is true.

**26.** Show that $\displaystyle\int_1^2 \frac{dx}{x+x^3} = \frac{1}{2}\log_e\frac{8}{5}$.

**27.** Show that $\displaystyle\int_0^{\sqrt[5]{2}} \frac{x^9}{(1+x^5)^3}\,dx = \frac{2}{45}$.

**28.** Prove that

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{dx}{(1+x^2)^{n-1}}$$

and hence compute $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^4}$.

**29.** Show that $\displaystyle\int_0^{1/\sqrt{3}} \frac{dx}{(2x^2+1)\sqrt{x^2+1}} = \mathrm{Tan}^{-1}\left(\frac{1}{2}\right)$.

**30.** Show that $\displaystyle\int_0^{-\log_e 2} \sqrt{1-e^{2x}}\,dx = \frac{\sqrt{3}}{2} + \log_e(2-\sqrt{3})$.

**31.** Evaluate $\displaystyle\int_{-\pi/2}^{-\pi/4} \frac{\cos^3 x}{\sqrt[3]{\sin x}}\,dx$ .

**32.** Compute $\displaystyle\int_0^{e-1} \log_e(1+x)\,dx$ .

**33.** Show that $\displaystyle\int_1^e (\log_e x)^3\,dx = 6-2e$.

**34.** Show that $\displaystyle\int_0^1 x\log(1+x^2)dx = \log_e 2 - \frac{1}{2}$.

**35.** Evaluate $\displaystyle\int_0^{\pi/4} (\cos 2x)^{3/2}\cos x\,dx$.

**Hint:** Put $\sqrt{2}\sin x = \sin\theta$.

**36.** If $\displaystyle\int_a^b \frac{x^n}{x^n+(16-x)^n}\;dx = 6,$ then prove that $a = 2$, $b = 14$ and $n$ is a natural number.

**37.** Show that $\displaystyle\int_0^1 \frac{\mathrm{Sin}^{-1}x}{x}\,dx = \frac{\pi}{2}\log_e 2$.

**38.** Evaluate $\displaystyle\int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx$.

**39.** Let $\displaystyle F(t) = \int_0^t \frac{x}{(1+x)(1+x^2)}\,dx$. Show that $\displaystyle\lim_{t\to\infty} F(t) = \frac{\pi}{4}$.

**40.** Show that $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\,dx = \frac{\pi}{2}(\pi-2)$.

**41.** Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin 8x\log(\cot x)}{\cos 2x}\,dx$ .

**42.** Evaluate $\displaystyle\int_{-1}^1 \frac{x\,\mathrm{Sin}^{-1}x}{\sqrt{1-x^2}}\,dx$ .

**43.** Show that $\int_{-2}^{2} \frac{3x^5+4x^3+2x^2+x+20}{x^2+4}\,dx = 3\pi + 8$.

**44.** Prove that $\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

**45.** The value of $\int_{-1}^{1} (\sqrt{1+x+x^2} - \sqrt{1-x+x^2})\,dx$ is ____.

**46.** Show that $F(x) = \int_{0}^{x} \log_e(t+\sqrt{1+t^2})\,dt$ is an even function.

**Hint:** Observe that $\log(t+\sqrt{1+t^2})$ is an odd function.

**47.** Show that $\int_{-1}^{3/2} |x\sin \pi x|\,dx = \frac{3\pi+1}{\pi^2}$.

**48.** If $f(x) = |2^x - 1| + |x-1|$, then show that

$$\int_{-2}^{2} f(x)dx = 5 + \frac{9}{4\log_e 2}$$

**49.** Show that

$$\int_{0}^{\pi} \frac{x}{(a^2\cos^2 x + b^2 \sin^2 x)^2}\,dx = \frac{\pi^2}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

where $a > 0, b > 0$.

**50.** Evaluate $\int_{0}^{\log_e 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx$.

**51.** If $S = \int_{0}^{\pi} \frac{\cos x}{(x+2)^2}\,dx$, then show that

$$\int_{0}^{\pi/2} \frac{\cos x \sin x}{(x+1)} = \frac{1}{4}\left(\frac{2}{2+\pi} + 1 - 2S\right)$$

**Hint:** Take $I = \int_{0}^{\pi/2} \frac{\cos x \sin x}{(x+1)}\,dx = \frac{1}{2}\int_{0}^{\pi/2} \frac{\sin 2x}{x+1} 2x.$ Use integration by parts.

**52.** If $F(x) = \int_{1/x}^{\sqrt{x}} \sin(t^2)dt$, then find $F'(1)$.

**53.** Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$.

**54.** Show that

$$\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \frac{n+3}{n^2+3^2} + \cdots + \frac{1}{n}\right] = \frac{\pi}{4} + \frac{1}{2}\log_e 2$$

**55.** Show that $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{e}$.

**56.** Show that

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{\frac{2}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}} \cdots \left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}\right] = \frac{4}{e}.$$

**Hint:** Assume that the limit is $l$ and take logarithm.

**57.** Show that

$$\lim_{n\to\infty}\left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \cdots + \frac{r^2}{r^3+n^3} + \cdots + \frac{1}{n}\right) = \frac{1}{3}\log_e 2.$$

**58.** Show that

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{1}{8n}\right) = \frac{3}{8}.$$

**59.** Show that $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \log 2$.

**60.** Show that $\lim_{n\to\infty}\sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2-r^2}} = \frac{\pi}{2}$.

**61.** Prove that the area common to the parabolas $y = 2x^2$ and $y = x^2 + 4$ is 32/3.

**62.** Show that the area included between the parabolas $y^2 = 4a(x+a)$ and $y^2 = 4b(x-b)$ is $\frac{8}{3}\frac{\sqrt{ab}(a+b)^{3/2}}{\sqrt{b-a}}$.

**63.** Compute the area bounded by the curves $y = x - 1$ and $(y-1)^2 = 4(x+1)$.

**64.** Find the area enclosed by the curves $3x^2 + 5y = 32$ and $y = |x-2|$.

**65.** Compute the area of the figure bounded by the parabola $(y-2)^2 = x - 1$, the tangent to it at the point with ordinate 3, and the $x$-axis.

**66.** Find the area of the figure which lies in the first quadrant inside the circle $x^2 + y^2 = 3a^2$ and bounded by the parabolas $x^2 = 2ay$ and $y^2 = 2ax (a > 0)$.

**67.** Compute the area given by $x + y = 6$, $x^2 + y^2 \le 6y$ and $y^2 \le 8x$.

**68.** A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at $P$ is $a(y-1) + x - 1 = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equations of the curve. Also obtain the area bounded by the $y$-axis, the curve and the normal to the curve at $P$.

**(IIT-JEE 1996)**

69. Find the area of the region bounded by the curve $C$: $y = \tan x$, the tangent drawn to $C$ at $x = \pi/4$ and the $x$-axis.

70. Compute the area enclosed between the parabolas $x = y^2, 3y^2 = 4(x - 1)$.

71. Find the area of the figure bounded by the curve $\sqrt{x} + \sqrt{y} = 1$ and the line $x + y = 1$.

72. Compute the area of the figure enclosed by the curve $y^2 = x^2 (1 - x^2)$.

73. Compute the area of the loop of the curve $y^2 = x^2 (1 + x)$.

74. Find the area of the figure contained between the parabola $x^2 = 4y$ and the curve $y = \dfrac{8}{x^2+4}$.

75. Compute the area of the figure bounded by the curve $y = \log_e x$, the $y$-axis and the straight lines $y = \log_e a$ and $y = \log_e b$.

76. Solve $(y^2 - 3x^2)dy + 2xy\, dx = 0$, given $y = 1$ when $x = 0$.

77. Solve $y\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} - y = 0$, given $y = \sqrt{5}$ when $x = 0$.

78. Find the curve such that the length of the polar radius of any point $M$ equals the distance between the point of intersection of the tangent at the point $M$, the $y$-axis, and the origin.

79. Solve $\dfrac{dy}{dx} = \dfrac{1}{2x - y^2}$.

80. Solve $(3y^2 + 3xy + x^2)dx = (x^2 + 2xy)dy$.

81. Solve $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x}$.

82. Solve $(2y)dx + (y^2 - 6x)dy = 0$.

83. Solve $\dfrac{dy}{dx} = \dfrac{e^y}{x^2} - \dfrac{1}{x}$.

84. Solve $\dfrac{dy}{dx} + 2y\tan x = \sin x$.

85. Solve $\dfrac{dy}{dx} - y\tan x = \dfrac{\sin x \cos^2 x}{y^2}$.

86. Solve $\dfrac{dy}{dx} + \left(\dfrac{x}{1-x^2}\right)y = x\sqrt{y}$.

87. Solve $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$.

88. Solve $x^2 y - x^3\dfrac{dy}{dx} = y^4 \cos x$, given that $y = \pi$ when $x = \pi$.

89. Solve $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$.

90. The gradient of a curve which passes through the point $(4, 0)$ is defined by the equation

$$\frac{dy}{dx} - \frac{y}{x} + \frac{5x}{(x+2)(x-3)} = 0$$

Find the equation of the curve and also the value of $y$ when $x = 5$.

91. Solve $\dfrac{dy}{dx} + (y - 1)\cos x = e^{-\sin x}(\cot^2 x)$.

92. Find the equation of the curve through the origin which satisfies the differential equation

$$\frac{dy}{dx} = (x - y)^2$$

93. Solve $\dfrac{dy}{dx} - xy^2 = \left(\dfrac{x-1}{x}\right)y$.

94. Solve $\left(\dfrac{x+y}{y^2}\right)\dfrac{dy}{dx} = \cos x + \dfrac{1}{y}$.

**Hint:** Put $y = vx$.

95. Solve $\dfrac{dy}{dx} - x\tan(y - x) = 1$.

96. Solve $\dfrac{dy}{dx} + \dfrac{y^2 + y + 1}{x^2 + x + 1} = 0$.

97. Solve $x^2 y\dfrac{dy}{dx} = (x + 1)\sec y$.

98. Form the differential equation representing of all parabolas having latus rectum $4a$ and whose axes are parallel to $x$-axis.

99. Form the differential equation representing all circles in the $xy$-plane with fixed radius $r$.

100. Solve $\sqrt{x^2y^2 + x^2 + y^2 + 1} + (xy)\dfrac{dy}{dx} = 0$.

## ANSWERS

**1.** $4(\sqrt{2}-1)$

**3.** $f(x)=e^x-\dfrac{e^2-1}{3}$

**5.** $\dfrac{n}{2}(e-1)$

**6.** $200\sqrt{2}$

**7.** 1

**8.** 14/15

**11.** $8+\dfrac{3\sqrt{3}}{2}\pi$

**15.** 0

**16.** 2

**21.** $x=2\log_e 2$

**28.** $\dfrac{11}{48}+\dfrac{5\pi}{64}$

**31.** –0.083

**32.** 1

**35.** $\dfrac{3\pi}{16\sqrt{2}}$

**38.** $\dfrac{\pi}{8}\log_e 2$

**41.** 0

**42.** 2

**45.** 0

**50.** $4-\pi$

**52.** $\dfrac{3}{2}\sin 1$

**53.** $\sqrt{5}-1$

**63.** 64/3

**64.** 33/2

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**95.** $\sin(y-x)=ce^{x^2/2}$

**96.** $\mathrm{Tan}^{-1}\frac{2x+1}{\sqrt{3}}+\mathrm{Tan}^{-1}\frac{2y+1}{\sqrt{3}}=c$

**97.** $y\sin y+\cos y=\log_e x-\frac{1}{x}+c$

**98.** $2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$

**99.** $r^2\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

**100.** $\sqrt{1+x^2}+\sqrt{1+y^2}+\frac{1}{2}\log_e\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|=c$

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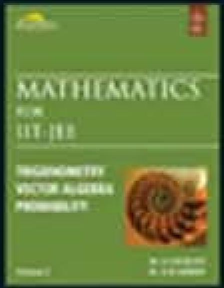


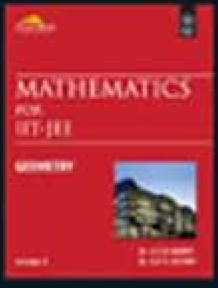


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ISBN 978-81-265-2184-5

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